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HAAR UNIQUENESS PROPERTIES AND MODELLING OF FINANCIAL MARKETS INTERPOLATIONS

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Abstract. This work is related to the modelling financial markets being subject to the buying up of stocks and to the task of transformation of arbitrage-free incomplete markets to complete ones. The survey shows how these two problems are intertwined. The main results that were obtained by a group of mathematicians headed by the author of this review over the course of two decades are demonstrated.

1. Introduction

The first half of this review article is devoted to the problem of modelling financial markets being subject to the buying up of stocks.

Aggressive buying up of stocks (the term "active buying up of stocks" is also used in the literature) is an essential part of the functioning of financial markets.

Aggressive buying up of stocks is actions of one or a group of investors for the intensive acquisition of stocks in a company. Often in buying up professional participants in the securities market are involved. This makes it difficult to quickly determine who are its actual initiators and how many there are. In this case, they are used methods of direct communication to shareholders, namely: publications in the media information with an offer to sell stocks, posting advertisements in the immediate proximity to enterprises or places of residence of its shareholders (company employees), sending letters to shareholders.

Aggressive buying up of stocks is carried out on the basis of many factors and combinations of various business indicators, the correct analysis of which is very difficult for an unprepared person. This is why novice investors go broke in many cases trying to independently play on the stock exchange and buying stocks en masse.

In the modern economy, in order to survive, you need to constantly develop, so large companies often buy up small ones. This is one of the reasons for aggressive buying up. Large companies buy stocks from many small shareholders for different purposes: collecting a sufficient number of shares to obtain a seat on the board of

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directors, to influence decision making in a company, to obtain blocking or control block of stocks, etc. Once the required number of shares have been purchased, the buying up stops. During aggressive buying, the stock price rises rapidly. As soon as the purchase stops, it happens that the shares cannot be sold at any price. Last thing primarily applies to shares of regional enterprises. Demand for these shares is very irregular. If you do not have time to sell stocks during the buying up, then the next opportunity may not present itself soon, or may not present itself at all.

Modelling financial markets which are subjects to aggressive buying of shares is a very difficult task. In this paper we will describe approaches to solving this problem proposed earlier by the author of this article.

First, we present basic information about the so-called (**B**,**S**)-markets. Then we present in detail the technique of Haar interpolations of financial (**B**,**S**) markets on finite probability spaces. This technique, based on the important properties of HUP and UHUP of martingale measures, was developed in the early 2000s by the author of this review and his graduate student M.N. Bogacheva. After this, we move on to the problems of modelling (**B**,**S**)-markets being subject to the buying up of stocks. We consider cases where aggressive buying is carried out by a finite number of buyers and a countable number of buyers. The results of this part of the article were also obtained by the author of the article and his students.

The second half of this article is devoted to problems of mainly theoretical significance. Here we consider rather difficult problems of Haar interpolation of static (**B**,**S**)-markets defined on a countable probability space. Here the properties of UHUP and SHUP of martingale measures were introduced by I.V. Pavlov; sufficient conditions for the existence of martingale measures satisfying UHUP were obtained by V.V. Shamraeva; sufficient conditions for the existence of martingale measures satisfying SHUP were obtained by I.V. Pavlov with his students and V.V. Shamraeva. Next, we give the formulation of the so-called inverse problem, solved in two out of three options. And finally, the concept of Haar interpolations of signed deflators is introduced and the results associated with the interpolation property SHUP of these deflators are presented.

2. Basic Concepts of Financial Mathematics

In stochastic financial mathematics, financial markets are called (**B**,**S**)-markets and modelled as follows. Introduce into consideration:

(1) risk-free asset $\mathbf{B} = (B_k)_{k=0}^N$ (bank account), represented (most often) as a deterministic sequence of positive numbers, where B_k expresses the price of a bank account at time k; for example, B_k may change according to the compound interest formula $B_k = B_0\,(1+r)^k$, where r is an interest rate;

(2) vector of risky assets (stocks) $\mathbf{S} = (S_1, S_2, ..., S_N)$, where $S_k = (S_k^i)_{i=1}^l$ (the superscript reflects the type of stock) and $S_k^{(i)}$ is positive random variable (r.v.), expressing the price of a stock of the i-th type at the moment time k.

The market $(\mathbf{1},\mathbf{Z})$, where $\mathbf{Z} = (Z_1, Z_2, ..., Z_N)$ and $Z_k = \frac{S_k}{B_k}$ $(k = 1, 2, ..., N)$, is called discounting of the (**B**,**S**)-market under consideration (resp., $Z_k^{(i)} = \frac{S_k^{(i)}}{B_k}$

is called discounting of the i-th type of stock at the moment k). Bank price of discounted market account is identically equal to one.

If $N < \infty$, we say that a market (**B**,**S**) has the finite horizon; otherwise the horizon is called infinite. In this work we mainly consider (**B**,**S**)-markets with finite horizons.

Since $S_k^{(i)}$ is a r.v., then it is defined on some measurable space $(\Omega, \mathcal{F})$, where Ω is the set of outcomes (situations) at the financial market, and $\mathcal{F}$ is a σ-algebra of events on Ω, which is interpreted as a set of all possible events that can happen in the market over the time of monitoring it (that is, in the time period from $k = 0$ to $k = N$). If not the opposite is said, then we believe that Ω is a finite set, and $\mathcal{F}$ is the totality of all subsets of the set Ω.

Let we are at time $n = 0$. At this moment the entire market situation is considered to be known to us. For example, we know the specific price $S_0^{(i)}$ of the i-th type, therefore we can qualify every statement about this price as true or false. As a result, the vector $S_0 = (S_0^{(1)}, S_0^{(2)}, ..., S_0^{(l)})$ is considered non-random and the trivial algebra $\mathcal{F}_0 = \{\Omega, \emptyset\}$ consisting only of true and false events is associated with this moment.

Now let us look at the next moment of time $n = 1$. This is the moment when new prices are announced for stocks. Since stock prices behave randomly (they can fall, stay the same, grow), then at this moment a number of new (non-trivial) events naturally arise. The σ-algebra of these events is denoted by $\mathcal{F}_1$. It describes the set of all events that can happen in the market at the moment $k = 1$. In the same way, when passing to $k = 2$ there arises the more eventful algebra $\mathcal{F}_2$, etc. Having reached $k = N$, we have $\mathcal{F}_N = \mathcal{F}$. As a result, we obtain an increasing sequence of algebras events:

$$\{\Omega, \emptyset\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_N = \mathcal{F}.$$

Such a flow of σ-algebras $\mathbf{F} = (\mathcal{F}_k)_{k=0}^N$ is called a filtration, and each σ-algebra $\mathcal{F}_k$ can be treated as a σ-algebra of events available for observations at time k, or as aggregate information about the market at that moment.

Since every σ-algebra $\mathcal{F}_k$ is finite, then it has an atomic structure, that is in $\mathcal{F}_k$ there is a set of events (atoms) $A_k^1, A_k^2, ..., A_k^{r_k}$ satisfying the conditions: $A_k^i \bigcap A_k^j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{r_k} A_k^i = \Omega$. The remaining events (excluding impossible event $\emptyset$) from $\mathcal{F}_k$ are representable as a sum of atoms. Thus, atoms are "basic" events in $\mathcal{F}_k$. We denote the set of all atoms from $\mathcal{F}_k$ via $\mathcal{D}_k$.

