Stochastic Modelling and Computational Sciences
Vol. 3 No. 2 (December, 2023)

METHODS FOR SOLVING CONTEMPORARY COMPUTATIONAL FINANCE PROBLEMS: APPLYING LÉVY MODELS AND MACHINE LEARNING

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Abstract. In the paper, we survey modern numerical methods for solving problems that arise in contemporary computational finance. We focus on option pricing problems in Lévy models of financial markets. We consider several categories of numerical techniques: backward induction methods, methods for solving partial integrodifferential equations (PIDEs), approximate Wiener-Hopf factorization methods, and Monte Carlo simulations. We briefly review the main ideas of the methods of each group and give extended details to the most efficient approaches that combine several numerical techniques. Special attention is paid to a new promising category of hybrid methods that combine standard numerical methods and machine learning tools. We justify that the core of the latter category should be computational techniques that admit the interpretability of artificial neural networks.

1. Introduction

In recent years, more and more attention has been paid to stochastic models of financial markets that depart from the traditional Black-Scholes model. At this time, a wide range of models are available. One of the tractable empirical models are jump diffusions or, more generally, Lévy processes. For an introduction to these applied models to finance, we refer to [19, 27]. More recently, Lévy processes have also been extensively used in modern insurance risk theory; see, for example, [5] and [74]. In insurance mathematics, it is the Lévy process itself that models the surplus wealth of an insurance company until its ruin. There are also extensive applications of Lévy processes in queuing theory, genetics, and mathematical biology as well as in stochastic differential equations (see, e.g. [23], [66]). A detailed review of recent developments in applications of Lévy processes can be found in the book [90], which provides an extensive bibliography on the subject. We restrict ourselves to the one-dimensional case.

Option pricing has a crucial priority in mathematical finance. Recall that options are financial derivatives that give their holders a right (not an obligation)

Date: Date of Submission September 01, 2023; Date of Acceptance September 25, 2023; Communicated by Igor V. Pavlov.

2000 *Mathematics Subject Classification.* Primary 60-08; Secondary 60G51.

Key words and phrases. Computational finance, numerical methods, option pricing, partial integro-differential equations, Lévy processes, Wiener-Hopf factorization, Fourier transform, Laplace transform, machine learning, neural networks, exotic options.

* This research is supported by Russian Science Foundation Grant No.23-21-00474.

to buy or sell the underlying asset under certain conditions (see details in [59]). Exotic options form a broad class of derivatives that differ from classic American and European options. Exotic derivatives include path-dependent options such as lookback or barrier options. The value of exotic continuous-time derivatives depends on the underlying asset's path, provided the price monitoring is permanent. From a probabilistic viewpoint, exotic option prices can be expressed in terms of conditional expectation on a payoff function that depends on the underlying stochastic process and its extrema. However, joint probability distributions of modeling process and its supremum and(or) infimum are unknown in explicit form as well as distributions of Lévy extrema themselves.

From an analytical point of view, the option pricing problem under Lévy processes can be reduced to solving a partial integrodifferential backward Kolmogorov equation [46] subject to appropriate boundary and initial conditions.

By now, there exist several large groups of relatively universal numerical methods for pricing path-dependent options under Lévy driven financial models. The number of publications in the field is so large that it is virtually impossible to make a full reference list. However, due to its complexity, pricing exotic options in exponential Lévy models remains a computational challenge.

Contemporary numerical methods in the literature can be categorized into several groups: backward induction methods, methods for solving partial integrodifferential equations (PIDEs), approximate Wiener-Hopf factorization methods, and Monte Carlo simulations. We also mention a new category of hybrid methods that combine standard numerical techniques and machine learning tools.

The remainder of the paper is organized as follows. In Section 2 we list the necessary facts of the theory of Lévy processes. The backward induction methods are reviewed in Section 3 with details of powerful tools of frame projection method for option pricing in Subsection 3.2. Section 4 deals with numerical methods for solving partial integrodifferential equations arisen in finance. Section 5 reviews the approximate Wiener–Hopf factorization methods with special attention given to the recently developed "Simplified Wiener–Hopf factorization method" in Subsection 5.3. Section 6 focuses on advanced Monte Carlo algorithms for pricing options under general Lévy models including numerical methods that combine the approximate Wiener–Hopf factorization technique and Monte Carlo simulations. In Section 7 we give a short introduction to machine learning methods and consider possibilities to combine standard numerical techniques with artificial neural networks (ANN). Special attention is paid to the interpretability of ANNs in the development of such hybrid methods in Subsection 7.2. Section 8 concludes.

2. Lévy processes: basic facts

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see, e.g. [11, 27, 113]). A Lévy process may have a Gaussian component and/or a pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$. If X_t has probability density p_t, then we

have

$$e^{-t\psi(\xi)} = \int_{-\infty}^{+\infty} e^{i\xi y} p_t(y) dy \tag{2.1}$$

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y|\leq 1}) F(dy), \tag{2.2}$$

where $\sigma^2 \geq 0$ is the variance of the Gaussian component, and the Lévy measure $F(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\} F(dy) < +\infty. \tag{2.3}$$

If the jump component is a process of finite variation, equivalently, if

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, |y|\} F(dy) < +\infty, \tag{2.4}$$

then the last term in the integrand in (2.2) can be integrated out and added to the drift term. Then we obtain

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y}) F(dy), \tag{2.5}$$

with a different μ, and the new μ is the drift of the Gaussian component.

Assume that under a risk-neutral measure chosen by the market, the price process has the dynamics $S_t = e^{X_t}$, where X_t is a certain Lévy process. Then we must have $E[e^{X_t}] < +\infty$, and therefore, ψ must admit the analytic continuation into a strip $\operatorname{Im}\xi \in (-1, 0)$ and the continuous continuation into the closed strip $\operatorname{Im}\xi \in [-1, 0]$.

The infinitesimal generator of X, denote it L, is an integro-differential operator which acts as follows:

$$Lu(x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x) + \mu\frac{\partial u}{\partial x}(x) + \int_{-\infty}^{+\infty} (u(x+y) - u(x) - y\mathbf{1}_{|y|\leq 1}\frac{\partial u}{\partial x}(x)) F(dy). \tag{2.6}$$

The infinitesimal generator L can also be represented as a pseudo-differential operator (PDO) with the symbol $-\psi(\xi)$: $L = -\psi(-i\partial_x)$. Recall that a PDO $A = a(-i\partial_x)$ acts as follows:

$$Au(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(\xi) \hat{u}(\xi) d\xi, \tag{2.7}$$

where $\hat{u}$ is the Fourier transform of a function u:

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} u(x) dx.$$

Note that the inverse Fourier transform in (2.7) is defined in the classical sense only if the symbol $a(\xi)$ and the function $\hat{u}(\xi)$ are sufficiently nice. In general, one defines the (inverse) Fourier transform by duality.

Furthermore, if the riskless rate, r, is constant and the stock pays dividends q, then the discounted price process must be a martingale. Equivalently, the following condition must hold

$$r - q + \psi(-i) = 0, \tag{2.8}$$

which can be used to express μ via the other parameters of the Lévy process:

$$\mu = r - q - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} (1 - e^y + y\mathbf{1}_{|y|\leq 1})F(dy). \tag{2.9}$$

Example 2.1 (Tempered stable Lévy processes)**.** The characteristic exponent of a pure jump KoBoL process of order $\nu \in (0,2), \nu \neq 1$ is given by

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu], \tag{2.10}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. Formula (2.10) is derived in [18, 19] from the Lévy-Khintchine formula with the Lévy densities of negative and positive jumps, $F_\mp(dy)$, given by

$$F_\mp(dy) = ce^{\lambda_\pm y}|y|^{-\nu-1}dy. \tag{2.11}$$

Later, the same class of processes was used in [25] under the name CGMY-model. The following relations between the parameters of the KoBoL model and the parameters C,G,M,Y parameters of CGMY-model is valid:

$$C = c,\ Y = \nu,\ G = \lambda_+,\ M = -\lambda_-.$$

More general version with $c_\pm$ instead of c, and the different exponents $\nu_\pm$ are known as a stable Tempered Lévy model [27]. In this case, we have for $\nu_+, \nu_- \in (0,2), \nu_+, \nu_- \neq 1$

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] + c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}], \tag{2.12}$$

where $c_+, c_- > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Example 2.2 (Normal Inverse Gaussian processes)**.** A normal inverse Gaussian process (NIG) [9] can be described by the characteristic exponent of the form

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}], \tag{2.13}$$

where $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbf{R}$.

Example 2.3 (Variance Gamma processes)**.** The Lévy density of a Variance Gamma process is of the form (2.11) with $\nu = 0$, and the characteristic exponent is given by (see [104])

$$\psi(\xi) = -i\mu\xi + c[\ln(\lambda_+ + i\xi) - \ln\lambda_+ + \ln(-\lambda_- - i\xi) - \ln(-\lambda_-)], \tag{2.14}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Example 2.4 (Kou model)**.** If $F_\mp(dy)$ are given by exponential functions on the negative and positive axis,

$$F_\mp(dy) = c_\pm(\pm\lambda_\pm)e^{\lambda_\pm y},$$

where $c_\pm \geq 0$ and $\lambda_- < 0 < \lambda_+$, then we obtain the Kou model. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}. \tag{2.15}$$

The version with one-sided jumps is due to [32], the two-sided version was introduced in [36], see also [77].

3. Backward induction methods

3.1. General approaches in backward induction methods. Backward induction methods are based on the fact that the risk neutral valuation formula for the European option can be seen as a convolution of the payoff function with the transition density. The key idea is to set up a time lattice and view the option as European type between two adjacent dates. Therefore, the backward induction methods [62,63,68,69] recurrently solve the option pricing problem step by step in short time intervals using an approximation for the probability density of a modeling Lévy process. Hence, the backward induction method requires the transition density to be known in closed form, which is the case, e.g. in the Black-Scholes model and Merton's jump-diffusion model.

The approximation proposed by [44] uses the discretization of the time parameter and the backward induction to price American options in the GBM model. The method was extended in [19] for some Lévy models, and its applications can be founded, e.g., in [86,98]. If there is no explicit formula for the probability density, it can be recovered by inverting the characteristic function, so the method can be used for a wide range of Lévy models.

Since convolutions can be handled very efficiently using the Fast Fourier Transform (FFT), the overall complexity of the method is $O(mn \ln n)$, where m and n are the numbers of points on the grid in time and space, respectively. The FFT-based backward induction method was applied in [63], see also [103]. In terms of the theory of pseudodifferential operators (PDOs), at each time step, the FFT-based backward induction method implements action of the PDO, which symbol is the characteristic function.

The method suggested in [60,61] solves the backward jump-diffusion PIDEs for the option prices by splitting the related operator into differential and jump parts. The key idea behind the approach involves representing a jump operator as a PDO with subsequent transformation into an operator exponential.

In a series of papers [67–69] a backward induction method based on the frame projection approach (PROJ) was developed. In particular, in [69] the approach was applied for discretely monitored barrier derivatives with robust pricing under exponential Lévy models. The coefficient functionals of the orthogonally projected transition density are given by its convolution with a dual B-spline scaling function of the first order, using the characteristic function of the underlying asset.

The method's efficiency is derived in part from the use of frame projected transition densities, which transform the problem into the Fourier domain and accelerate the convergence of intermediate expectations. These expectations are approximated by Toeplitz matrix-vector multiplications, resulting in a fast implementation by means of the Fast Fourier Transform. Additionally, the method includes proper truncating support of the transition density. In [68], the frame projection approach is generalized to the case of B-spline scaling functions of arbitrary order.

3.2. The frame projection method. We briefly describe the numerical framework of [69] to value exotic options in exponential Lévy models. Throughout, the riskless rate r and the dividend rate q are assumed to be constant. We consider

here the special case of frame projection onto a linear spline basis, and more theoretical details and general results for the B-spline basis can be found in [67, 72]. After reviewing the implementation for European options, we consider the case of knock-out barrier options in Section 3.2.2, and Bermudan options.

Let T, K be the contract maturity and strike, and the stock price $S_t = S_0 e^{X_t}$ be an exponential Lévy process under a chosen risk-neutral measure (see (2.8)). We seek to value an option with European-style payoff of $G(x)$ at time T. Recall that for the call option $G(x) = (S_0 e^x - K)_+$, and for the put option $G(x) = (K - S_0 e^x)_+$.

Then the price of the European option without arbitration at time $t_0 = 0$ is given by

$$f(x, t_0) = E^x \left[e^{-rT} G(X_T) \right] \tag{3.1}$$

For consistency in notation with exotic options below, let $\Delta\tau$ denote the increase in time between *monitoring dates* of the contract. For a European option, $\Delta\tau = T - t_0$.

To calculate the price in (3.1), note that in the general case, $p_{\Delta\tau}$ can be expressed in terms of the characteristic exponent $\psi(\xi)$, using the Fourier transform

$$p_{\Delta\tau}(\nu) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\xi - \Delta\tau\psi(\xi)} d\xi. \tag{3.2}$$

For a fixed resolution $a > 0$ and a generator $\phi(\nu) = (1 - |\nu|)\mathbf{1}_{[-1,1]}$, we obtain the following analytical representation of the orthogonally projected density

$$p_{\Delta\tau}(\nu) \approx \sum_{k=1}^{N} \Big(\int_{-\infty}^{+\infty} p_{\Delta\tau}(y) \widetilde{\phi}_{a,k}(y) dy \Big) \phi_{a,k}(\nu) \tag{3.3}$$

onto a space of compactly supported basis elements $\phi_{a,k}(\nu) := a^{1/2}\phi(a(\nu - \nu_k))$, where ν_k are the points on a uniformly spaced grid of width $\Delta\nu = 1/a$. Using the Fourier transform technique and (3.2), one may rewrite (3.3) as follows

$$\begin{aligned} p_{\Delta\tau}(\nu) &\approx \sum_{k=0}^{N-1} \frac{a^{-1/2}}{\pi} \operatorname{Re}\Big(\int_0^{+\infty} \exp[i\nu_k\xi - \Delta\tau\psi(\xi)] h_a(\xi) d\xi \Big) \phi_{a,k}(\nu) \\ &\approx \frac{a^{5/2}}{N} \sum_{k=0}^{N-1} \beta_{a,k} \phi_{a,k}(\nu), \end{aligned} \tag{3.4}$$

where $\nu_k = \nu_1 + k\Delta\nu$, $k = 0, \ldots, N-1$,

$$h_a(\xi) = \frac{\sin^2(\xi/2a)}{\xi^2(2 + \cos(\xi/2a))}.$$

We choose ν_1 in order to cover the support of the transition density (see [68] for several viable approaches).

To evaluate European options, we have the valuation formula

$$f(x, t_0) \approx e^{-rT} \frac{a^{5/2}}{N} \sum_{k=0}^{N-1} \beta_{a,k} \int_{\nu_{k-1}}^{\nu_{k+1}} G(\nu) \phi_{a,k}(\nu) d\nu,$$

where the coefficients $\int_{\nu_{k-1}}^{\nu_{k+1}} G(\nu)\phi_{a,k}(\nu)$ are easy to evaluate numerically for arbitrary payoffs (see [68]), and are available in closed form for many standard payoffs (see [67]).

3.2.1. *Projection Coefficients by FFT.* Approximation for the coefficients $\beta_{a,k}$ can be efficiently computed by using the Fast Fourier Transform (FFT). Consider the algorithm (the discrete Fourier transform (DFT)) defined by

$$G_k = DFT[g](k) = \sum_{j=0}^{N-1} g_j e^{-2\pi ikj/N}, \quad k = 0, ..., N-1. \tag{3.5}$$

The DFT maps N complex numbers (the g_j's) into N complex numbers (the G_k's) (see [109] Press, W. et al. (1992) for technical details). The formula for the inverse DFT which recovers the set of g_j's exactly from G_k's is:

$$g_j = iDFT[G](j) = \frac{1}{N}\sum_{k=0}^{N-1} G_k e^{2\pi ikj/N}, \quad j = 0, ..., N-1. \tag{3.6}$$

In our case, the input data consist of the following complex-valued array $\{g_j\}_{j=0}^{M}$:

$$g_0 = 1/24a^2, g_j = \exp(-i\nu_1\xi_j)\exp[-\Delta\tau\psi(\xi_j)]h_a(\xi_j), j > 0. \tag{3.7}$$

Then we obtain

$$\beta_{a,k} = \mathrm{Re}(DFT[g](k)), k = 0, \ldots, N-1. \tag{3.8}$$

3.2.2. *Barrier Options.* As a basic example to illustrate the method we consider pricing discretely monitored down-and-out call and put options under the CGMY (KoBoL) model. Let T, K, H be the maturity, strike, and barrier, and the stock price $S_t = S_0 e^{X_t}$ be an exponential Lévy process under a chosen risk-neutral measure (see (2.8)). Denote by M the number of equally spaced monitoring dates t_k, $k = 0, 1, \ldots, m$, where $t_0 = 0$ and $t_M = T$.

Set $h = \ln H/S_0$ and $\Delta\tau := T/M$. Then the no-arbitrage price of the barrier option at time $t_0 = 0$ and $X_t = x > h$ is given by

$$f(x, t_0) = E^x\left[e^{-rT}\mathbf{1}_{m_1>h}\mathbf{1}_{m_2>h}\ldots\mathbf{1}_{m_M>h}G(X_T)\right], \tag{3.9}$$

where $m_n = \min_{k=0,1,\ldots,n} X_{t_k}$ is the processes of the minimum up to the nth monitoring date, $G(x)$ is the payoff at maturity. Recall that for the call option $G(x) = (S_0e^x - K)_+$, and for the put option $G(x) = (K - S_0e^x)_+$.

We have

$$f(x, t_M) = G(x), \ x > h, \tag{3.10}$$

and for all m,

$$f(x, t_m) = 0, \ x \le h. \tag{3.11}$$

For $m = M-1, M-2, \ldots, 0$, and $x > 0$, the price $f(x, t_m)$ can be found as the price of the European option with the terminal payoff $f(X_{t_{m+1}}, t_{m+1})$ and the expiry date t_{m+1}:

$$f(x, t_m) = E[e^{-r\Delta\tau}f(X_{t_{m+1}}, t_{m+1}) \mid X_{t_m} = x], \ x > h. \tag{3.12}$$

If an explicit formula for the probability density $p_{\Delta\tau}$ of $X_{\Delta\tau}$ under EMM is known (e.g., GBM or NIG model), we can use it to write (3.12) in the form

$$f(x,t_m) = e^{-r\Delta\tau}\int_{-\infty}^{+\infty} p_{\Delta\tau}(y-x)f(y,t_{m+1})dy, x > h. \tag{3.13}$$

We will evaluate $f(x,t_m)$ along a grid of points in log asset space, $x_n = \ln H/S_0 + n\Delta x$, $n = 0,\dots,N/2-1$, where $\Delta x = \Delta\nu$, using the frame projection approximation of $p_{\Delta\tau}$ defined in (3.4).

If the payoff G decays at $+\infty$, then by truncating the integration domain in (3.13) above by $u = x_{N/2-1}$ (see details in [69]), we can rewrite (3.13) by using (3.4), and for x_n, $n = 0,\dots,N/2-1$:

$$\begin{aligned} f(x_n,t_m) &\approx \frac{24a^2e^{-r\Delta t}}{N}\sum_{k=0}^{N/2-1}\beta_{a,N/2+k-n}a^{1/2}\int_h^u f(y,t_{m+1})a^{1/2}\phi(a(y-y_k))dy \\ &= C\sum_{k=0}^{N/2-1}\beta_{a,N/2+k-n}\theta_{m,k} \end{aligned} \tag{3.14}$$

where

$$\theta_{m,k} = a^{1/2}\int_h^u f(y,t_{m+1})a^{1/2}\phi(a(y-y_k))dy, \quad C = \frac{24a^2e^{-r\Delta t}}{N}. \tag{3.15}$$

Convolution (3.14) can be computed quickly by using the fast Fourier transform and the Toeplitz matrix theory. Set $a_{-N/2} = 0$, and

$$a_j = \beta_{a,N+l-1}, j = -N/2+1\dots,-1; \quad a_j = \beta_{a,N/2-j-1}, j = 0,\dots N/2-1.$$

The sequence $\{a_j\}_{j=-N/2+1}^{N/2-1}$ generates the truncated Toeplitz matrix $T(a)$:

$$T_{N/2}(a) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & ... & a_{-N/2+1} \\ a_1 & a_0 & a_{-1} & ... & a_{-N/2+2} \\ a_2 & a_1 & a_0 & ... & a_{-N/2+3} \\ ... & ... & ... & ... & ... \\ a_{N/2-1} & a_{N/2-2} & a_{N/2-3} & ... & a_0 \end{pmatrix}. \tag{3.16}$$

Then

$$\sum_{k=0}^{N/2-1}\beta_{a,N/2+k-n}\theta_{m,k} = T_{N/2}(a)\widetilde{\theta}_m,$$

where $\widetilde{\theta}_m = \{\theta_{m,0},\theta_{m,1},...\theta_{m,N/2-1},\underbrace{0,0,...,0,0}_{N/2 \text{ times}}\}$.

The symbol $a(\eta) = \sum_{j=-N/2+1}^{j=N/2-1} a_j e^{i\eta j}$ of the Toeplitz matrix $T(a)$ can be computed at the points $\eta_k = -2\pi k/N, k = 0,...,N-1$ using the discrete Fourier transform (3.5):

$$a(\eta_k) = DFT[\widetilde{a}](k) = \sum_{j=0}^{N/2} a_j e^{-2\pi ikj/N} + \sum_{j=N/2+1}^{N/2-1} a_{j-N}e^{-2\pi ikj/N}, \quad k = 0,...,N-1. \tag{3.17}$$

where $\widetilde{a} = \{a_0, a_1, ...a_{N/2}, a_{-N/2+1}, a_{-N/2+2}, ..., a_{-2}, a_{-1}\}$.

It is easy to show that

$$T_{N/2}(a)\widetilde{\theta}_m = iDFT[DFT[\widetilde{a}] * DFT[\widetilde{\theta}_m]],$$

where $u * v$ is the element-wise product of vectors u with v.

Notice that $\theta_{M,k}$ could be computed explicitly, while the other coefficients $\theta_{m,k}$, $m < M$, are computed using polynomial interpolation (see details in Kirkby (2017a)).

3.2.3. *Treatment for Unbounded Payoffs.* In the case of an unbounded payoff function $G(x)$ in $+\infty$ (e.g., in the case of the call option), as in [81] we choose real ω in a way such that $e^{\omega x}G(x)$ is absolutely integrable. In the case of the down-and-out call option and typical parameters of the Lévy model $\omega = -2$ is a good choice.

Then we can rewrite the algorithm in terms of new functions:

$$f_\omega(x, t_m) = e^{\omega x} f(x, t_m), m = 0, 1, \ldots, M. \tag{3.18}$$

In this case, the frame projection method should be applied to the weighted transition density $e^{-\omega x} p_{\Delta\tau}(x)$ instead of the function $p_{\Delta\tau}(x)$.

Then taking into account that due (2.1)

$$\int_{-\infty}^{+\infty} e^{i\xi y} e^{-\omega x} p_t(y) dy = e^{-t\psi(\xi + i\omega)},$$

we may rewrite the formulas for (3.7) in (3.8) as follows:

$$g_0 = 1/24a^2 \exp[-\Delta\tau\psi(i\omega)], g_j = \exp(-ix_1\xi_j) \exp[-\Delta\tau\psi(\xi_j + i\omega)]h_a(\xi_j), j > 0.$$

If in the cross-barrier event, the knock-out option provides the rebate $R > 0$ to holders, one can represent $f(x, t_m)$ as $v(x, t_m) + R$ and adjust the algorithm accordingly.

3.2.4. *American Options.* The frame projection method is extended to Bermudan/American options in [70] for Lévy processes, and [73] for stochastic volatility models. The approach is based on a value recursion, in terms of the frame projected approximation of the transition density $p_{\Delta\tau}$ as follows:

$$\begin{aligned} f(x, t_M) &= G(x) \\ \mathcal{C}(x, t_m) &= e^{-r\Delta\tau} \int f(y, t_{m+1}) p_{\Delta\tau}(y - x) dy, \quad m = M - 1, ..., 0 \\ f(x, t_m) &= \begin{cases} \max\{\mathcal{C}(x, t_m), G(x)\} & m = M - 1, ..., 1 \\ \mathcal{C}(x, t_m) & m = 0 \end{cases} \end{aligned} \tag{3.19}$$

where $\mathcal{C}(x, t_m)$ is the *continuation value* at time t_m. We will again use the frame projection method with a discrete *log-asset grid* $\{x_k\}_{k=1}^N$ over $[l, u]$, which is chosen to ensure that $x_{\bar{k}} = \ln(K/S_0)$ for some fixed index $\bar{k}$, so that the payoff kink is aligned with the grid. The recursive valuation proceeds in a similar manner as for Barrier options described previously and exploits the Toeplitz structure of the convolution with respect to the projected transition density.

To compute the recursive valuation, we use formula (3.14) where the value coefficients $\theta_{m,k}$ are computed in a similar manner as in (3.15), using

$$\begin{aligned}\theta_{m,k} &= a^{1/2}\int_l^u f(y,t_{m+1})a^{1/2}\phi(a(y-y_k))dy \\ &= a\int_l^u \max\{\mathcal{C}(y,t_{m+1}),G(x)\}\phi(a(y-y_k))dy. \end{aligned} \tag{3.20}$$

To avoid a loss in convergence order due to the non-smoothness introduced by the $\max\{\cdot\}$ operation (and to determine the explicit optimal exercise policy), we estimate the *early-exercise* point directly before computing the integral. This point, x_*^m, should satisfy $G(x_*^m) = \mathcal{C}(x_*^m, t_m)$. For a Bermudan put option, we first note that the left bracketing index and grid point can be found easily (by binary search at a cost of $\mathcal{O}(\log(N))$) using

$$k_* = \max\{1 \le k \le \bar{k} : G(x_k) - \mathcal{C}(x_k,t_m) \ge 0\}, \qquad x_{k_*} = x_1 + (k_*-1)\Delta, \tag{3.21}$$

so the early exercise point satisfies $x_{k_*} \le x_*^m < x_{k_*+1}$. Once k_* is found, we approximate x_*^m by

$$x_*^m \approx x_{k_*} + \Delta\frac{G(x_{k_*}) - \mathcal{C}(x_{k_*},t_m)}{(G(x_{k_*}) - \mathcal{C}(x_{k_*},t_m)) - (G(x_{k_*+1}) - \mathcal{C}(x_{k_*+1},t_m))}. \tag{3.22}$$

We can then compute the integral in (3.20) with high accuracy by splitting the domain at the point x_*^m and preserve the natural convergence rate (see [70] for more details and closed-form algebraic expressions for $\theta_{m,k}$). Note that this procedure extends naturally to multi-early exercise contracts, as demonstrated in [71] for swing option pricing.