This paper mainly considers a "truncated" (**B**,**S**)-market, consisting from a bank account and stocks of the same type. In this case, the flow of event σ-algebras $\mathbf{F} = (\mathcal{F}_k)_{k=0}^N$ is generated, generally speaking, by the entire development of the situation on the market (in particular, it can be generated by the prices of all l types of stocks). Economically the need to study such "truncated" (**B**,**S**)-markets is justified, for example, by the fact that many individuals (for example, participants of the privatization in Russia) received owned stocks of the enterprises where they worked, and subsequently had the ability to operate only with these specific stocks of your company, and also with a (relatively) risk-free bank account in Sberbank of Russia (case of storage money "under the pillow" corresponds to the value $r = 0$). In passing, we note that many of the indicated individuals subsequently experienced the buying up of stocks in their enterprises by aggressive buyers.

So, let S_k denote the price of a stock of one specific type at time k (further, this stock will be subject to aggressive buying). Since at the moment k the events of σ-algebras $\mathcal{F}_k$ are known to us, it is natural to assume that the price S_k is also known. Mathematically, this fact means that the r.v. S_k is measurable relative to $\mathcal{F}_k$, that is, for any value s that can take r.v. S_k, the event $\{S_k = s\}$ is included in $\mathcal{F}_k$. In view of the fact that this is true for any $k = 0, 1, 2, ..., N$, we say that the sequence of prices $(S_k)_{k=0}^N$ is adapted to the filtration $\mathbf{F} = (\mathcal{F}_k)_{k=0}^N$. The probability space $(\Omega, \mathcal{F}, P)$ together with filtration $\mathbf{F} = (\mathcal{F}_k)_{k=0}^N$ will be called *stochastic basis* $\left(\Omega, \mathbf{F} = (\mathcal{F}_k)_{k=0}^N, \mathcal{F}, P\right)$. Recall that $\mathcal{F}_0 = (\Omega, \emptyset)$ and $\mathcal{F}_N = \mathcal{F}$. If S_k are r.v. on $(\Omega, \mathcal{F}, P)$ and the sequence $(S_k)_{k=0}^N$ adapted to the filtration $\mathbf{F} = (\mathcal{F}_k)_{k=0}^N$, we say that (**B**,**S**)-market under consideration is determined on a stochastic basis $\left(\Omega, \mathbf{F} = (\mathcal{F}_k)_{k=0}^N, \mathcal{F}, P\right)$. In this work we will assume that the probability measure (=probability) P loads all elementary events (atoms) $\omega \in \Omega$, i.e. $\forall \omega \in \Omega \; P(\omega) > 0$. By default, we will assume that other considered here probability measures satisfy this property. All such measures are designated by the letter $\mathcal{P}$.

Let us denote by β_k the number of units of a bank account, and by γ_k the number of stocks at time k. The quantities β_k and γ_k can take both positive and negative values, which means borrowing from a bank account and the ability to "short sell" a stock.

Stochastic predictable sequence $\pi = (\beta_k, \gamma_k)_{k=0}^N$ (i.e. β_k and γ_k are $\mathcal{F}_{k-1}$-measurable $\forall\, k = 0, 1, ..., N$, $\mathcal{F}_{-1} := \mathcal{F}_0$) is called *investment strategy* or *portfolio* of securities. $\mathcal{F}_{k-1}$-measurability of β_k and γ_k means that the state of the portfolio at time k is completely determined by the information available at the moment $k-1$.

The capital of a portfolio π is a sequence of r.v. $(X_k^\pi)_{k=0}^N$ given by the formula:

$$X_k^\pi = \beta_k B_k + \gamma_k S_k. \tag{2.1}$$

The following theorem composes of results that can be found in [1].

Theorem 2.1. *Let us consider portfolio $\pi = (\beta_k, \gamma_k)_{k=0}^N$ of securities with capital* (2.1)*. Then the following conditions are equivalent:*

(a) $X_{k-1}^\pi = \beta_k B_{k-1} + \gamma_k S_{k-1}, \quad 0 < k \le N$ *(self-financing condition);*

(b) $B_{k-1}\Delta\beta_k + S_{k-1}\Delta\gamma_k = 0, \;\; 0 < k \le N$ *(balance property);*

(c) $\Delta X_k^\pi = \beta_k \Delta B_k + \gamma_k \Delta S_k, \;\; 0 < k \le N$ *(capital increment formula);*

(d) $\Delta\left(\frac{X_k^\pi}{B_k}\right) = \gamma_k \Delta\left(\frac{S_k}{B_k}\right), \;\; 0 < k \le N$ *(formula for increment of discounted capital).*

The self-financing condition (a) means that before the portfolio composition changes in the interval between time points $k-1$ and k, the portfolio does not experience any inflow additional capital, nor capital outflow (for example, for consumption).

Let us explain the financial meaning of the balance property (b). If $\Delta\gamma_k > 0$ (i.e. between time moments $k-1$ and k we ***buy*** stocks at price S_{k-1}), then from (b) it follows that $\Delta\beta_k < 0$, and therefore this purchase can be made only by withdrawing from the bank account an amount $-B_{k-1}\Delta\beta_k$ equal to $S_{k-1}\Delta\gamma_k$. If $\Delta\gamma_k < 0$, that is, the share is being sold, then the amount $B_{k-1}\Delta\beta_k = -S_{k-1}\Delta\gamma_k > 0$ is credited to the bank account.

Formula (c) shows that in the case under consideration ***real*** change of the capital occurs only due to ***real*** changes of the values ΔB_k and ΔS_k (expressing the change in bank account price and stock prices), and not due to changes in $\Delta\beta_k$ and $\Delta\gamma_k$ (figuratively speaking, from simply "shifting money from one pocket to another" a real increase of the capital can not be received).

Relation (d) means that when moving from time $k-1$ to time k the entire increase of discounted capital is determined only by the increase in discounted stock price.

All portfolios we consider here will be self-financing.

"Absence of arbitrage" in a market means that it is "fair", "rationally structured" in the sense that there is no possibility of making a profit without risk. More accurately (see [1]), we say that a self-financing portfolio π implements arbitrage opportunity if $X_0^\pi = 0$, $X_N^\pi \geq 0$ and $P\left(X_N^\pi > 0\right) > 0$. (**B,S**)-market in which there are no arbitrage opportunities, is called *arbitrage-free*.

Let us now move on to the concept of hedging. Assume that financial activity on the market is limited by moments $k = 0, 1, 2, ...N$. Let f_N be some $\mathcal{F}_N$-measurable r.v. called contingent claim. Self-financing portfolio $\pi = (\beta_k, \gamma_k)_{k=0}^N$ such that at the moment of time N its total capital X_N^π majorizes the contingent claim f_N (i.e. for which $X_N^\pi \geq f_N$) is called *hedging portfolio* (hedge) of this contingent claim or *hedging strategy*. If $X_N^\pi = f_N$, then the hedge is called *perfect*. Procedure building a hedging strategy is called *hedging*. (**B,S**)-market is called *complete* if $\forall\ f_N$ there exists a self-financing portfolio $\pi = (\beta_k, \gamma_k)_{k=0}^N$ and initial capital x such that $X_0^\pi = x$ and $X_N^\pi = f_N$, that is, any contingent claim f_N is achievable (replicable).

The econometric concepts of arbitrage-free and completeness have a direct connection with the theory of martingales. Let us introduce the concept of martingale and martingale measure. Designate once and for all by $A \in \mathcal{D}_k$ an arbitrary atom and consider its representation:

$$A = \bigcup_{i=1}^{m} B_i, \tag{2.2}$$

where B_i are atoms from $\mathcal{D}_{k+1}$ (m depends on A). Let us also put $Z_k|_A = a$, $Z_{k+1}|_{B_i} = b_i$. Then the fact that P^* is a martingale probability measure means that the following equality holds: $a = \frac{1}{P^*(A)}\sum_{i=1}^m b_i P^*(B_i)$, $\forall k\ (0 \leq k < N)$, $\forall A \in D_k$. In this case, the process $Z = (Z_k, \mathcal{F}_k, P^*)_{k=0}^N$ itself is called *martingale*.