4. Option pricing by solving partial integrodifferential equations

4.1. Numerical methods for solving PIDEs in finance. The next large group deals with numerical methods to solve the generalized Black–Scholes equation or approximate computing of the corresponding expectation. There are three main approaches to solving PIDEs: finite difference schemes, Galerkin methods, and integral transform methods. The methods start by reducing a boundary problem for the generalization of the Black–Scholes equation (backward Kolmogorov equation); in the case of American options, a free boundary problem arises. In [19], the equation for the price of derivative security is derived in the sense of the theory of generalized functions. Later, in [28], prices of European and barrier options are expressed as solutions of partial integrodifferential equations (PIDEs) that involve, in addition to a (possibly degenerate) second-order differential operator, a nonlocal integral term (see also [27]).

The theory of pseudo-differential operators (PDO) extends the notion of a differential operator and is widely used to solve integrodifferential equations. The essential idea is that a differential operator with constant coefficients can be represented as a composition of a Fourier transform, multiplication by a polynomial function, and an inverse Fourier transform. Moreover, the PDO technique based on the Fourier transform and the operator form of the Wiener–Hopf method is much more powerful than the method based on the study of the PIDE kernel.

This was why the theory of PDO was invented in the first place; see, e.g., [38] and [55].

The straightforward idea of applying PDO theory in the context of option pricing has been systematically pursued in a series of publications summarized in three monographs [19,21,62]. However, the general formulas for the prices involve a double Fourier inversion (and one more integration needed to calculate the factor in the Wiener–Hopf factorization formula). Hence, it is not easy to implement them in practice except in the particular case where there are explicit formulas for the factors.

If the characteristic exponent of the underlying Lévy process is rational, then one can apply Laplace transform methods. The primary examples of this are Brownian motion, Kou's model, and its generalization, the hyperexponential jump-diffusion model (HEJD).

First, we find the Laplace transform of a given option's value function with respect to the time to maturity. In [6, 8, 78, 100, 114], the Laplace transform is derived from the distribution of the first passage time; the distribution is calculated applying the Wiener–Hopf factorization method in the form used in probability. See also [93] and [3, 52]. Once the Laplace transform is calculated, a suitable numerical Laplace inversion algorithm is used to recover the option price. In other cases, one can approximate the initial process by Kou's model or by the HEJD process, and then use the Laplace transform method (see, e.g., [29, 64]).

However, the problem of the Laplace transform inversion is non-trivial from the computational point of view. There are many different numerical Laplace inversion methods, but some procedures, such as the popular Gaver-Stehfest algorithm in computational finance, usually require high precision. Notice that the latter is based on the Post–Widder inversion formula, which involves differentiation instead of integration.

For more information on a general framework for numerical Laplace inversion that includes an optimized version of the one-dimensional Gaver–Stehfest method, refer to [2]. Notice that [29, 81, 114] found that the choice of 12–14 terms in the Gaver–Stehfest formula may result in satisfactory accuracy for Kou and HEJD models. In this case, the standard double-precision gives good results.

Another feature that often slows down the calculations is the fact that the values of the Laplace transform must first be found at several (at least a dozen) different points. Apart from a few cases where an explicit formula gives the transform function, the calculation of these values is time-consuming.

Galerkin methods are based on the variational formulation of PIDEs. While implementation of finite-difference methods requires only moderate programming knowledge, Galerkin methods use specialized toolboxes [27]. Finite difference schemes use less memory than Galerkin methods since there is no overhead for managing grids, but a refinement of the grid is more complicated. A wavelet Galerkin method is constructed to price American options under exponential Lévy processes in [106]. Notice that applying variational methods for finite variation processes, one can prove convergence in the H^s–norm only, where $s < 1/2$. Therefore, convergence cannot be guaranteed in the C–norm.

In a finite difference scheme, derivatives are replaced by finite differences. In the presence of jumps, one needs to discretize the integral term as well. Finite difference schemes are applied to pricing barrier options in [28, 80] and to pricing American options in [26, 53, 80, 98].

Construction of any finite difference scheme involves discretization in space and time, truncation of large jumps, and small jumps approximation. Truncation of large jumps is necessary because we cannot calculate an infinite sum; approximation of small jumps is needed when the Lévy measure diverges at zero. The result is a linear system that needs to be solved at each time step, starting from the payoff function. In the general case, the system solution in each time step by a linear solver requires $O(m^2)$ operations (m is the number of space points), which is too time consuming. In [26, 28, 53], the integral part is calculated using the solution of the previous time step, while the differential term is treated implicitly. This leads to the explicit-implicit scheme, with a tridiagonal system that can be solved in $O(m \ln m)$ operations. The authors of [98] use the implicit scheme and the iteration method at each time step.

The methods in [26,53,98] apply to processes of infinite activity and finite variation; the part of the infinitesimal generator corresponding to small jumps is approximated by a first-order differential operator (additional drift component). In [28], an approximation using a differential operator of second order (additional diffusion component) was used. The Wiener–Hopf factorization method for Toeplitz matrices is suggested for implicit schemes in [80].

As a theoretical background behind the procedure, the methods of the third group use the method of horizontal lines [111] which includes a time discretization while a space variable remains continuous. P. Carr suggested an important probabilistic interpretation of the method in [24], which we call a time randomization technique or Carr randomization.

After the time discretization, a sequence of certain stationary boundary problems arises for integrodifferential equations on a half-line. To solve them, one may apply finite-difference methods like in [28, 80] or the Wiener-Hopf factorization method (see, for example, [15, 81, 87]). We will consider the Wiener-Hopf factorization technique in Section 5. In the case of continuously monitored options, one can also reduce the initial Kolmogorov backward equation to the Wiener-Hopf one applying the Laplace transform in time variable (see, e.g. [81, 96, 108]). It can be shown (see, e.g. [79, 88, 95]) that Carr's randomization or a time discretization is equivalent to the Laplace transform inversion in the Post-Widder formula. A discrete monitoring case can be treated analogously using the z-transform (see, e.g., [42, 43]).

The method suggested in [60, 61] solves the backward jump-diffusion PIDEs for the option prices by splitting the related operator into differential and jump parts. The key idea behind the approach involves representing a jump operator as a PDO with subsequent transformation into an operator exponential. The method combines backward induction approach and finite difference schemes techniques. We give the details of this approach in the next subsection.

4.2. The method of pseudodifferential operators. We shortly describe a numerical framework of [60,61] to value options in exponential Lévy models. As a basic example to illustrate the method, we consider pricing European put options under Tempered stable Lévy processes. Let T, K be the maturity, strike, and the stock price $S_t = e^{X_t}$ be an exponential Lévy process under a chosen risk-neutral measure. The riskless rate is assumed constant $r > 0$.

Then the payoff at maturity is $G(x) = (K - e^x)_+$, and the no-arbitrage price of the European option at time $t < T$ and $X_t = x$ is given by

$$V(t,x) = V(T, G; t, x) = E^{t,x}\left[e^{-r(T-t)}G(X_T)\right]. \tag{4.1}$$

It is well known that $V(t,x)$ is the solution to the following problem.

$$(\partial_t + L - r)V(t,x) = 0, t < T; \tag{4.2}$$

$$V(T,x) = G(x). \tag{4.3}$$

It follows from (2.6), that the infinitesimal generator of a Lévy process is the sum of the infinitesimal generator of the diffusion component (with drift) and the pure jump component, which we denote by L_G and L_J, respectively. Then we can rewrite (2.6) as

$$Lu = L_G u + L_J u. \tag{4.4}$$

Consider equally spaced dates t_k, $k = 0, 1, \ldots, m$, where $t_0 = 0$, $t_m = T$. Set $\Delta\tau := T/m$. Using the splitting technique (for further reading see [105]) as described in [60,61], we approximate $V(x,t)$ in the correspondent discrete time model (4.1) as follows. We have

$$V(x, t_m) = (K - e^x)_+. \tag{4.5}$$

For $k = m-1, m-2, \ldots,$ the numerical scheme includes three steps.

$$V_1(x, t_{k+1}) = \exp\left(\frac{\Delta\tau}{2}(L_G - r)\right) V(x, t_{k+1}); \tag{4.6}$$

$$V_2(x, t_{k+1}) = \exp\left(\Delta\tau L_J\right) V_1(x, t_{k+1}); \tag{4.7}$$

$$V(x, t_k) = \exp\left(\frac{\Delta\tau}{2}(L_G - r)\right) V_2(x, t_{k+1}). \tag{4.8}$$

Thus, instead of an unsteady PIDE, we obtain one PIDE with no drift and diffusion (the second equation in (4.7) and two unsteady PDEs ((4.6) and (4.8)). Recall that $\exp(\tau L)$ is the exponential operator, which acts exactly like the Taylor series expansion of $\exp(\tau L)$ around $\tau = 0$. Let ψ_G and ψ_J be the Gaussian and jump parts of the characteristic exponent ψ in (2.2). Hence, we have

$$L_G u(x) = -\psi_G(-i\partial_x)u(x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x) + \mu\frac{\partial u}{\partial x}(x), \tag{4.9}$$

$$L_J u(x) = -\psi_J(-i\partial_x)u(x). \tag{4.10}$$

The steps (4.6)–(4.8) can be numerically implemented by using the finite difference method. Let ∇_x denote the discrete analogue of ∂_x obtained by discretizing the finite difference of ∂_x in the space grid $\mathbf{x} = \{x_l\}$. Consequently, let us define the matrix $A_G = -\psi_G(-i\nabla_x)$ and $A_J = -\psi_J(-i\nabla_x)$ to be the discrete analogues of the operators L_G and L_J, respectively.

Let A be a matrix that represents a differential or jump operator. It follows from [61] that the finite difference scheme

$$V(\mathbf{x},t) = \exp(\Delta\tau A)\, V(\mathbf{x}, t+\Delta\tau)$$

is unconditionally stable in time τ and preserves the non-negativity of the vector $V(\mathbf{x},t)$ if there exists a M-matrix B such that $\Delta\tau A = -B$, where τ is the time step of the scheme. Once the discretization is performed, we need to compute a matrix exponential $\exp(\Delta\tau A)$, and then a product of this exponential with $V(\mathbf{x}, t+\Delta\tau)$.

This statement gives us a recipe for the construction of the appropriate discretization of the operators L_G and L_J. Notice that (4.6) and (4.8) can be reduced to implicit finite difference schemes

$$\left(1 + \frac{\Delta\tau}{2}(r - A_G)\right) V_1(x, t_{k+1}) = V(x, t_{k+1}); \tag{4.11}$$

$$\left(1 + \frac{\Delta\tau}{2}(r - A_G)\right) V(x, t_k) = V_2(x, t_{k+1}); \tag{4.12}$$

Since a constant time step is used for computations, the matrix $\exp(\Delta\tau A_J)$ can be precomputed once the space grid.

In order to reach the unconditional stability of the finite difference scheme in time τ in (4.11)-(4.12), we need to approximate ∂_x^2 using the central difference and choose an approximation for the first spatial derivative depending on the drift sign. If $\mu > 0$, we use the forward differences; otherwise, we use the backward ones.

Let $A_G = (d_{ij})$ and h be the uniform space step. In the case of tempered stable Lévy models (see Example 1), $\sigma = 0$, therefore, we need to approximate in L_G only the first spatial derivative. In particular, if $\mu > 0$, we set

$$\begin{aligned} d_{i,i} &= 1 + \mu\frac{\Delta\tau}{2h} + r\frac{\Delta\tau}{2}; \\ d_{i,i+1} &= -\mu\frac{\Delta\tau}{2h}; \\ d_{i,i+j} &= 0, j \neq 0, j \neq 1; \end{aligned}$$

otherwise, we set

$$\begin{aligned} d_{i,i} &= 1 - \mu\frac{\Delta\tau}{2h} + r\frac{\Delta\tau}{2}; \\ d_{i,i-1} &= \mu\frac{\Delta\tau}{2h}; \\ d_{i,i+j} &= 0, j \neq 0, j \neq -1; \end{aligned}$$

Consider a finite difference approximation for L_J in the case of $\nu_\pm \in (0,1)$. . We represent the correspondent matrix A_J as follows

$$A_J = c_+\Gamma(-\nu_+)[(\lambda_+ I + A^B)^{\nu_+} - \lambda_+^{\nu_+} I] + c_-\Gamma(-\nu_-)[(-\lambda_- I - A^F)^{\nu_-} - (-\lambda_-)^{\nu_-} I], \tag{4.13}$$

where A^B and A^F are the backward and forward first order differences, respectively. It can be shown that A_J is the negative of an M-matrix and gives a $O(h)$ approximation of the operator L_J.

Calculating a real power of a matrix A^ν by definition uses the formula $A^\nu = \exp(\nu \ln A)$, which involves the implementation of exponential matrix and logarithmic matrix functions and can be computationally expensive.

5. Wiener-Hopf factorization methods

5.1. The Wiener-Hopf factorization. A property that is common to all Lévy processes is the so-called Wiener-Hopf factorization. There are several forms of Wiener-Hopf factorization. Suppose that for any $q > 0$, $T_q \sim \operatorname{Exp} q$ is an independent of X exponentially distributed random variable with mean q^{-1}. Recall that $\overline{X}_t = \sup_{s \le t} X_s$ and let $\underline{X}_t := \inf_{s \le t} X_s$. The following useful relations hold for any $t > 0$ (see [113]):

$$\bar{X}_t \overset{d}{\sim} X_t - \underline{X}_t \tag{5.1}$$

$$\underline{X}_t \overset{d}{\sim} X_t - \bar{X}_t. \tag{5.2}$$

Here, we use the notation $\overset{d}{\sim}$ to mean equality in distribution.

It follows that the random variables $\overline{X}_{T_q}$ and $X_{T_q} - \overline{X}_{T_q}$ are independent and $X_{T_q} - \overline{X}_{T_q}$ is of the same distribution as $\underline{X}_{T_q}$.

The Wiener-Hopf factorization formula used in probability reads:

$$E[e^{i\xi X_{T_q}}] = E[e^{i\xi \bar{X}_{T_q}}] E[e^{i\xi \underline{X}_{T_q}}], \quad \forall\, \xi \in \mathbf{R}. \tag{5.3}$$

Equivalently,

$$X_{T_q} \overset{d}{\sim} S_q + I_q, \tag{5.4}$$

where S_q and I_q are independent and equal in distribution to $\overline{X}_{T_q}$ and $\underline{X}_{T_q}$, respectively.

Introducing the notation

$$\begin{aligned} \phi_q^+(\xi) &= qE\left[\int_0^\infty e^{-qt} e^{i\xi \bar{X}_t} dt\right] = E\left[e^{i\xi \bar{X}_{T_q}}\right], \\ \phi_q^-(\xi) &= qE\left[\int_0^\infty e^{-qt} e^{i\xi \underline{X}_t} dt\right] = E\left[e^{i\xi \underline{X}_{T_q}}\right] \end{aligned}$$

we can rewrite (5.3) as

$$\frac{q}{q + \psi(\xi)} = \phi_q^+(\xi) \phi_q^-(\xi). \tag{5.5}$$

Introduce the normalized resolvent of X or the expected present value operator (EPV–operator) under X. The name of the latter operator was suggested in [20] due to the observation that, for a stream $g(X_t)$,

$$\mathcal{E}_q g(x) = E\left[\int_0^{+\infty} q e^{-qt} g(X_t) dt \mid X_0 = x\right]. \tag{5.6}$$

Replacing in (5.6) process X with the supremum and infimum processes $\bar{X}$ and $\underline{X}$, we obtain the EPV operators $\mathcal{E}_q^\pm$ under the supremum and infimum process. Equivalently,

$$\mathcal{E}_q^+ u(x) = E[u(x + \bar{X}_{T_q})],\ \mathcal{E}_q^- u(x) = E[u(x + \underline{X}_{T_q})]. \tag{5.7}$$

Hence, $\mathcal{E}_q$ and $\mathcal{E}_q^{\pm}$ admit interpretation as expectation operators:

$$\mathcal{E}_q g(x) = \int_{-\infty}^{+\infty} g(x+y) P_q(dy), \quad \mathcal{E}_q^{\pm} g(x) = \int_{-\infty}^{+\infty} g(x+y) P_q^{\pm}(dy),$$

where $P_q(dy)$, $P_q^{\pm}(dy)$ are probability distributions with the characteristic functions $q(q+\psi(\xi))^{-1}$ and $\phi_q^{\pm}(\xi)$, respectively. Notice that

$$P_q^{\pm}(y) = 0, \ \forall \pm y < 0.$$

5.2. Overview of approximate Wiener–Hopf factorization methods. The Wiener-Hopf method is a general tool for solving integrodifferential equations with convolution-type kernels on a half-line. In application to finance, as we mentioned above, the Wiener-Hopf method was widely used to solve 2-dimensional initial boundary value problems for pricing path-dependent options under Lévy processes. However, in the case of general Lévy models, the Wiener-Hopf factors are not available in a closed form and should be approximated by using special numerical tricks. In particular, an approximate Wiener-Hopf factorization was suggested in [87] as the main ingredient of the fast, accurate and universal numerical method for pricing barrier options under Lévy models. We will refer to that method as "The Fast Wiener-Hopf factorization method" (FWHF method). In [81] the approximate factorization was generalized; convergence of the method was accelerated. Alternative methods that use various approximate techniques for Wiener-Hopf factorization can be found in [15, 17, 37, 42, 96, 97, 108] among others.

In [87], a fast and accurate numerical method was developed to price barrier options in a broad class of Lévy processes by using an efficient approximation of the Wiener–Hopf factors in the exact formula for the solution. In contrast to finite difference schemes, where the method's application involves a detailed analysis of the underlying Lévy model, the FWHF method deals with the process's characteristic exponent. The method starts with discretization of time, which can be interpreted as Carr's randomization (see [24]). A sequence of stationary boundary problems for a PDO on the line results. Each problem is solved by using the Wiener–Hopf approach.

The methods developed in [79, 81, 82] generalize and enhance the approach of approximate factorization introduced in [87]. The author of the papers prove the convergence of Carr's randomization for different types of option and suggest an acceleration of the method convergence. The enhanced methods can be applied to pricing options with barrier and look-back features under broad classes of Lévy processes.

The idea behind the approaches in [79, 81] is to transform the problem to a space where the solution is relatively easy to obtain by using the FWHF method. Apart from particular cases where an explicit expression gives the Laplace transform, one can apply the methods developed in the paper for the general case. The Laplace transform maps the generalized Black–Scholes equation with the appropriate boundary conditions into the one-dimensional problem on the half-line, parametrically dependent on the transform parameter.

The first approach (see, e.g., [79]) is based on the Post–Widder formula. Find the nth derivative of the transformed function at the value of the specific transform

parameter by using an iterative procedure, which is simply Carr's randomization in the FWHF method. Repeat the operation several times for different values of n and apply the convergence acceleration algorithm of [1].

In the second approach (see, e.g. [81]), we solve the problems obtained using the FWHF-method at real positive values of the transform parameter specified by the Gaver–Stehfest algorithm (see, e.g. [1]). Then the option prices are recovered via the numerical inversion formula.

In the recent paper [88], a new simple approximate Wiener–Hopf factorization method was developed. The new approach can be applied to pricing barrier options under pure non-Gaussian Lévy models with jumps of finite variation (2.5). The key idea of the method is to represent an Lévy process as a difference of two subordinators (increasing Lévy processes). Applying such a splitting rule to the process at exponentially distributed randomized time moments, we can find the option price by solving a cascade of simple Wiener–Hopf equations explicitly. We will refer to the mentioned simplified approximate Wiener–Hopf factorization method as the SWHF-method.

In the next Subsection, we consider details of the SWHF-method to pricing barrier options under Lévy processes.

5.3. A simplified Wiener–Hopf factorization method. Let T, K, H be maturity, strike and barrier, and the price of the stock $S_t = S_0 e^{X_t}$ under a chosen risk neutral measure (see (2.8)) is an exponential Lévy process that has no diffusion component ($\sigma = 0$) and only jumps of finite variation (see (2.5)). As a basic example to illustrate our method, we consider pricing continuously monitored down-and-out put options without rebate under the Tempered Stable Lévy model with jumps of finite variation (see Example 2.1). This implies that the parameters ν_+ and ν_- which characterize the activity of jumps should satisfy the following inequalities:

$$0 < \nu_+ < 1,\ 0 < \nu_- < 1. \tag{5.8}$$

The risk-free rate r and the dividend rate d are assumed to be constant. Set $h = \ln H/S_0$.

Consider the no-arbitrage price of the barrier option at the beginning of a period under consideration ($t = 0$), see e.g. [87])

$$V(T,x) = E^x\big[e^{-rT} G(X_T) \mathbf{1}_{\underline{X}_T > h}\big], \tag{5.9}$$

where T is the final date, and $G(x) = (K - S_0 e^x)_+$ is the payoff at time T. Shorthand notation $E^x[\cdot]$ means that we take the expectation conditioned on the event $X_0 = \underline{X}_0 = \bar{X}_0 = x$.

The following theorem follows from the results of [88] and justifies the iterative numerical scheme to calculate $V(T,x)$.

Theorem 5.1. *Let N be a positive integer. For $q > 0$ and $n = 1, 2, \ldots, N$, we define the the following sequence of functions*

$$V_n(q,x) = \frac{1}{(1 + r/q)} E^x[V_{n-1}(q, X_{T_{q+r}}) \mathbf{1}_{\underline{X}_{T_{q+r}} > h}], \tag{5.10}$$

where $V_0(q,x) = G(x)$. Then $v_N(N/T, x)$ converges to $V(T,x)$ defined by (5.9) as $N \to +\infty$.

Proof. Denote by $\hat{V}(q,x)$ the Laplace transform of $V(T,x)$ w.r.t. T. Applying Fubini's theorem, we obtain that $\hat{V}(q,x)$ is the discounted expected value of the payoff function $G(X_t)\mathbf{1}_{\underline{X}_t>h}$ at exponentially distributed time T_{q+r}:

$$\begin{aligned}\hat{V}(q,x) &= \int_0^{+\infty} e^{-qt}E^x\left[e^{-rt}G(X_t)\mathbf{1}_{\underline{X}_t>h}\right]dt \\ &= E^x\left[\int_0^{+\infty} e^{-(q+r)t}G(X_t)\mathbf{1}_{\underline{X}_t>h}dt\right] \qquad (5.11)\\ &= \frac{1}{q+r}E^x\left[G(X_{T_{q+r}})\mathbf{1}_{\underline{X}_{T_{q+r}}>h}\right]. \qquad (5.12)\end{aligned}$$

Differentiating $n-1$ times the expression (5.11) w.r.t q and multiplying it by $\frac{(-1)^{n-1}q^n}{(n-1)!}$, we obtain

$$\begin{aligned}v_n(q,x) &:= \frac{(-1)^{n-1}q^n}{(n-1)!}\partial_q^{n-1}\hat{V}(q,x) \qquad (5.13)\\ &= \frac{q^n}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-(q+r)t}E^x\left[G(X_t)\mathbf{1}_{\underline{X}_t>h}\right]dt \\ &= \frac{1}{(1+r/q)^n}E^x[G(X_{\Gamma(n,q+r)})\mathbf{1}_{\underline{X}_{\Gamma(n,q+r)}>h}], \qquad (5.14)\end{aligned}$$

where $\Gamma(n,q)$ is a Gamma random variable with the shape parameter $n>0$ and the rate parameter $q>0$.

Using the relation $\Gamma(n,q)\sim\Gamma(n-1,q)+T_q$, and taking into account that

$$\mathbf{1}_{\underline{X}_{\Gamma(n,q+r)}>h} = \mathbf{1}_{X_{T_{q+r}}+\underline{X}_{\Gamma(n-1,q+r)}>h}\mathbf{1}_{\underline{X}_{T_{q+r}}>h},$$

we conclude that for $n=1,2,\ldots$

$$v_n(q,x) = \frac{1}{(1+r/q)}E^x[v_{n-1}(q,X_{T_{q+r}})\mathbf{1}_{\underline{X}_{T_{q+r}}>h}], \qquad (5.15)$$

where $v_0(q,x)=G(x)$. Comparing (5.10) and (5.15), we see that the sequences $\{V_n\}$ and $\{v_n\}$ coincide.

If $f(t)$ is a non-negative function of a real variable t, and it's Laplace transform $\mathcal{L}f(q)=\int_0^\infty e^{-qt}f(t)\,dt$, is known, then the approximate Post-Widder formula for $f(t)$ reads as follows (see e.g. [2])

$$\begin{aligned}f(t) &= \lim_{N\to\infty} f_N(t); \qquad (5.16)\\ f_N(t) &:= \frac{(-1)^N}{N!}\left(\frac{N}{t}\right)^{N+1}\mathcal{L}f^{(N)}\left(\frac{N}{t}\right),\end{aligned}$$

where $\mathcal{L}f^{(N)}(q)$ – Nth derivative of $\tilde{f}$ at the point q.

Hence, we see that due to the Post-Widder formula, for a fixed x, $v_N(N/T,x)$ converges to $V(T,x)$ as $N\to+\infty$. □

The procedure (5.15) with $q=N/T$ is equivalent to Carr's randomization, introduced in [24] for the case of American call options. Later, it was generalized to a wider class of stochastic control problems in the paper [13]. In [16, 87] the technique was successfully used to price barrier options in Lévy models. The proofs

of the convergence of Carr randomization in the case of similar problems for Lévy processes are presented in [79, 95]. Notice that in [88] the different technique was applied.

The state-of-the-art implementation of the Wiener-Hopf method in option pricing (see, e.g. [16, 87, 96, 108]) leads to factorization (5.5) of $(q+r)/(q+r+\psi(\xi))$, where $\psi(\xi)$ is the characteristic exponent of the Lévy process X_t. Then using (5.1)-(5.2) and (5.7) one can calculate the sequence (5.15) with $q = N/T$ as follows: for $n = 1, \ldots, N$

$$\begin{aligned} v_n(q,x) &= \\ &\frac{1}{(1+r/q)} E[v_{n-1}(q,(x + X_{T_{q+r}} - \underline{X}_{T_{q+r}}) + \underline{X}_{T_{q+r}})\mathbf{1}_{x+\underline{X}_{T_{q+r}}>h}] \\ &= \frac{1}{(1+r/q)} \mathcal{E}^-_{q+r}\mathbf{1}_{(h,+\infty)}\mathcal{E}^+_{q+r} v_{n-1}(q,x). \end{aligned} \tag{5.17}$$

Alternatively, we can calculate the Laplace transform $\hat{V}(q,x)$ using the formula (5.12) similar to (5.17) at a number of points q specified by a numerical algorithm of Laplace transform inversion:

$$\hat{V}(q,x) = \frac{1}{(q+r)} \mathcal{E}^-_{q+r}\mathbf{1}_{(h,+\infty)}\mathcal{E}^+_{q+r} G(x). \tag{5.18}$$

Then we can recover $V(T,x)$ from (5.18) using the inversion algorithm chosen.