The martingale measure is often called a risk-neutral measure. In the future, by $\mathcal{P}\left(Z, \mathbf{F}\right)$ we will denote the set of measures P^* from $\mathcal{P}$, with respect to whose process $Z = (Z_k, \mathcal{F}_k, P^*)_{k=0}^N$ is a martingale. Thus, the set $\mathcal{P}\left(Z, \mathbf{F}\right)$ is the set of martingales measures of process Z. It is easy to see that the set $\mathcal{P}\left(Z, \mathbf{F}\right)$ is a convex subset of the set $\mathcal{P}$. It follows that there are three possibilities: 1) $\mathcal{P}(Z, \mathbf{F}) = \emptyset$; 2) $\mathcal{P}(Z, \mathbf{F})$ consists of one element; 3) $\mathcal{P}(Z, \mathbf{F})$ consists of an infinite number of elements.

Two fundamental theorems of financial mathematics play a very important role (see [1]).

Theorem 2.2. *(**B,S**)-market is arbitrage-free if and only if $\mathcal{P}(Z, \mathbf{F}) \neq \emptyset$, where $Z_k = \frac{S_k}{B_k}$ $(k = 1, 2, ..., N)$.*

Theorem 2.3. *An arbitrage-free (**B,S**)-market is complete if and only if $\mathcal{P}(Z, \mathbf{F})$ consists of one element.*

Thus, if $\mathcal{P}(Z, \mathbf{F})$ consists of an infinite number of elements, then the (**B,S**)-market is arbitrage-free, but incomplete. Most of real financial markets are like that.

3. Haar Interpolations of Financial Markets on Finite Probability Spaces

Let us consider first the general concept of interpolation of financial markets. Let $\mathbf{G} = (\mathcal{G}_n)_{n=0}^{L}$ $(L > N)$ be some other filtration on the probability space $(\Omega, \mathcal{F}, P)$. $\mathbf{G}$-adapted process $Y = (Y_n, \mathcal{G}_n)_{n=0}^{L}$ will be called an interpolation of the process $Z = (Z_k, \mathcal{F}_k)_{k=0}^{N}$ if there is a sequence of indices $0 = n_0 < n_1 < ... < n_N = L$, for which $\mathcal{G}_{n_k} = \mathcal{F}_k$ and $Y_{n_k} = Z_k$ $\forall k$ $(0 \le k \le N)$. Let us define a new $(\tilde{\mathbf{B}}, \tilde{\mathbf{S}})$-market by the following relations: $\tilde{B}_n = B_k$ for $n_k \le n < n_{k+1}$ $(0 \le k < N)$, $\tilde{B}_L = B_N$; $\tilde{S}_n = \tilde{B}_n Y_n$ at $0 \le n \le L$ (the first relation corresponds to the natural assumption that interest is accrued only at times $k = 1, 2, ..., N$). Obviously, for $n = n_k$ $\tilde{B}_n = B_k$ and $\tilde{S}_n = S_k$, that is, the constructed $(\tilde{\mathbf{B}}, \tilde{\mathbf{S}})$-market interpolates initial $(\mathbf{B}, \mathbf{S})$-market.

It is clear that by means of interpolation it is possible to predict "market" stock prices in intervals between the announcement of new prices for them. If the interpolation is constructed successfully, then we can always benefit from this (for example, by quickly getting rid of some stocks and buying others or more effectively constructing hedging portfolios). Moreover, using interpolations we can improve some market properties.

We will deal primarily with Haar interpolations (these interpolations are universal in many cases). Let us describe the method of Haar interpolations (see [2], [3]).

Let us first consider the one-step model $(N = 1)$. We will proceed from the fact that new events (for simplicity let these be atoms from $\mathcal{D}_1$) arise not on the market at the same time. Thus, when moving from $k = 0$ to $k = 1$, one event from $\mathcal{D}_1$ arises first, then another, etc. Without loss of generality, we can assume that the order of their appearance is: $A_1^1, A_1^2, ..., A_1^{r_1}$. These events occur at intermediate times between $k = 0$ and $k = 1$. Enter intermediate times. We can set intermediate times in different ways, as long as they are ordered. Taking into account market assumptions it is best to change the timeline scaling (see Figure 1).

First, we define the σ-algebras of events : $\mathcal{H}_0 = \mathcal{F}_0$, $\mathcal{H}_1 = \left\{\Omega, \emptyset, A_1^1, \bar{\bar{A}}_1^1\right\}$ (that is, the σ-algebra generated by the atom A_1^1), $\mathcal{H}_2$ (σ-algebra, generated by the atoms

A_1^1 and A_1^2),...,$\mathcal{H}_{r_1-1}$ (σ-algebra, generated by the atoms A_1^1,A_1^2,...,$A_1^{r_1-1}$). It is clear, that $\mathcal{H}_{r_1-1} = \mathcal{F}_1$. Let's denote $n_1 = r_1 - 1$. So $\mathcal{H}_{n_1} = \mathcal{F}_1$. Resulting filtration is: $\{\Omega, \emptyset\} = \mathcal{H}_0 \subset \mathcal{H}_1 \subset ... \subset \mathcal{H}_{n_1} = \mathcal{F}_1$. We call such filtrations special interpolating Haar filtrations (SIHF, c.f. [4]). Since the extreme σ-algebras of the constructed Haar filtration coincide with the initial σ-algebras $\mathcal{F}_0$ and $\mathcal{F}_1$, then it is natural to say that filtration $(\mathcal{H}_n)_{n=0}^{n_1}$ interpolates filtration $(\mathcal{F}_k)_{k=0}^{1}$. In this case, the initial timeline is "stretched":

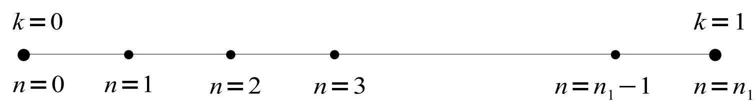

Figure 1. Stretched timeline

Note that interpolating Haar filtration (IHF) can be constructed in a more general way (c.f. [5]). For example, you can put $\mathcal{H}_1 = \{\Omega, \emptyset, A, B\}$, where $A = A_1^1 \bigcup A_1^2$, $B = \overline{A_1^1 \bigcup A_1^2}$, and at the next step split A or B. General definition (when N is any natural number) looks like this (see [3, p. 143]): Haar filtration $H = (\mathcal{H}_n)_{n=0}^{L}$ will be called interpolating Haar filtration of the initial filtration $\mathbf{F}$ if there is a sequence natural numbers $0 = n_0 < n_1 < ... < n_N = L$, for which $\mathcal{H}_{n_k} = \mathcal{F}_k$, $\forall k$ $(0 \le k \le N)$.

Assume that $\mathcal{P}(Z, F) \neq \emptyset$ and fix a measure $P^* \in \mathcal{P}(Z, F)$. Using the martingale $Z = (Z_k,\ \mathcal{F}_k,\ P^*)_{k=0}^{N}$ we construct *martingale Haar interpolation* $Y = (Y_n,\ \mathcal{H}_n, P^*)_{n=0}^{L}$, applying the formula: $Y_n = E^*[Z_N|\mathcal{H}_n]$ (probabilistic solution to the Dirichlet problem), where $E^*[Z_N|\mathcal{H}_n]$ means taking conditional mathematical expectation with respect to the measure P^*. If an interpolating Haar filtration $\mathbf{H}$ of filtration $\mathbf{F}$ is fixed, then the martingale Y is determined according to martingale Z uniquely.