Let the characteristic exponent $\psi(\xi)$ of the Tempered Stable Lévy model X_t be defined by (2.12) and let the parameters $\nu_\pm$ satisfy (5.8). The new approach to calculating (5.15) requires the following steps. First, we represent X_t as a difference between two subordinators X^+_t and $-X^-_t$:

$$X_t = X^+_t - (-X^-_t).$$

Recall that a subordinator is a Lévy process with sample paths being almost surely non-decreasing. According to [27, Proposition 3.10], a subordinator has no diffusion component, only positive jumps of finite variation and non-negative drift.

X^+_t and X^-_t are Lévy processes with the characteristic exponent $\psi_+(\xi)$ and $\psi_-(\xi)$, respectively. If $\mu \geq 0$ we define $\psi_+(\xi)$ and $\psi_-(\xi)$ as follows

$$\begin{aligned} \psi_+(\xi) &= -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}], \\ \psi_-(\xi) &= c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}], \end{aligned}$$

otherwise

$$\begin{aligned} \psi_+(\xi) &= c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}], \\ \psi_-(\xi) &= -i\mu\xi + c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}]. \end{aligned}$$

Notice that X^-_t almost surely has nonincreasing sample paths. Hence we have that

$$\bar{X}^+_t = X^+_t, \tag{5.19}$$

$$\underline{X}^-_t = X^-_t. \tag{5.20}$$

Second, we approximation $X_{T_{q+r}}$ in (5.15) as subsequent movements: upward by $X^{+,1}_{T_{q+r}/2}$, then downward by $X^{-,1}_{T_{q+r}}$, and finally upward by $X^{+,2}_{T_{q+r}/2}$. where $X^{+,1}_t$, $X^{-,1}_t$, $X^{+,2}_t$ are independent, $X^{-,1}_t \overset{d}{\sim} X^-_t$, and $X^{+,1}_t, X^{+,2}_t \overset{d}{\sim} X^+_t$.

This representation can be considered as similar to the operator splitting method suggested in [62] where the backward jump-diffusion PIDE for option prices is solved by splitting the related operator into differential and jump parts. Details of the approach are presented in Section 4.2.

Then we may approximate $\mathbf{1}_{x+\underline{X}_{T_{q+r}}>h}$ with $\mathbf{1}_{x>h}\mathbf{1}_{X^{+,1}_{T_{q+r}/2}+X^{-,1}_{T_{q+r}}>h}$.

Notice that $T_{q+r}/2$ is also an exponentially distributed random variable, but with the intensity parameter equal to $2(q+r)$.

Introduce the following operators:

$$\mathcal{E}^+_{2(q+r)}u(x) = E[u(x+\bar{X}^+_{T_{q+r}/2})], \tag{5.21}$$

$$\mathcal{E}^-_{q+r}u(x) = E[u(x+\underline{X}^-_{T_{q+r}})]. \tag{5.22}$$

Set

$$\phi^+_{2(q+r)}(\xi) = \frac{2(q+r)}{2(q+r)+\psi_+(\xi))}, \tag{5.23}$$

$$\phi^-_{q+r}(\xi) = \frac{q+r}{q+r+\psi_-(\xi))}. \tag{5.24}$$

Taking into account (5.19)-(5.20), we can rewrite the operators $\mathcal{E}^+$ and $\mathcal{E}^-$ in (5.21)-(5.22) with symbols (5.23)-(5.24) as follows

$$\begin{aligned}\mathcal{E}^+_{2(q+r)}u(x) &= E[u(x+X^+_{T_{2(q+r)}})] = \mathcal{F}^{-1}_{\xi\to x}\phi^+_{2(q+r)}\mathcal{F}_{x\to\xi}u(x),\\ \mathcal{E}^-_{q+r}u(x) &= E[u(x+X^-_{T_{q+r}})] = \mathcal{F}^{-1}_{\xi\to x}\phi^-_{q+r}\mathcal{F}_{x\to\xi}u(x).\end{aligned}$$

Now, we may approximate $v_n(q,x)$ in (5.15)

$$\begin{aligned}v_n(q,x) &= \\ &\frac{1}{(1+r/q)}E[v_{n-1}(q,x+X^{+,1}_{T_{q+r}/2}+X^{-,1}_{T_{q+r}}+X^{+,2}_{T_{q+r}/2})\mathbf{1}_{x+\underline{X}_{T_{q+r}}>h}]\\ &\approx \frac{1}{(1+r/q)}\mathbf{1}_{(h,+\infty)}\mathcal{E}^+_{2(q+r)}\mathcal{E}^-_{q+r}\mathbf{1}_{(h,+\infty)}\mathcal{E}^+_{2(q+r)}v_{n-1}(q,x).\end{aligned} \tag{5.25}$$

For technical details on the derivation of (5.25), we refer the reader to [88].

EPV operators $\mathcal{E}^-$ and $\mathcal{E}^+$ can be efficiently implemented using the fast Fourier transform (FFT) for functions with real value. Recall that the discrete Fourier transform (DFT) is defined by

$$F_l = DFT[f](l) = \sum_{k=0}^{M-1} f_k e^{2\pi ikl/M}, \quad l = 0, ..., M-1.$$

The inverse DFT recovers the set of f_k's exactly from G_l's. The correspondent formula reads:

$$f_k = iDFT[F](k) = \frac{1}{M}\sum_{l=0}^{M-1} F_l e^{-2\pi ikl/M}, \quad k = 0, ..., M-1.$$

If the data consist of a real-valued array $\{f_k\}_{k=0}^{M}$, then the resulting transform satisfies $F_{M-l} = \bar{F}_l$. Since F_0 and $F_{M/2}$ are real, the transformed complex-valued array has the same "degrees of freedom" as the original real data set $\{f_l\}$. In this case, it is efficient to use the FFT algorithm for real-valued functions (see [109] for technical details). To distinguish DFT of real functions we will use the notation RDFT.

Fix the spatial step $h > 0$ and the number of space points $M = 2^m$. Define the partitions of the normalized log-price domain $[-\frac{Mh}{2}; \frac{Mh}{2})$ by points $x_k = -\frac{Mh}{2} + kh$, $k = 0, ..., M-1$, and the frequency domain $[-\frac{\pi}{h}; \frac{\pi}{h}]$ by points $\xi_l = \frac{2\pi l}{hM}$, $l = -M/2, ..., M/2$. Then the Fourier transform of a function u on the real line can be approximated as follows:

$$\hat{u}(\xi_l) \approx h(-1)^l \overline{RDFT[u](l)}, \quad l = 0, ..., M/2.$$

Here and below, $\overline{z}$ denotes the complex conjugate of z. Using the notation (5.23)-(5.24), we can approximate $\mathcal{E}^{\pm}$ as follows:

$$(\mathcal{E}^{\pm}u)(x_k) = iRDFT[\overline{\phi_{\pm}}. * RDFT[u]](k), \ k = 0, ..., M-1,$$

where $.*$ is the element-wise multiplication of arrays that represent the functions.

The main parameters of the algorithm for evaluating the function $V(T, x)$ is the number of time steps N, the step h of the mesh and the localization interval $(-L\ln(2); L\ln(2))$, where L is the scaling factor (for more details about the choice of the corresponding parameters of the algorithm, see [81], [87]).

We will refer to the method developed as "The simplified Wiener-Hopf factorization method" (SWHF method); see details in [88].

If in the cross-barrier event, the down-and-out option provides the rebate $R > 0$ to the holders, one can represent $v_n(q, x)$ as $u_n(q, x) + R$ and adjust the algorithm accordingly.

6. Monte Carlo methods

6.1. Overview of Monte Carlo methods: standard and advanced approaches. Monte Carlo methods are well known to perform well to price path-dependent options in jump-diffusion models when jump activity is finite. In this case, one can control the behavior of the process between the jump times when the log price follows a Browning bridge process (for details, see [107] or [27]). In the infinite activity case, Monte Carlo methods are much less accurate and more time-consuming. Evaluation of American option prices by Monte Carlo simulation faces additional difficulties: it involves the computation of conditional expectations, see, e.g., [102]. The reader can find an overview of Monte Carlo-based methods for option pricing under Lévy processes in [7, 12, 14, 45, 94]. Generally, Monte Carlo methods consume much more time than other numerical methods.

Recent developments in advanced Monte Carlo techniques in the framework of traditional computational mathematics can be found in [10, 22, 40, 41, 82, 92]. The authors of [92] suggest a more effective method for simulating the joint distribution of the terminal position of the Lévy process and its supremum. Their approach involves randomizing the time to expiry and representing it as a sum of

independent and identically distributed exponential random variables. At randomized time points, Wiener–Hopf factorization is applied to determine the positions of the supremum and infimum and then construct the desired joint distribution. However, one can only apply this approach to a limited class of models that admit an explicit factorization (the Black–Scholes model, Kou [77], β-class of Lévy processes [91]). On the other hand, [82] shows that many time steps are still needed to achieve good computational accuracy. In the papers [40] and [41], the method [92] was improved using the multilevel Monte Carlo algorithm. The article [82] suggests a theoretical basis for a general approach to the construction of Monte Carlo methods to price exotic options under a broad class of Lévy processes.

6.2. Approximate Wiener–Hopf factorization Monte Carlo method. It is shown in [82] that the Monte Carlo method constructed in [92] admits an interpretation as the Laplace transform inversion with the Post–Widder formula (5.16), where N is the number of independent and identically distributed exponential random variables whose sum approximates the time to expiration T. It is well known (see [2]) that convergence in the Post–Widder formula $f_N(\tau)$ to $f(\tau)$ is rather slow – $O(N^{-1})$.

The paper [82] proposes two generalized approaches to construct the Monte Carlo method for general Lévy models that do not admit explicit Wiener–Hopf factorization.

The first uses, as in [92], the randomization of time, but the simulation of the supremum and infimum processes at exponentially distributed moments is based on the inversion of their cumulative distribution functions. The latter can be estimated with approximate Wiener–Hopf factorization formulas for one-touch options (see, e.g., [81]).

The second approach developed in [82] involves a direct simulation of the terminal values of the infimum (supremum) process. It will be more efficient in calculating the expected payoff value, which depends on the maximum or minimum process. Using the second approach, we do not need to divide a sample path of the Lévy process into parts (for example, for lookback options with fixed or floating strikes). The method provides an efficient estimate of the joint distribution function of the values of the supremum and infimum processes at a given time. One can also use this approach to simulate the joint distribution of the Lévy process's terminal position and the corresponding supremum (infimum) process.

Let $\tilde{X}$ be a random variable identically distributed with $\underline{X}_T$, and $F_-(x) = \mathbf{P}(\tilde{X} < x)$, $p_- = \frac{d}{dx}F_-(x)$ stand for the cumulative distribution function (cdf) and the probability distribution function (pdf) of $\tilde{X}$, respectively. If $F_-(x)$ is continuous, then

$$F_-(x) = \int_{-\infty}^{x} p_-(y)dy, x \leq 0. \tag{6.1}$$

If the variable X is continuous, then the cumulative distribution function F_- has an inverse $F_-^{-1} : (0,1) \to \mathbf{R}$. Once we have approximated the c.d.f. F_-, we may simulate $\tilde{X}$ by using samples from $F_-^{-1}(U)$, where U is a uniform distribution on $(0,1)$. To approximate $F_-^{-1}(U)$ with the p.d.f. supported on a half-line, we choose a uniformly spaced grid $x_0, \ldots, x_M \in \mathbf{R}$ subject to conditions $F_-(x_0) < \epsilon$,

$x_M = 0$ (since p_- is supported at a subset of $(-\infty, 0]$), where ϵ is the desired precision. Then for arbitrary $u \in (0,1)$, we define $F_-^{-1}(u)$ using linear or quadratic interpolation.

If the probability density p_- is known, one can apply a quadrature rule to (6.1) to numerically compute the c.d.f. F_-. However, in the case of infinitely divisible distributions, explicit analytical formulas for the p.d.f. are not available as a rule. To recover the p.d.f. p_-, one can use approximate Wiener-Hopf factorization formulas for the characteristic function $\phi^-_{\underline{X}_T}(\xi)$ (see, e.g. [81,82]). Recall that the p.d.f. $p_-(x)$ can be expressed in terms of the characteristic function $\phi^-_{\underline{X}_T}(\xi)$, by using the Fourier transform

$$p_-(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\xi} \phi^-_{\underline{X}_T}(\xi) d\xi. \tag{6.2}$$

The formula (6.2) as well as (3.2) can be efficiently implemented by employing the FFT algorithm. The approximation of the cdf of $\overline{X}_{T_T}$ is treated analogously.

6.3. Approximate Wiener–Hopf factorization Monte Carlo method. Let us consider the algorithm of the approximate Wiener–Hopf factorization Monte Carlo (AWHF&MC method) method for the case of seasoned European lookback options developed in [85]. Let the stock price $S_t = e^{X_t}$ be an exponential Lévy model. A seasoned European floating strike lookback put is defined as follows:

$$V(T_1, T_2; S, H) = E_{T_1}\big[e^{-r(T_2-T_1)}(e^{\bar{X}_{T_2}} - e^{X_{T_2}})|X_{T_1} = \log S, \bar{X}_{T_1} = \log H\big],$$

where S is a spot price at time T_1, $H(= Se^h)$ is a predefined maximum.

Set $T = T_2 - T_1$. Then the option and can be priced as the function $V(T, S, h)$:

$$V(T, S, h) = E\big[e^{-rT} S(e^{\max\{\bar{X}_T, h\}} - e^{X_T})\big]. \tag{6.3}$$

Notice that due to (2.8),

$$E\big[e^{-rT} e^{X_T}\big] = 1.$$

Hence, we may efficiently apply the AWHF&MC method to (6.3) to simulate $\bar{X}_T$ directly.

Let $F_+(x,T) = \mathbb{P}(\bar{X}_T < x)$ be the cdf's of $\bar{X}_T$. Applying the Laplace transform $\mathcal{L}$ with respect to time T to $F_+(-x,T) = \mathbb{P}(x + \bar{X}_T < 0)$, we have, for $x \le 0$,

$$\hat{F}_+(x,q) := \mathcal{L}F_+(-x,\cdot) = q^{-1} E\big[\mathbf{1}_{(-\infty,0)}(x + \bar{X}_{T_q})\big]. \tag{6.4}$$

Then we apply the algorithm of the AWHF&MC method as follows:

- input a set of stock prices $S_1, S_2, ..., S_L$ and fix the number n of sample paths;
- use the method of approximate factorization (see [81,82,85] or Section 5.3) to estimate the characteristic function $\phi_q^+(\xi)$;
- define a fine uniform mesh $\{x_j\}$ in the log-price space;
- evaluate the function

  $$\hat{F}_+(x_j, q_k) = q_k^{-1} \mathcal{E}^+_{q_k} \mathbf{1}_{(-\infty,0)}(x_j), x_j \le 0$$

  at points q_k, as specified by the Gaver–Stehfest algorithm or the Post–Widder formula (5.16);

- recover the c.d.f. $F_+(x_j, T)$ using a numerical algorithm of Laplace transform inversion (the Gaver–Stehfest algorithm, the Post–Widder formula, or others);
- simulate $X^{(i)}$, $i = 1, ..., n$, from the law $\bar{X}_T$, using $F_+^{-1}(U, T)$, where U is an independent uniform distribution on $(0, 1)$ (see details in Subsection 6.1;
- calculate the values $S_j e^{\max\{X^{(i)}, h\}}$, $j = 1, ..., L$;
- find an estimate to $V(T, S_j, h)$ as $e^{-rT}\Big(\frac{1}{n}\sum_k S_j e^{\max\{X^{(i)}, h\}} - S_j\Big)$.

Numerical experiments [81] show that functions similar to $F_+(x, T)$ can be evaluated on a fine uniform mesh in fractions of a second.

The algorithm of the AWHF&MC method for pricing options whose pay-off depends on the terminal value of the infimum process can be written analogously. The values of $\bar{X}_T$ are simulated on the same mesh on which the probabilities were evaluated; here, we can consider only those x_j such that $F_+(x_j, T) \le 1 - 1/N$.

Note that by simulating $\underline{X}_T$, $\bar{X}_T$, we simultaneously simulate $Se^{\underline{X}_T}$, $Se^{\bar{X}_T}$ for any spot price S. The key advantage of this approach over the modifications of the methods from [92] and the standard Monte Carlo methods is that the former does not require simulation of the path of the process's extrema by splitting it into parts.

7. Hybrid methods: machine learning and standard numerical methods

7.1. Machine learning in finance. Feedforward neural networks (FF) are a form of supervised machine learning that uses hierarchical layers of abstraction to represent high-dimensional, nonlinear predictors. These networks are quite straightforward as they feed information from the front to the back, i.e., input and output, respectively. A simple example of the architecture of the neural network of the FF can be found in Figure 1.

The primary decision-making units of neural networks are activation functions. Therefore, it is critical to choose the most appropriate activation function. The chosen activation function has to be (piecewise) differentiable, since the backpropagation algorithm [112] uses the derivatives of the activation function.

Further, we give the list of commomnly used activation functions:

- Step function

$$\phi(x) = \begin{cases} 1, x > 0; \\ 0, x \le 0. \end{cases}$$

- Linear activation function:

$$\phi(x) = ax + b, a \in R.$$

- Sigmoid activation function:

$$\phi(x) = \frac{1}{1 + e^{-x}}.$$

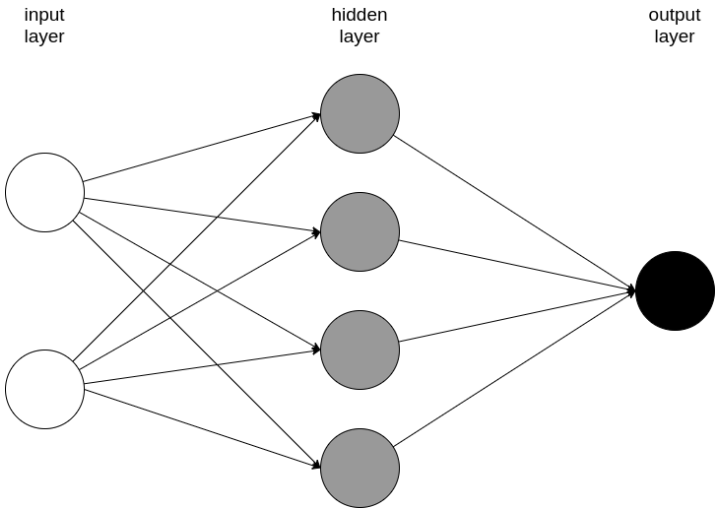

FIGURE 1. Simple network structure.

- Hyperbolic tangent activation function:

$$\phi(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

- Softsign activation function:

$$\phi(x) = \frac{x}{1 + |x|}.$$

- Basic rectified linear unit (ReLU):

$$\phi(x) = \max(x, 0).$$

- Leaky (ReLU):

$$\phi(x) = \max(x, \alpha x), \quad 0 < \alpha < 1.$$

- Softplus activation function:

$$\phi(x) = \ln(1 + e^x).$$

The advances in machine learning and artificial intelligence are having a major impact on the way researchers and engineers work in this field, and machine learning models are increasingly being incorporated into computational finance. In earlier applications, the researchers applied artificial neural networks and other machine learning tools directly in almost any field of computational finance [34,51], including asset price prediction [35,99], market calibration [33], and portfolio management [39,50]. The finance industry is increasingly dependent on computational methods. One of the reasons that computer algorithms for learning called "machine learning" have been successful is that they can model complex and high-dimensional data generation processes, operate through millions of model configurations, and then quickly and robustly evaluate and correct the models in response

to new information. This fact makes them a valuable instrument in addition to the existing algorithms and established practices. However, machine learning methods suffer from a lack of interpretability.

The idea that artificial neural networks can be efficiently used to solve model calibration problems, which describe asset price behavior, is presented, e.g., in [57]. The approach proposed is as follows. After the appropriate model has been chosen, the calibration problem essentially becomes a search for optimal model parameters, in the sense that the optimal parameters can be used to minimize the value of an error functional, which is interpreted as a distance between the historical market prices and the prices generated by the model. With respect to ANNs, the idea is to train the ANN using the available historical data and synthetic dataset, if needed, replace the existing calibration method with the trained ANN, and compare the results in terms of speed and accuracy. In some cases, a trained ANN can outperform traditional calibration methods in speed by an order of magnitude [57]. Among the drawbacks of this approach, there is low accuracy when the training data set is not large enough and does not represent important market patterns, the ability to generate jumps in continuous derivative pricing models, and the need to specifically handle non-arbitrage conditions.

In [58], a special ANN is developed to predict the parameters of the Lévy process and methods to increase the volume of training datasets. In papers [57,101] ANNs are used for option price evaluation in Black-Scholes and Heston models.

7.2. Hybrid methods. In state-of-the-art machine learning approaches to option pricing, artificial neural networks are used as function approximators (see, e.g. [49,54,58,101]). In particular, the authors of [54] apply supervised deep neural networks to price vanilla and exotic options under the Lévy processes. The labeled dataset for machine learning model training can be obtained via standard numerical methods (Fast Fourier Transform algorithm using the characteristic function of the log-price process, Monte Carlo simulations, etc.).

The theoretical background for using neural networks for functions approximation is given by the Kolmogorov–Arnold representation theorem [4, 75]. A naive formulation of the result reads: If f is a multivariate continuous function, then f can be written as a finite composition of continuous functions of a single variable and the binary operation of addition. In [56] the authors show how neural networks with one hidden layer are universal approximators to nonlinear functions. FF ANNs are often described as having layers in which each layer consists of input, hidden, or output cells in parallel. Two adjacent layers are fully connected (every neuron forms one layer with every neuron in another layer). The simplest, somewhat practical network has two input cells and one output cell, which can be used to model logic gates. The Kolmogorov-Arnold result gave the basis for developing the so-called universal approximation theorems that justify approximating properties of neural networks of different architecture (see, e.g. [30, 56].

From a theoretical point of view, functions can be modeled using machine learning methods due to Cybenko's theorem presented in [30]. According to this theorem, a feedforward artificial neural network with one hidden layer and the same-type sigmoidal activation functions can approximate any continuous function of

multiple variables with any accuracy. This theorem belongs to the class of universal approximation theorems that establish the approximation capabilities of different neural networks (see also [55]). Let us reformulate this theorem (Theorem 2, [30]) for the simplest univariate case.

Theorem 7.1. *Let $s(x)$ be an arbitrary continuous sigmoidal function, and the real numbers a, b be such that $a < b$. For a given $\epsilon > 0$ and a given $F(x) \in C[a, b]$ there is a sum of the form*

$$G(x) = \sum_{j=1}^{N} \omega_j s(\alpha_j x + \beta_j), \quad \omega_j, \alpha_j, \beta_j \in \mathbf{R}, \tag{7.1}$$

such that

$$|G(x) - F(x)| < \epsilon, \text{ for all } x \in [a, b]. \tag{7.2}$$

On the other hand, artificial neural networks can be efficient in solving computational finance problems as an auxiliary ingredient of a numerical method that deals with a repeatable typical part of the algorithm. Natural candidates for implementing ANNs in this sense are Monte Carlo algorithms for pricing options. Recall that from the probabilistic viewpoint, an option price is the expectation of a certain pay-off function dependent on the modeling process final position and/or its extrema. Concerning hybrid Monte Carlo methods Monte Carlo methods (i.e. combined wialgorithms), we would like to mention the work [48], in which Gaussian process regression was used to evaluate American options in multivariate Markov and non-Markov models.

For example, using Theorem 7.1, we can approximate the cumulative distribution functions of continuous random variables with a simple ANN of the form (7.1) in any closed interval. Since any increasing continuous sigmoidal function $s(x)$ is a cumulative distribution function itself, we expect that the number of terms in (7.1) will be reasonable for such $s(x)$.

The recent paper [47] studies the expression rates of deep learning neural networks with the ReLU activation function for pricing options written in baskets of several risky assets whose logarithmic returns are modeled by a multivariate Lévy process with a general correlation structure of jumps. As a particular case, the pricing of the European option in the univariate exponential Lévy model is considered.

In the article [115], deep learning neural networks are used to solve the Black-Scholes equation by minimizing the highly nonlinear loss function defined as a weighted sum of the L_2-errors for the PDE and the boundary/initial conditions. For standard activation functions, the derivatives in the loss function are evaluated by the automatic differentiation provided by the Tensorflow library. The hybrid methods constructed in [47, 115] combine neural networks and the P(I)DE technique.

We briefly describe an example of the hybrid method that combines the approximate Wiener-Hopf technique and ANNs for model calibration presented in [110]. The paper considers a method for calibrating a CGMY model by fitting the frequencies of crossing a set of fixed barriers by the process associated with a cryptocurrency rate to theoretical probabilities. Such probabilities can be interpreted

as prices of synthetic one-touch digital options. To prepare synthetic historical data, one should generate the parameters of the CGMY model and calculate the probabilities of crossing barriers as prices of one-touch digital options by using the Wiener-Hopf factorization method developed in [81]. Hence, the commonly used approach based on simulating random-walk trajectories is avoided, and the computational time required to prepare synthetic data is avoided. The choice of model and the technique for preparing the input data makes the approach significantly different from similar research in the area [58,101]. As a result, a calibration scheme is obtained that allows evaluating the risks of price movements above and below certain defined levels.

The following theorem proved in [85] will allow us to simulate the supremum of a Lévy process that has a continuous distribution.

Theorem 7.2. *Let $s(x)$ be an arbitrary sigmoid function. For any $\epsilon > 0$, given confidence level $\gamma \in (0,1)$ and given cdf $F(x)$ jf continuous nonnegative random variable X, there exists a finite sum.*

$$G(u) = \sum_{j=1}^{N} \omega_j s(\alpha_j u + \beta_j), \quad \omega_j, \alpha_j, \beta_j \in \mathbf{R}, \tag{7.3}$$

such that

$$\Pr(|G(U) - F^{-1}(U)| < \epsilon) \geq \gamma, \tag{7.4}$$

where U – uniformly distributed on $(0,1)$.