Definition 3.1. We will say that a martingale measure $P^* \in \mathcal{P}(Z, \mathbf{F})$ satisfies Haar uniqueness property (HUP) if for the initial filtration $\mathbf{F}$ ***it is possible to construct*** such a Haar interpolation $\mathbf{H}$ that for the corresponding martingale interpolation $Y = (Y_n,\ \mathcal{H}_n)_{n=0}^{L}$ the set $\mathcal{P}(Y, \mathbf{H})$ consists of one element (that is, ***only*** with respect to the initial measure P^* the process Y is a martingale).

Much more important for applications is the following definition.

Definition 3.2. We will say that a martingale measure $P^* \in \mathcal{P}(Z, \mathbf{F})$ satisfies universal Haar uniqueness property (UHUP), if ***for any*** Haar interpolation $\mathbf{H}$ of the initial filtration $\mathbf{F}$ the set $\mathcal{P}(Y, \mathbf{H})$ consists of one element, where $Y = (Y_n,\ \mathcal{H}_n)_{n=0}^{L}$ is the corresponding martingale interpolation of the process Z.

Definitions 3.1 and 3.2, as well as a number of results on this topic are given in [2] (for a detailed description, see [3]). Let us present the main results from [2] and [3]. We use the notation (2.2).

Theorem 3.3. *1) If a measure $P^* \in \mathcal{P}(Z, \mathbf{F})$ satisfies* HUP*, then $\forall k$ $(0 \le k < N)$ and for any atom $A \in \mathcal{D}_k$ for $m > 1$ the following inequality holds:*

$$\min_{1 \le i \le m} b_i < a < \max_{1 \le i \le m} b_i, \tag{3.1}$$

and for $m = 1$ the following equality holds:

$$a = b_1. \tag{3.2}$$

2) If $\forall k$ $(0 \le k < N)$ and for any atom $A \in \mathcal{D}_k$ for $m > 1$ the inequality (3.1) *holds and for $m = 1$ the equality* (3.2) *holds, then **any** measure $P^* \in \mathcal{P}(Z, \mathbf{F})$ satisfies* HUP.

So, the fulfilment of HUP denies, for $m > 1$, the fulfilment of the equality $a = b_1 = b_2 = ... = b_m$, which means local trivialization of this market. So the fulfilment of HUP denies a local trivialization.

Theorem 3.4. *A measure $P^* \in \mathcal{P}(Z, \mathbf{F})$ satisfies* UHUP *iff $\forall\ k\,(0 \le k < N)$ and for any atom $A \in \mathcal{D}_k$ the set of numbers $\{b_1, b_2, \ldots, b_m\}$ supplied weights $p_1, p_2, \ldots, p_m$ (where $p_i = P^*(B_i)$, $i = 1, ..., N$) satisfies the non-coincidence barycentres condition* (NBC)*: for any two disjoint index subsets $I = \{i_1, i_2, \ldots, i_\alpha\}$ and $J = \{j_1, j_2, \ldots, j_\beta\}$ of the set $\{1, 2, \ldots, m\}$ the following inequality is fulfilled:*

$$\frac{b_{i_1}p_{i_1} + b_{i_2}p_{i_2} + \ldots + b_{i_\alpha}p_{i_\alpha}}{p_{i_1} + p_{i_2} + \ldots + p_{i_\alpha}} \ne \frac{b_{j_1}p_{j_1} + b_{j_2}p_{j_2} + \ldots + b_{j_\beta}p_{j_\beta}}{p_{j_1} + p_{j_2} + \ldots + p_{j_\beta}}.$$

Theorem 3.5. *Let $\mathcal{P}(Z, \mathbf{F}) \ne \emptyset$. In the set $\mathcal{P}(Z, \mathbf{F})$ there are measures satisfying* UHUP *if and only if $\forall\ k\,(0 \le k < N)$ and for any atom $A \in \mathcal{D}_k$ the numbers $b_1, b_2, \ldots, b_m$ are different and none of them coincides with the number a.*

Theorem 3.6. *Let $\mathcal{P}(Z, \mathbf{F}) \ne \emptyset$ and $\forall\ k$ $(0 \le k < N)$ in the transition from k to $k+1$ any atom A from $\mathcal{F}_k$ split up into no more than 3 atoms (i.e. $m \le 3$). Then, if the conditions of Theorem 3.5 are satisfied, **any** martingale measure $P^* \in \mathcal{P}(Z, \mathbf{F})$ satisfies* UHUP.

Unfortunately, Theorem 3.6 becomes false if at least one atom A is split into more than 3 parts.

Theorem 3.7. *Let $\mathcal{P}(Z, \mathbf{F}) \ne \emptyset$ and $\exists\ k$ $(0 \le k < N)$ and the atom A from $\mathcal{F}_k$, which upon transition from moment k to moment $k+1$ is split into more than 3 atoms (i.e. $m \ge 4$). Then there exists $P^* \in \mathcal{P}(Z, \mathbf{F})$ that does not satisfy* UHUP.

Note, however, there are martingale measures from $\mathcal{P}(Z, \mathbf{F})$ satisfying UHUP significantly more than measures that do not satisfy UHUP. Let us explain it more precisely. Obviously, the convex set $\mathcal{P}(Z, \mathbf{F})$ can be embedded in some finite-dimensional space. In this space we consider the subspace of the minimum dimension containing $\mathcal{P}(Z, \mathbf{F})$. Let us introduce in the selected subspace Lebesgue measure. Then the set of measures from $\mathcal{P}(Z, \mathbf{F})$ that do not satisfy UHUP has Lebesgue measure zero. From this, in particular, it follows that any measure from $\mathcal{P}(Z, \mathbf{F})$, which does not satisfy UHUP, can be approximated with any degree of accuracy by a measure that satisfies UHUP. This circumstance is effectively used in computational procedures (for example, when calculating options).

4. Stochastic Bases and Models of (B,S)-Markets Subjected to Aggressive Buying Up of Stocks

Since Ω is finite, it is convenient to describe filtrations by trees. Let us start from the most simple tree (Figure 2). According to it, the stochastic basis is defined as follows: $\Omega = \{B_1, B_2, ..., B_N, G_N\}$, $\mathcal{F}$ is the collection of all subsets of the set Ω, P is some probability loading all points of the set Ω, $\mathcal{F}_k = \sigma\{B_1, ..., B_k, G_k\}$ is the σ-algebra of events generated by elementary events $B_1, ..., B_k, G_k$ (i.e. the collection of all possible sums of these elementary events plus the empty set). The resulting filtration $(\mathcal{F}_k)_{k=0}^N$ is called *special Haar filtration.* On this stochastic basis, (**B**,**S**)-markets are considered, where the bank account and stock price can evolve in arbitrary ways. Event G_k means that at time k the stock continues to trade on the market. The event B_k is interpreted as follows:: if this event occurs, it means that the stock has been bought up at time n by one (or average) aggressive buyer. After that it begins to evolve just like a bank account (in the discounted case under some assumptions its price is frozen).