Using Theorem 7.2, one can develop hybrid Monte Carlo methods that combine fast computation of cdf and its approximation with ANN for fast simulation. However, as the numerical experiments in [85] show, an approximation of the inverse function of a cdf with ANNs is involved rather than the approximation of the cdf itself. The next theorems help to handle this problem.

In [84] a new probabilistic form of universal approximation theorems (similar to [31, 65]) is given.

Theorem 7.3. *Let X be an arbitrary continuous random variable distributed on $[a,b]$, $a < b$. For a given $\epsilon > 0$ and a given continuous random variable Y there is a random variable of the form*

$$Z = \begin{cases} \alpha_1 Y_1 + \beta_1, & \text{with probability } p_1; \\ \dots \\ \alpha_j Y_j + \beta_j, & \text{with probability } p_j; \\ \dots \\ \alpha_N Y_N + \beta_N, & \text{with probability } p_N, \end{cases} \tag{7.5}$$

where $Y_j \overset{d}{\sim} Y$ are independent, $p_j > 0$, $\alpha_j > 0$, $\beta_j \in \mathbf{R}$, such that

$$\sum_{j=1}^{N} p_j = 1; \tag{7.6}$$

$$|F_X(x) - F_Z(x)| < \epsilon, \quad \text{for all } x \in \mathbf{R}, \tag{7.7}$$

$F_X(x)$ and $F_Z(x)$ are cdfs of X and Z, respectively.

In the following theorem, the result of Theorem 7.3 is extended to the case of an arbitrary continuous random variable X (see the proof in [84]).

Theorem 7.4. *Let X be an arbitrary continuous random variable. For a given $\epsilon > 0$ and a given continuous random variable Y there is a random variable of the form*

$$Z = \begin{cases} \alpha_1 Y_1 + \beta_1, & \text{with probability } p_1; \\ \dots \\ \alpha_j Y_j + \beta_j, & \text{with probability } p_j; \\ \dots \\ \alpha_N Y_N + \beta_N, & \text{with probability } p_N, \end{cases} \tag{7.8}$$

where $Y_j \overset{d}{\sim} Y$ are independent, $p_j > 0$, $\alpha_j > 0$, $\beta_j \in \mathbf{R}$, such that

$$\sum_{j=1}^{N} p_j = 1; \tag{7.9}$$

$$|F_X(x) - F_Z(x)| < \epsilon, \quad \text{for all } x \in \mathbf{R}, \tag{7.10}$$

$F_X(x)$ and $F_Z(x)$ are cdfs of X and Z, respectively.

Theorems 7.3 and 7.4 can be considered as universal approximation theorems in probabilistic form. By using these theorems, we can develop hybrid Monte Carlo methods combined with artificial neural networks. Thus, in practice, for a given precision, we can approximate the cdf of a Lévy process with a monotonous feedforward ANN with one input neuron, one output neuron, and one hidden layer with N neurons and with the standard logistic function as an activation function.

As the required ANN is constructed, we may extract from its weight structure the probabilistic characteristics of each random component in the mix (7.8). Therefore, we do not need an involved approximation of the inverse cdf with neural networks as in straightforward approaches to hybrid Monte Carlo methods, but just an inversion of the activation function that is known in explicit form. It makes this approach rather more efficient.

The progress in computational power and open-access availability of special libraries like Keras and Tensorflow made it possible to solve high-dimensional problems of this kind numerically in a reasonable amount of time.

8. Conclusion

The review of modern numerical methods for solving contemporary computational finance problems shows the trend to develop hybrid approaches that combine elements of different techniques. During the last 10 years, many efficient hybrid numerical methods were designed for pricing various options under Lévy processes. For instance, the combination of frame projection, integral transforms of chfs and Toeplitz matrices computation (Subsection 3.2) led to a family of PROJ-methods applied for a wide class of option pricing problems.

The approximate Wiener-Hopf factorization methods described in Subsection 5.2 can be efficiently combined with Monte Carlo methods, as shown in Subsection 6.3. We also consider possible combinations of an approximate Wiener-Hopf factorization with a frame projection method to be quite promising.

We consider a recently developed SWHF method [88] described in Subsection 5.3 as a prospective approach to pricing exotic options whose payoff depends on the infimum or supremum of Lévy processes at expiry. Mixing the Wiener-Hopf factorization method with splitting a Lévy process into positive and negative jumps has made it easy to implement such a complex tool as the Wiener-Hopf factorization for general Lévy models with jumps of finite variation. The key ideas of the simplified Wiener-Hopf factorization can be extended in several directions. First, the efficiency of the method can be improved by increasing the number of terms in the splitting rule or changing the order of the ones. Second, the approach can be extended to the problem of pricing double-barrier options and other exotic options (the first steps in this direction can be found in [83]). Finally, the Wiener-Hopf factorization procedure can be generalized for the case of general Lévy models with jumps of infinite variation.

The most recent development in computational finance is related to incorporating machine learning tools into numerical methods. In particular, we considered new approaches to designing hybrid Monte Carlo methods based on probabilistic analogues of universal approximation theorems (see Theorems 7.3, 7.4). In particular, Theorems 7.3 and 7.4 show that any continuous infinitely divisible random variable can be successfully approximated with a mix of logistic distributions or other one-type continuous distributions. In contrast to a direct approximation of the solution to the option pricing problem with neural networks, the approach described in Subsection 7.2 admits a clear probabilistic interpretation. Note that this method can be applied to Lévy processes whose increments have probability density. It would be interesting to extend the results of Theorems 7.3, 7.4 (with some modifications) to more general random variables.

Therefore, in the near future we can expect the development of hybrid numerical methods with interpretable machine learning elements.

References

1. Abate, J., Whitt, W.: Numerical inversion of Laplace transforms of probability distributions, *ORSA Journal on Computing* **20(7)** (1995) 36–43.
2. Abate, J., Whitt, W.: A unified framework for numerically inverting Laplace transforms, *INFORMS Journal on Computing* **18 (4)** (2006) 408–421.
3. Alili, L., Kyprianou, A.: Some remarks on first passage of Levy processes, the American put option and pasting principals, *Annals of Applied Probability* **15 (3)** (2005) 2062–2080.
4. Arnold, V.: On functions of three variables, *Amer. Math. Soc. Transl.* **28** (1963) 51–54.
5. Asmussen, S., Albrecher, H-J.: *Ruin Probabilities (2nd Edition)* World Scientific Press, River Edge, NJ, 2010.
6. Asmussen, A., Avram, F., Pistorius, M.R.: Russian and American put options under exponential phase-type Lévy models, *Stoch. Proc. Appl.* **109** (2004) 79–111.
7. Asmussen, S., Rosiński, J.: Approximations of small jumps of Lévy processes with a view towards simulation, *J. Appl. Prob.* **38 (2)** (2001) 482–493.

8. Avram, F., Chan, T., Usabel, M.: On the valuation of constant barrier options under spectrally one-sided exponential Lévy models and Carr's approximation for American puts, *Stoch. Proc. Appl.* **100** (2002) 75–107.
9. Barndorff-Nielsen, O.E.: Processes of Normal Inverse Gaussian Type, *Finance and Stochastics* **2** (1998) 41–68.
10. Beliavsky, G., Danilova, N., Ougolnitsky, G.: Calculation of Probability of the Exit of a Stochastic Process from a Band by Monte-Carlo Method: A Wiener-Hopf Factorization, *Mathematics* **7(7)** (2019) 581.
11. Bertoin, J.: *Lévy Processes. Cambridge Tracts in Mathematics* **121**, Cambridge Univer, Press, Cambridge, 1996.
12. Bondesson, L.: On Simulation from Infinitely Divisible Distributions, *Adv. Appl. Prob.* **14(4)** (1982) 855–869.
13. Bouchard, B., Karoui, N.,Touzi, N.El.: Maturity randomization for stochastic control problems, *Annals of Applied Probability* **15 (4)** (2005) 2575–2605.
14. Broadie, M., Detemple, J.: Option pricing: valuation models and applications, *Management Science* **50 (9)** (2004) 1145–1177.
15. Boyarchenko, M., de Innocentis, M., Levendorskiĭ, S. Z.: Prices of barrier and first-touch digital options in Lévy-driven models, near barrier, *International Journal of Theoretical and Applied Finance* **14(7)** (2011) 1045–1090.
16. Boyarchenko, M., Levendorskiĭ, S.: Prices and sensitivities of barrier and First-Touch Digital Options in Levy-Driven Models, *International Journal of Theoretical and Applied Finance* **12(8)** (2009) 1125–1170.
17. Boyarchenko, M., Levendorskiĭ, S. : Valuation of continuously monitored double barrier options and related securities, *Mathematical Finance* **21: no. doi: 10.1111/j.1467-9965.2010.00469.x** (2011)
18. Boyarchenko, S.I., Levendorskiĭ, S.Z.: Option pricing for truncated Lévy processes, *International Journal of Theoretical and Applied Finance* **3** (2000) 549–552.
19. Boyarchenko, S.I., Levendorskiĭ, S.Z.: *Non-Gaussian Merton-Black-Scholes theory*, World Scientific, New Jercey, London, Singapore, Hong Kong, 2002.
20. Boyarchenko, S.I., Levendorskiĭ, S.Z.: American options: the EPV pricing model, *Annals of Finance* **1 (3)** (2005) 267–292.
21. Boyarchenko, S.I. Levendorskiĭ, S.Z.: *Irreversible Decisions under Uncertainty (Optimal Stopping Made Easy). Series: Studies in Economic Theory,* **27**, Springer-Verlag, Berlin, 2007.
22. Boyarchenko, S., Levendorskiĭ, S.: Sinh-Acceleration: Efficient Evaluation Of Probability Distributions, Option Pricing, And Monte Carlo Simulations, *International Journal of Theoretical and Applied Finance* **22(3)** (2019) 1–49.
23. Caballero, M-E., Lambert, A., Uribe, G.: Proof(s) of the Lamperti representation of continuous-state branching processes, *Probab. Surv.* **6** (2009) 62–89.
24. Carr, P.: Randomization and the American put", *Review of Financial Studies* **11** (1998) 597–626.
25. Carr, P., Geman, H., Madan, D.B., Yor, M.: The fine structure of asset returns: an empirical investigation, *Journal of Business* **75** (2002) 305–332.
26. Carr, P., Hirsa, A.: Why be backward?, *Risk* **26(January)** (2003) 103–107.
27. Cont, R., Tankov, P.: *Financial modelling with jump processes* **2nd ed.**, Chapman & Hall/CRC, USA: Boca Raton, FL, 2008.
28. Cont, R., Voltchkova, E.: A finite difference scheme for option pricing in jump diffusion and exponential Lévy models, *SIAM Journal on Numerical Analysis* **43 (4)** (2005) 1596–1626.
29. Crosby, J., Le Saux, N., Mijatović, A.: Approximating Lévy Processes With A View To Option Pricing,*International Journal of Theoretical and Applied Finance* **13(01)** (2010) 63–91.
30. Cybenko, G.: Approximation by superpositions of a sigmoidal function, *Mathematics of Control. Signals and Systems* **2** (1989) 303-314.
31. Daniels, H., Velikova, M.: Monotone and partially monotone neural networks, *IEEE Transactions on Neural Networks* **21(6)** (2010) 906–917.

32. Das, S., Foresi,S.: Exact solutions for bond and option pricing with systematic risk, *Review of Derivatives Research* **1** (1996) 7–24.
33. Dimitroff, G., Roder, D., Fries, C.: Volatility model calibration with convolutional neural networks *Working Paper* (2018) Available at SSRN: https://ssrn.com/abstract=3252432
34. Dixon, M.F., Halperin, I., Bilokon, P.: *Machine Learning in Finance: From Theory to Practice*, Springer International Publishing, Germany, 2020.
35. Dixon, M., Klabjan, D., Bang, J.: *Classification-based financial markets prediction using deep neural networks*, Social Science Electronic Publishing, 2016.
36. Duffie, D., Pan, J., Singleton, K.: Transform analysis and options pricing for affine jump diffusions, *Econometrica* **68** (2000) 1343–1376.
37. Eberlein, E., Glau, K., Papapantoleon, A.: Analyticity of the Wiener–Hopf factors and valuation of exotic options in Lévy models, in: *Di Nunno, G., Oksendal, B. (Eds.), Advanced Mathematical Methods for Finance* (2011) 223–245, Springer .
38. Eskin, G.I.: *Boundary Problems for Elliptic Pseudo-Differential Equations*, Nauka, Moscow, 1973.
39. Feng, G., He, J.,Polson, N.: Deep learning for predicting asset returns, *arXiv e-prints* (2018) arXiv:1804.09314.
40. Ferreiro-Castilla, A., Kyprianou, A.E., Scheichl, R., Suryanarayana, G.: Multilevel Monte Carlo Simulation for Levy Processes Based on the Wiener-Hopf Factorisation, *Stochastic Processes and their Applications* **124 (2)** (2014) 985–1010.
41. Ferreiro-Castilla, A., van Schaik, K.: Applying the Wiener-Hopf Monte Carlo simulation technique for Levy processes to path functionals such as first passage times, undershoots and overshoots, *Journal of Applied Probability* **52 (1)**(2015) 129–148.
42. Fusai, G., Germano, G., Marazzina, D.: Spitzer identity, Wiener-Hopf factorization and pricing of discretely monitored exotic options, *European Journal of Operational Research*, **251 (1)** (2016) 124–134.
43. Phelan, C. E., Marazzina, D., Fusai, G., Germano, G.: Hilbert transform, spectral filters and option pricing, *Annals of Operations Research* **282** (2018) 1–26.
44. Geske, R., Johnson, H.: The American put valued analytically, *J. of Finance* **39** (1984) 1511–1542.
45. Glasserman, P.: *Monte Carlo Methods in Financial Engineering*, Springer, New York, 2003.
46. Glau, K.: Classification of Lévy processes with parabolic Kolmogorov backward equations, *Theory of Probability and its Applications* **60(3)** (2016) 383–406.
47. Gonon, L., Schwab, C.: Deep ReLU network expression rates for option prices in high-dimensional, exponential Lévy models, *Finance Stoch* **25** (2021) 615–657.
48. Goudenège, L., Molent, A., Zanette, A.: Machine learning for pricing American options in high-dimensional Markovian and non-Markovian models, *Quantitative Finance* **20(4)** (2020) 573–591.
49. Grohs, P., Hornung, F., Jentzen, A., von Wurstemberger, P.: *A proof that artificial neural networks overcome the curse of dimensionality in the numerical approximation of Black-Scholes partial differential equations* **284**, American Mathematical Society, 2023.
50. Heaton, J., Polson, N., Witte, J.: Deep learning for finance: deep portfolios, *Applied Stochastic Models in Business and Industry* **33 (1)** (2017) 3–12.
51. Hertz J.: *Introduction to the theory of neural computation*, CRC Press, Boca Raton, 2018.
52. Hieber, P.: Pricing exotic options in a regime switching economy: a fourier transform method, *Review of Derivatives Research* **21** (2018) 231–252.
53. Hirsa, A., Madan, D.B.: Pricing American options under Variance Gamma, *Journal of Computational Finance* **7(2)** (2003).
54. Hirsa, A., Karatas, T., Oskoui, A.: Supervised deep neural networks (DNNS) for pricing / calibration of vanilla / exotic options in various different processes, (2019) arXiv:1902.05810.
55. Hörmaner, L.: *Analysis of Partial Differential Opertors.* **III**, Springer-Verlag, Berlin Heidelberg, New York, 1985.
56. Hornik, K.: Approximation capabilities of multilayer feedforward networks, *Neural Networks* **4(2)** (1991) 251–257.

57. Horvath, B., Muguruza, A., Tomas, M.: Deep learning volatility: a deep neural network perspective on pricing and calibration in (rough) volatility models, *Quantitative Finance* **21(1)** (2021) 11–27.
58. Huh, J.: Pricing Options with Exponential Lévy Neural Network, *Expert Systems with Applications* **127** (2019) DOI: 10.1016/j.eswa.2019.03.008.
59. Hull, J.C.: *Options, futures, and other derivatives, ninth ed.*, Pearson Education, Inc, New York, 2015.
60. Itkin, A.: Splitting and matrix exponential approach for jump-diffusion models with inverse normal Gaussian, hyperbolic and Meixner jumps, *Algorithmic Finance* **3** (2014) 233-250.
61. Itkin, A.: Efficient solution of backward jump-diffusion PIDEs with splitting and matrix exponentials, *J. Comput. Finance* **19** (2016) 29–70.
62. Itkin, A.: *Pricing Derivatives Under Levy Models, vol. 12 of Pseudo-Differential Operators*, Birkhauser, New York, 2017.
63. Jackson, K.R., Jaimungal, S., Surkov, V.: Fourier space time-stepping for option pricing with Lévy models, *Journal of Computational Finance* **12 (2)** (2008) 1–28.
64. Jeannin, M., Pistorius, M.: A transform approach to compute prices and Greeks of barrier options driven by a class of Lévy processes, *Quantitative Finance* **10(6)** (2010) 629–644.
65. Kay, H., Ungar, L.H.: Estimating monotonic functions and their bounds, *AIChE J.* **46(12)** (2000) 242–2434.
66. Kella, O., Whitt, W.: Useful martingales for stochastic storage processes with Lévy input, *J. Appl. Probab.* **29** (1992) 396–403.
67. Kirkby, J. L.: Efficient Option Pricing by Frame Duality with the Fast Fourier Transform, *SIAM Journal on Financial Mathematics* **6(1)** (2015) 713–747.
68. Kirkby, J. L.: Robust option pricing with characteristic functions and the B-spline order of density projection, *Journal of Computational Finance* **21(2)** (2017) 61–100.
69. Kirkby, J. L.: Robust barrier option pricing by frame projection under exponential Lévy dynamics, *Applied Mathematical Finance* **24(4)** (2017) 337–386.
70. Kirkby, J.L.: American and exotic option pricing with jump diffusions and other Lévy dynamics, *Journal of Computational Finance* **22(3)** (2018) 89–148.
71. Kirkby, J.L., Deng, S.J.: Efficient Swing Option Pricing by Dynamic Programming with B-spline density projection, *International J. of Theoretical and Applied Finance* **22(8)** (2019) 1950038.
72. Kirkby, J.L., Deng, S.J.: Static Hedging and Pricing of Exotic Options with Payoff frames, *Mathematical Finance* **29(2)** (2019) 612–658.
73. Kirkby, J. L., Nguyen, D., Cui, Z.: A unified approach to Bermudan and barrier options under stochastic volatility models with jumps, *J. Economic Dynamics and Control* **80** (2017) 75–100.
74. Klüppelberg, C., Kyprianou, A.E., Maller, R.: Ruin probabilities and overshoots for general Lévy insurance risk process, *Ann. Appl. Probab.* **14**(2004) 1766–1801.
75. Kolmogorov, A.: On the representation of continuous functions of several variables by superpositions of continuous functions of a smaller number of variables, *Proceedings of the USSR Academy of Sciences* **108** (1956) 179–182; English translation: *Amer. Math. Soc. Transl.* **17** (1961) 369–373.
76. Koponen, I.: Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process, *Physics Review E* **52** (1995) 1197–1199.
77. Kou, S.G.: A jump-diffusion model for option pricing, *Management Science* **48** (2002) 1086–1101.
78. Kou, S., Wang, H.: First passage times of a jump diffusion process, *Advances in Applied Probability* **35** (2003) 504–531.
79. Kudryavtsev, O.Ye.: An efficient numerical method to solve a special class of integro-differential equations relating to the Levy models, *Math Models Comput Simul* **3** (2011) 706–711.
80. Kudryavtsev, O.: Finite Difference Methods for Option Pricing under Levy Processes: Wiener-Hopf Factorization Approach, *The Scientific World Journal* (2013) Article ID 963625.

81. Kudryavtsev, O.: Advantages of the Laplace transform approach in pricing first touch digital options in Lévy-driven models, *Boletín de la Sociedad Matemática Mexicana* **22(2)** (2016) 711–731.
82. Kudryavtsev, O.E.: Approximate Wiener–Hopf Factorization and Monte Carlo Methods for Levy Processes, *Theory of Probability & Its Applications* **64(2)**(2019) 186–208.
83. Kudryavtsev, O.: A Simple Wiener-Hopf Factorization Approach for Pricing Double-Barrier Options, in: *Karapetyants, A.N., Pavlov, I.V., Shiryaev, A.N. (eds) Operator Theory and Harmonic Analysis. OTHA 2020. Springer Proceedings in Mathematics & Statistics* **358**, (2021) 273–291, Springer, Cham.
84. Kudryavtsev, O., Danilova N.: Applications of artificial neural networks to simulating Lévy processes, *J Math Sci to appear* (2023).
85. Kudryavtsev, O.E., Grechko, A.S., Mamedov, I.E.: Monte Carlo method for pricing look-back options, *Theory of Probab ility & Its Applications to appear*
86. Kudryavtsev, O.E., Levendorskiĭ, S.Z.: Pricing of first touch digitals under normal inverse Gaussian processes, *International Journal of Theoretical and Applied Finance,* **9 (6)** (2006) 915–949.
87. Kudryavtsev, O.E., Levendorskiĭ, S.Z.: Fast and accurate pricing of barrier options under Levy processes, *J. Finance and Stochastics* **13 (4)** (2009) 531–562.
88. Kudryavtsev, O., Luzhetskaya, P.: The Wiener-Hopf Factorization for Pricing Options Made Easy, *Engineering Letters* **28(4)** (2020) 1310–1317.
89. Kudryavtsev, O., Zanette, A.: Efficient pricing of swing options in Lévy-driven models, *Quantitative Finance* (2013) **13(4)** 627–635.
90. Kudryavtsev, O., Zanette, A.: *Applications of Lévy Processes // Mathematics Research Developments*, Nova Science Publishers, Incorporated, NY, 2021. ISBN: 978-1-53619-525 -5
91. Kuznetsov, A.: Wiener-Hopf factorization and distribution of extrema for a family of Lévy processes, *Ann.Appl.Prob.* **20** (2010) 1801–1830.
92. Kuznetsov, A., Kyprianou, A.E., Pardo, J.C., van Schaik, K.: A Wiener-Hopf Monte Carlo simulation technique for Lévy process, *Ann. App. Probab.* **21(6)** (2011) 2171–2190.
93. Kyprianou, A.E., Pistorius,M.R.: Perpetual options and Canadization through fluctuation theory, *Annals of Applied Probability* **13** (2003) 1077–1098.
94. Lemieux, C.: *Monte Carlo and Quasi-Monte Carlo Sampling, Series: Springer Series in Statistics*, Springer, New York, 2009.
95. Levendorskiĭ, S.Z: Convergence of Carr's Randomization Approximation Near Barrier, *SIAM FM*, **2 (1)** (2011) 79–111.
96. Levendorskiĭ,S.Z.: Method of paired contours and pricing barrier options and CDS of long maturities, *International Journal of Theoretical and Applied Finance* **17(5)** (2014) 1–58.
97. Levendorskiĭ,S.Z.: Ultra-fast pricing barrier options and CDSs, *International Journal of Theoretical and Applied Finance* **20(5)** (2017) 1750033.
98. Levendorskii, S., Kudryavtsev, O., Zherder, V.: The relative efficiency of numerical methods for pricing American options under Lévy processes, *Journal of Computational Finance* **9 (2)** (2006) 69–97.
99. Liang,Q., Rong,W., Zhang,J., Liu,J., Xiong,Z.: Restricted boltzmann machine based stock market trend prediction, in: *2017 International Joint Conference on Neural Networks (IJCNN, Anchorage, AK)*, (2017) 1380–1387.
100. Lipton, A.: Assets with jumps, *Risk* **September** (2002) 149—153.
101. Liu, S., Borovykh, A., Grzelak, L., Oosterlee, C.W.: A neural network-based framework for financial model calibration, *Journal of Mathematics in Industry*, **9(9)** (2019) 1–9.
102. Longstaff, F.A., Schwartz, E.S.: Valuing American options by simulation: a simple least-squares approach, *The Review of Financial Studies* **14 (1)** (2001) 113—147.
103. Lord, R., Fang, F., Bervoets, F., Oosterlee, C. W.: A fast and accurate FFT-based method for pricing early-exercise options under Lévy processes, *SIAM Journal on Scientific Computing* **30(4)** (2008) 1678–1705.
104. Madan, D.B., Carr, P., Chang, E.C.: The variance Gamma process and option pricing, *European Finance Review* **2** (1998) 79–105.
105. Marchuk, G.I.: *Methods of Numerical Mathematics*, Springer, New York, 1975.

106. Matache, A.-M., Nitsche, P.-A., Schwab, C.: Wavelet Galerkin Pricing of American Options on Lévy Driven Assets, *Quantitative Finance* **5(4)** (2005) 403–424.
107. Metwally, S., Atiya, A.: Using Browning bridge for fast simulation of jump-diffusion processes and barrier options, *Journal of Derivatives* **10** (2002) 43–54.
108. Phelan, E., Marazzina, D., Fusai, G., Germano, G.: Fluctuation identities with continuous monitoring and their application to price barrier options, *European Journal of Operational Research* **271(1)** (2018) 210–223.
109. Press, W., Flannery, B., Teukolsky, S., Vetterling, W.: *Numerical recipes in C: The Art of Scientific Computing*, Cambridge Univ. Press, available at www.nr.com, 1992.
110. Rodochenko V., Kudryavtsev, O.: A Machine Learning Approach to Option Pricing Under Lévy Processes, in: *Kudryavtsev, O., Zanette, A. (eds) Applications of Lévy Processes, Mathematics Research Developments*, (2021) 33–81, Nova Science Publishers, Inc., New York, USA.
111. Rothe, E.: Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensinaler Randwerttaufgaben, *Math. Ann.* **102 (5)** (1930).
112. Rumelhart, D.E., Hinton, G.E., Williams, R.J.: Learning representations by backpropagating errors, *Nature* **323 (6088)** (1986) 533–536.
113. Sato, K.: *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 1999.
114. Sepp, A.: Analytical Pricing of Double-Barrier Options under a Double-Exponential Jump Diffusion Process: Applications of Laplace Transform, *International Journal of Theoretical and Applied Finance* **7 (2)** (2004) 151–175.
115. Wang, X., Li, J., Li, J.: A Deep Learning Based Numerical PDE Method for Option Pricing, *Comput Econ* **62** (2023) 149–164. https://doi.org/10.1007/s10614-022-10279-x

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