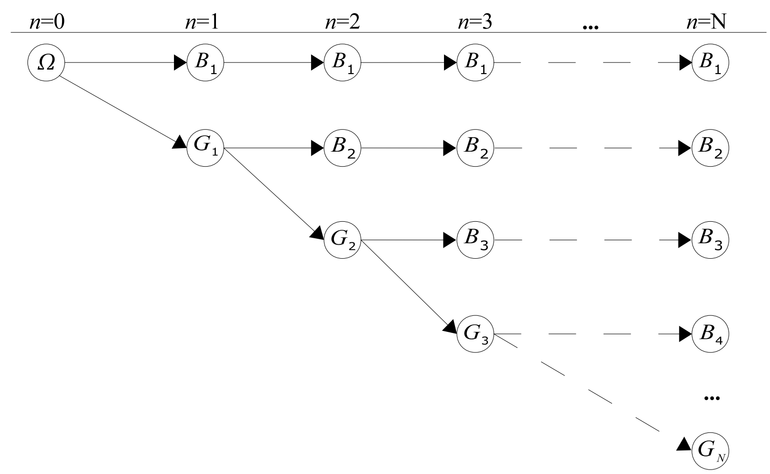

FIGURE 2. One buyer scheme

This circumstance adequately reflects the process of aggressive buying up: the purchased stock is not returned to the market, but is shelved until the end of aggressive buying. It is the time period of market functioning that can be modelled using the above scheme. This model was first proposed in the report [6], and then studied in works [7]–[16].

Let us now move on to a more complex model. Consider a tree in Figure 3. A distinctive feature of this model, proposed and studied in the works [9], [10] and [12] is that it separates the moments of time when the announcement occurs new stock prices, and moments when purposeful buying up of shares are made. This circumstance is formalized as follows: it is assumed that at even moments of time new stock prices are announced, and at odd ("intermediate") moments the buying up takes place. It is also natural to assume that information about the state of the market in odd moments is not publicly available and that at odd moments fluctuations in stock prices also occurs (perhaps, however, not with such an amplitude as in even-numbered ones).

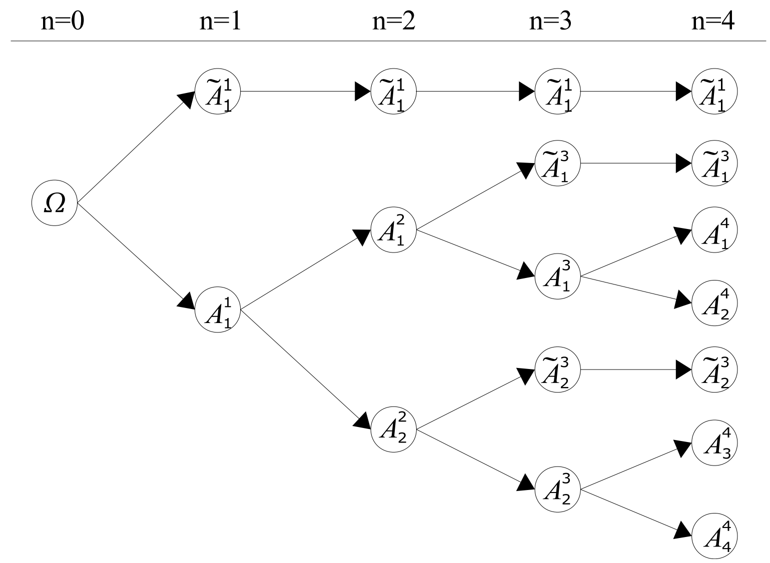

FIGURE 3. Advanced scheme

In connection with the described division of the roles of even and odd times σ-algebras of events making the filtration have different structures at even and odd times. Namely, if n is odd, then

$$\mathcal{F}_n := \sigma\left\{\tilde{A}_1^1;\ \tilde{A}_1^3, \tilde{A}_2^3; ....;\ \tilde{A}_1^n, \tilde{A}_2^n, ..., \tilde{A}_{2^{(n-1)/2}}^n;\ A_1^n,\ A_2^n, ..., A_{2^{(n-1)/2}}^n\right\};$$

for even n

$$\mathcal{F}_n := \sigma\left\{\tilde{A}_1^1;\ \tilde{A}_1^3, \tilde{A}_2^3; ....;\ \tilde{A}_1^{n-1}, \tilde{A}_2^{n-1}, ..., \tilde{A}_{2^{(n-2)/2}}^{n-1};\ A_1^n,\ A_2^n, ..., A_{2^{n/2}}^n\right\}.$$

Here the atoms $\tilde{A}_k^i,\ i = 1, 2, \ldots, n$, are considered as events, consisting in the fact that at time i the share is bought and its price, starting from this moment, evolves like a bank account. The atoms A_k^n correspond to the event, that at the moment in time n the stock has not yet been bought up and its price continues to evolve on the market. Since the purchased shares do not enter the market in the future, only atoms A_k^n split. This diagram (in which it is natural to consider a (**B**,**S**)-market model with one aggressive buyer of shares too) is more complex than the market model with respect to special Haar filtration, however it is more flexible and adequate. Moreover, it is a forerunner of the next models, the study of which requires additional mathematical apparatus.

Consider the tree (Figure 4):

This scheme successfully incorporates a model with two aggressive stock buyers. Let σ-algebra $\mathcal{F}_k$ is generated by partition of Ω into atoms $A_1, A_2, ..., A_{2k}, B_k$, where the event A_{2k-1} means that at time moment k the stock was bought by the first buyer, A_{2k} by the second buyer, and the event B_k is that the stock was

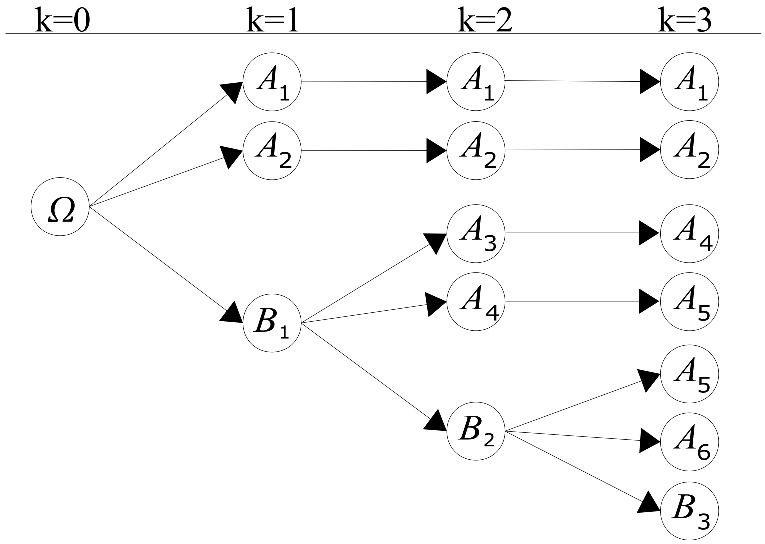

FIGURE 4. Two buyers scheme

not purchased. This model is no longer so favourable for calculations. The fact is that considered within its framework (**B**,**S**)-market is ***always*** incomplete (this follows from the fact that during the transition from moment $k-1$ to moment k, the atom B_{k-1} is split into ***three*** atoms: A_{2k-1},A_{2k} and B_k. The question arises: is it possible, having some additional information about this market, to convert it to complete on? Taking into account the results of works [2]–[3], it can be done for a fairly large class of arbitrage-free financial markets.

For simplicity, let the market under consideration be discounted and $(\mathcal{F}_k)$-adapted random sequence (Z_k) reflects the evolution of the price of stocks that are subject to purchase. By $Z_k\left(A_i\right)$ (resp., $Z_k\left(B_k\right)$) we will denote the value of the random variable Z_k on the atom $A_i,\ i=1,2,...,2n$ (resp. value Z_k on atom B_k). Let us assume that this (**B**,**S**)-market is arbitrage-free and for any $k \leq N$ numbers $Z_k\left(A_{2k-1}\right)$, $Z_k\left(A_{2k}\right)$ and $Z_k\left(B_k\right)$ are different, and none of them coincides with $Z_{k-1}\left(B_{k-1}\right)$. It follows from Theorem 3.6 that all martingale measures of this market satisfy UHUP. Let us explain that stems of this property. Let, for example, we know that the first buyer is always ahead of the second. Let us introduce the following σ-algebras of events: $\mathcal{H}_0 = \mathcal{F}_0$; $\mathcal{H}_1$ is generated by the events A_1 and $A_2{+}B_1$; $\mathcal{H}_2 = \mathcal{F}_1$, $\mathcal{H}_3$ is generated by events $A_1, A_2, A_3, A_4{+}B_2$; $\mathcal{H}_4 = \mathcal{F}_2$; etc., up to $\mathcal{H}_{2N} = \mathcal{F}_N$. We obtained a special Haar filtration $\mathbf{H} = (\mathcal{H}_n)_{n=0}^{2N}$, interpolating the initial filtration $\mathbf{F} = (\mathcal{F}_k)_{k=0}^{N}$. Consider **H**-adapted random sequence (Y_n) satisfying the conditions: for $0 \leq k \leq N$ $Y_{2k} = Z_k$ (i.e. the sequence (Y_k) interpolates (Z_n)) and (B,S)-market on a stochastic basis $\left(\Omega, (\mathcal{H}_k)_{k=0}^{2N}, \mathcal{F}, P\right)$, consisting of a single bank account and stocks $(Y_n)_{n=0}^{2N}$, is arbitrage-free. It is easy to show that in this case this interpolating market is complete. It is clear that the random variables Y_n for odd n can be define in many ways. Therefore, an effective forecast is very important here. If there is such a forecast, then it makes it possible to choose a martingale measure $P^* \in \mathcal{P}(Z, \mathbf{F})$, regulating the behaviour of this market, and obtain: $Y_n = E^*[Z_N|\mathcal{H}_n]$. If you need to fulfil some contingent claim F_N, then using well-known calculation formulas for a complete and arbitrage-free market we get a perfect hedge $(\beta_n, \gamma_n)_{n=0}^{2N}$, so that $x = \beta_0 + \gamma_0 Z_0$ is fair price of this contingent claim and $\beta_{2N} + \gamma_{2N} Z_{2N} = F_N$. Note that here in the hedging process not only the main moments of time $k = 0, 1, ..., N$ are involved, but also intermediate times $n = 1, 3, 5, ...$

The next more difficult tree is the tree in Figure 5.

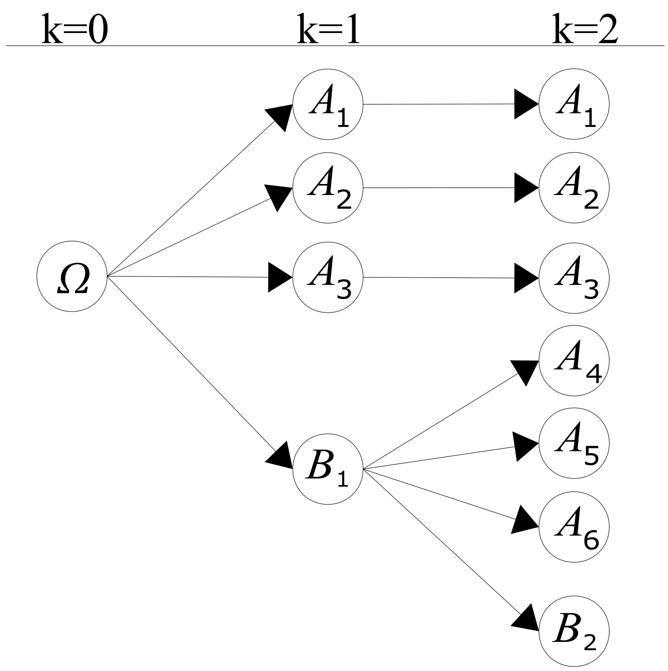

FIGURE 5. Three buyers scheme

Consider the stochastic basis generated by this tree, and a non-arbitrage discounted market on it, satisfying the conditions of Theorem 3.5 (see [13]). By Theorem 3.7, there exist martingale measures $P^* \in \mathcal{P}(Z, \mathbf{F})$ not satisfying UNUP. Thus, if a regulatory measure of a given market is a measure $P^* \in \mathcal{P}(Z, \mathbf{F})$ that does not satisfy UHUP, then, using the interpolation formula $Y_n = E^*[Z_N|\mathcal{H}_n]$, we may not get the complete market (for some Haar interpolations $\mathbf{H} = (\mathcal{H}_n)_{n=0}^{L}$ completeness can be obtained if HUP is satisfied, see Theorem 3.3). Thus, we again find ourselves in conditions of uncertainty. An exit is like this: it is necessary to approximate the regulatory measure P^* with the required degree of accuracy by the martingale measure $\tilde{P}$ satisfying UHUP, and carry out the interpolation process using this new measure. Relevant procedures are described in [14].

Note that in Figure 5 there are three stabilizing events: A_1, A_2, A_3. This corresponds to the case of aggressive buying with three aggressive stock buyers. Further generalization of this model to any finite number of aggressive buyers does not require new ideas: everything is done in the same way as in the case of three buyers.

5. Models of Aggressive Stock Buyers in the Case of an Infinite Number of Aggressive Buyers

In the last months of 2008 and at the beginning of 2009 as the crisis in Russia develops and during the fall exchange rate of the ruble and the growth of the dollar and euro (Figure 6), a significant part of the Russian population began buy currency and keep it "under the carpet". It can be considered an indicator of aggressive buying up by a large number of buyers. The works [17], [18] were devoted to the consideration of such models.

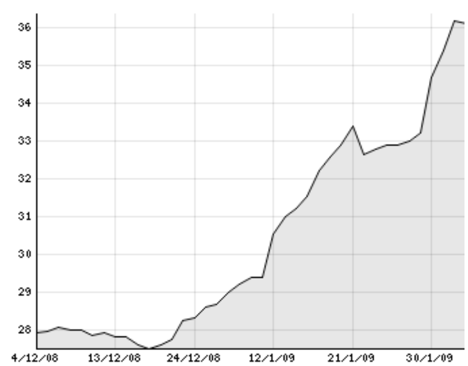

FIGURE 6. Evolution of dollar price during buying up

The question arises: how to act when there are a lot of buyers or their number is unknown in advance? In this case, from a mathematical point of view, it is most natural to assume that an infinite (more precisely, countable) number of aggressive buyers act at the market. Filtration tree of such models is shown in Figure 7.

The first model of this kind was proposed in [17]. The model of the work [17] is an infinite-dimensional (arbitration-free, but not complete) version of the Cox-Ross-Rubinstein model (see [1]). Under certain conditions using the Haar interpolation method, with the help of any initial martingale measure P^* this model can be converted to a complete and arbitrage-free market model. In this chapter we will touch upon results that are not related to this model.

It is clear that the influence of some buyers on the financial market is more significant than others. Thus, sometimes at each moment of time it is possible to make partitions of buyers into two classes: strong (priority) and weak (non-priority). Priority buyers work more energetically in the market, operate in it faster than others and, thus, have the opportunity to buy shares at a lower price (in conditions of fairly pronounced growth the last one). With such a division, the

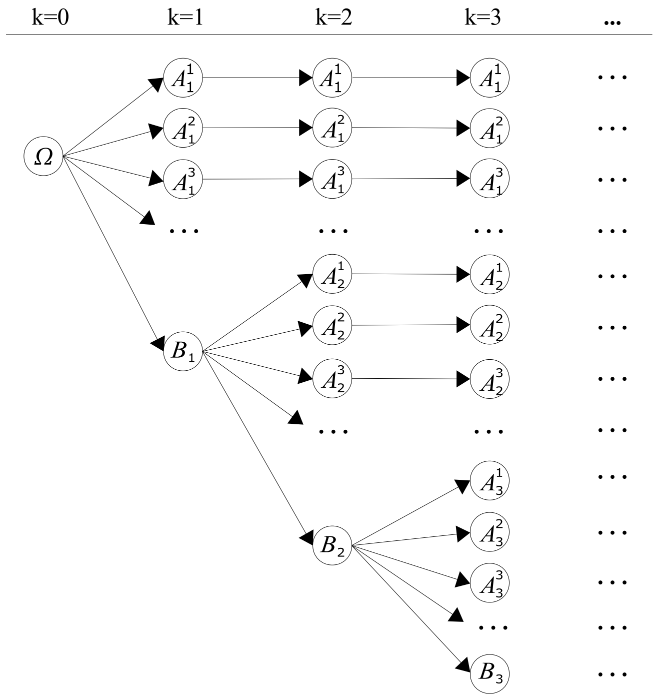

FIGURE 7. Infinite buyers scheme

number of non-priority buyers will be infinite, and the influence of each of them on the market is significantly less than each of the priority ones. It is natural to combine all non-priority buyers into one (average), which buys stocks at a certain average price. As a result of applying this method, it becomes possible to approximate the financial market under consideration with the help of a (**B**,**S**)-market with a finite number of aggressive stock buyers. Let's show how it is done.

Let Ω be a countable set. Consider the filtration $(\mathcal{F}_k)_{k=0}^N$ $(N < \infty)$ of the following form:

$$\mathcal{F}_0 = \{\Omega, \emptyset\}, \mathcal{F}_k = \sigma\left\{B_k; A_k^1, A_k^2, \ldots, A_k^i, \ldots; \right.$$
$$\left. A_{k-1}^1, A_{k-1}^2, \ldots, A_{k-1}^i, \ldots; \ldots; A_1^1, A_1^2, \ldots, A_1^i, \ldots\right\},$$

where A_k^i $(k = 1, 2, \ldots, N;\ i = 1, 2, \ldots)$ is an event consisting in the fact that the stock was bought by the ith buyer at time k; event B_k $(k = 1, \ldots, N)$ is that by time k the stock was not purchased. As always, we assume that $\mathcal{F} = \mathcal{F}_N$.

Since computer technology can only operate with finite-dimensional structures data (and in our model Ω is a countable set), the question arises: how in this way the initial market model can be approximated by a model with a finite number of states. It turns out that this problem can be solved if for each moment $k = 1, 2, \ldots, N$ we apply the following procedure:

(1) determine the procedure for buyers' access to the market;

(2) set stock prices;
(3) identify priority buyers and identify non-priority buyers who cannot be ignored; from these latter, form an average of non-priority buyer, and ignore the remaining non-priority buyers; as a result, obtain a (**B**,**S**)-market model with a finite number of states;
(4) determine the interpolation properties of the resulting model.

Let us briefly describe this procedure. Let $Z = (Z_k, \mathcal{F}_k)_{k=0}^N$ be adapted random process, by which we mean the discounted value of stock of a certain type, subject to aggressive buying by a countable number of aggressive buyers. Let d_i ($i = 1, 2, \dots;\ d_1 > -1$) be infinitely increasing bounded sequence of interest rates that determines discounted stock prices Z_k as follows way:

$$\begin{cases} Z_k(B_k) & = Z_{k-1}(B_{k-1})(1+d_\infty) \\ Z_k(A_k^{n_k^{(i)}}) & = Z_{k-1}(B_{k-1})(1+d_i) \qquad (i = 1, 2, \dots) \\ Z_k(A_j^{n_j^{(i)}}) & = Z_{k-1}(A_j^{n_j^{(i)}}) \qquad (j = 1, 2, \dots, k-1; \quad i = 1, 2, \dots) \\ & \end{cases},$$

where $(n_k^{(1)}, n_k^2, \dots, n_k^{(i)}, \dots)$ is a permutation sequences of natural numbers $(1, 2, \dots, i, \dots)$, $k = 1, 2, 3, \dots, N$. The number $n_k^{(i)}$ means that in the interval between moments of time $k-1$ and k the i^{th} buyer gained access to the market under number $n_k^{(i)}$.

The sequence d_i is strictly increasing, and $\lim\limits_{i\to\infty} d_i \le d_\infty < \infty$, $d_i \neq 0$, $\forall i$.

The market is arbitrage-free, i.e. one of two conditions is satisfied:

1) $\exists\, l < \infty$ that $d_1 < d_2 < \dots < d_l < 0 < d_{l+1} < d_{l+2} < \dots$;

2) $d_1 < d_2 < \dots < 0 < d_\infty$.

Let us define in some way the number c ($d_1 < c < \lim\limits_{i\to\infty} d_i$), which we will call the criterion interest rate for selection priority buyers. The number of priority buyers m is determined as follows:

$$\tilde{m} = max\,\{i : d_i < c\}\,.$$

It is clear that $1 \le \tilde{m} < \infty$. We will consider all other buyers as non-priority. We will divide these latter into two classes: into a finite set of non-priority buyers, which cannot be ignored, and a countable number of other non-priority buyers, which we will ignore.

Let us describe the procedure for finding $\tilde{\tilde{m}}$ non-priority buyers who cannot be ignored. Let ε ($0 < \varepsilon < 1$) be some number, reflecting the accuracy of the transition from a model with an infinite number of states to a finite model. We put:

$$\tilde{\tilde{m}} := max\left\{ i - \tilde{m} | i > \tilde{m}, d_i - d_{i-1} > \varepsilon, \frac{d_i - d_{i-1}}{d_\infty - d_{i-1}} > \varepsilon \right\}.$$

It is clear that $0 \le \tilde{\tilde{m}} < \infty$. We will assume that the stock is being bought by the "averaged" non-priority buyers at an average interest rate

$$d = \sum_{i=\tilde{m}+1}^{\tilde{m}+\tilde{\tilde{m}}} \frac{d_i}{\tilde{\tilde{m}}}. \tag{5.1}$$

Depending on the value of the interest rate d, two situations are possible:

(1) $d = 0$; set the new value of $\tilde{\tilde{m}}$ equal to $\tilde{\tilde{m}}_{new} = \tilde{\tilde{m}} + 1$ and recalculate d using the formula (5.1); in this case d_{new} becomes strictly positive;
(2) $d \neq 0$; no correction required.

Note that if $d_\infty > \lim\limits_{i\to\infty} d_i$, then always $d < 0$ (no correction required).

Theorem 5.1. *With the described choice of priority and non-priority buyers the* $(\boldsymbol{B}, \boldsymbol{S})$*-market under consideration can be naturally approximated by a market with a finite number of states satisfying the property of universal Haar uniqueness.*

Described procedure is the basis of the "Hedging by Interpolation" software package. During the development of this software package, a number of algorithms were created with its own input and output data, sequence of steps and options of developments of events. Let us list the main ones.

Algorithm 1. General algorithm of the software package.

Algorithm 2. Algorithm according to which the complex constructs (**B**,**S**)-market model with an arbitrary number of aggressive buyers.

Algorithm 3. Procedure for making financial calculations.

Algorithm 4. Determination of priority, non-priority and average buyers.

Detailed description of these and other algorithms is presented in [18], [19].

The software package is divided into the main core and modules. The core of the program provides basic functionality required to work with discrete structures of financial markets. The modules implement algorithms for working with specific (B, S)-market models. ***Transparent*** interaction between the kernel and modules by the "Abstract Model" class is implemented. With this approach, the core of the complex ***is not knows*** which specific model it works with. This makes expansion of the complex with new models of financial markets much easier. After the user has designed model, the complex begins to interact with a specific module (by the abstract model).

To construct models, a special extensible "wizard" has been created, allowing carry out a step-by-step dialogue with the user of the program. Optional expansion of the functionality of the "wizard" is performed using special external ***directories***. Each directory is an XML-file that contains algorithms for determining priority and non-priority buyers. These algorithms are implemented using technology QtScript (allowing to execute external scripts in the main application environment) libraries Qt4. The directories also contain rules for specifying parameters that are for these algorithms source data, and set the values of the resulting variables. Thus, the logic associated with the transition from an infinite-dimensional model to a finite-dimensional one is taken out from the main program. The latter easily permits to expand the functionality of the software package by adding new directories and makes it possible to use the developed technology when implementing new models of financial markets.

The software package was implemented using a cross-platform library Qt4 and its extensions. A third-party library GLPK is used to solve optimization problems. Object-oriented language C++ was chosen as the main algorithmic language. Thus, it was possible to obtain a software package that meets modern requirements for the software, such as: operation on a number of software platforms (Windows, Linux, MacOS X), high performance, convenient and ergonomic user interface. The software package successfully copes with calculations within the framework of the considered models of financial markets and can be expanded by new models.

6. UHUP for Static Financial Markets with a Countable Number of States

In Chapter 5, the study of financial market models with a countable number of states was reduced to the transition to models with a finite number of states. This chapter and further demonstrates the development of interpolation techniques directly on countable probability spaces.

Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_n, \dots\}$ be a countable set, $\mathcal{F}_0 = \{\Omega, \emptyset\}$, $\mathcal{F}_1$ be the σ-algebra of all subsets of Ω, $\{\Omega, \mathbf{F} = (\mathcal{F}_0, \mathcal{F}_1)\}$ be a one-step filtration. Consider a static $(\mathbf{1},\mathbf{Z})$-market, where $Z = (Z_k, \mathcal{F}_k)_{k=0}^{1}$ is an $\mathbf{F}$-adapted stochastic process (a discounted value of a stock). Denote $Z_0 = a, Z_1(\omega_i) = b_i$, $(i \in \mathbb{N} = \{1, 2, 3, \dots\})$.

We identify P and the vector $(p_1, p_2, ...)$, where $p_i = P(\omega_i)$, and consider only probability measures $P = (p_1, p_2, ...)$ with strictly positive components (non-degenerate probability measures). We denote by $\mathcal{P}(Z, \mathbf{F})$ the set of non-degenerate martingale measures P of $(\mathbf{1},\mathbf{Z})$-market under consideration. It is obvious that $\mathcal{P}(Z, \mathbf{F})$ coincides with the set of solutions of the system

$$\begin{cases} \sum\limits_{i=1}^{\infty} p_i = 1 \\ \sum\limits_{i=1}^{\infty} b_i p_i = a \\ p_i > 0, \ i \in \mathbb{N}. \end{cases} \tag{6.1}$$

We assume that $\inf\limits_i b_i < a < \sup\limits_i b_i$. This condition provides the solvability of the system (6.1) and hence the absence of arbitrage possibilities at the financial market. The incompleteness of this market is obvious.

For the passage from incomplete markets to complete ones let us use (as in Chapter 3) such important property of the measure $P \in \mathcal{P}(Z, \mathbf{F})$ as UHUP. Recall that a filtration $\mathbf{H} = (\mathcal{H}_n)_{n=0}^{\infty}$, $\mathcal{H}_0 = \{\Omega, \emptyset\}$, is Haar filtration if, when passing from the time moment n to the moment $n+1$, exactly one atom from $\mathcal{H}_n$ is split into two parts (atoms of $\mathcal{H}_{n+1}$) and the other atoms remain unchanged. Such a filtration $\mathbf{H}$ is said interpolating Haar filtration (IHF) for the (B, S)-market filtration $\mathbf{F}$ if $\sigma\{\mathcal{H}_n, n \geq 0\} = \mathcal{F}_1$. We say that a measure $P \in \mathcal{P}(Z, \mathbf{F})$ satisfies universal Haar uniqueness property (UHUP) if for any IHF $\mathbf{H}$ of the filtration $\mathbf{F}$ the interpolating process $Y = (Y_n, \mathcal{H}_n)_{n=0}^{\infty}$, where $Y_n = E^P[Z_1|\mathcal{H}_n]$, admits the unique martingale measure (coinciding with the initial measure P). In [20] it is shown that this property (as in Chapter 3) is equivalent to the non-coincidence barycentre condition (NBC). In fact, the following proposition is true.

Proposition 6.1. *A martingale measure* $P = (p_1, p_2, ...)$ *satisfies* UHUP *if and only if for any two disjoint subsets* $I, J \subset \mathbb{N}$ *of indices the following inequality holds:*

$$\frac{\sum\limits_{i\in I} b_i p_i}{\sum\limits_{i\in I} p_i} \neq \frac{\sum\limits_{j\in J} b_j p_j}{\sum\limits_{j\in J} p_j}. \tag{6.2}$$

In work [21],very restrictive conditions were obtained for the market under consideration, ensuring the existence of martingale measures satisfying UHUP (the author of this revue is not aware of any other results in this direction). Let us formulate the corresponding theorem.

Theorem 6.2. *Let the* (B, S)*-market under consideration be such that the sequence* $\{b_i,\, i \geq 1\}$ *is strictly positive,* $b_1 < a < b_2$ *and*

$$b_i \geq 2b_{i-1}, \quad \forall i \geq 2. \tag{6.3}$$

Then there exists a measure $P \in \mathcal{P}(Z, \boldsymbol{F})$ *such that it satisfies* NBC.

When studying specific models of financial markets, a weaker property can often be used instead UHUP, namely, the special Haar uniqueness property (SHUP). The papers [22]–[30] are devoted to this topic.

7. SHUP for Static Financial Markets with a Countable Number of States

In this chapter we continue to use the notation from Chapter 6.

Let us first outline the main points of the theory of special Haar interpolations. For the passage from incomplete markets to complete ones we use in this Chapter so-called SHUP, namely, special Haar uniqueness property of martingale measures. Let $\{k_1, \ldots, k_n, \ldots\}$ be a permutation of $\mathbb{N} = \{1, 2, \ldots, n, \ldots\}$. Let us construct the filtration with an infinite horizon:

$$\mathcal{H}_0 = \mathcal{F}_0,\, \mathcal{H}_1 = \sigma\{B_{k_1}\},\, \ldots,\, \mathcal{H}_n = \sigma\{B_{k_1}, B_{k_2}, \ldots, B_{k_n}\},\, \ldots,\, \mathcal{H}_\infty = \mathcal{F}_1.$$

This filtration is called special Haar filtration interpolating the initial filtration $\mathbf{F}$. For $P \in \mathcal{P}(Z, \mathbf{F})$ let us define (as in Chapter 6) Dirichlet martingale $Y_n := E^P[Z_1|\mathcal{H}_n]$. It is obvious that $Y_0 = Z_0, \quad Y_\infty = Z_1$. The process $Y = (Y_n, \mathcal{H}_n)_{n=0}^\infty$ is called special Haar interpolation of Z.

Definition 7.1. Let $P \in \mathcal{P}(Z, \mathbf{F})$. If for any permutation $\{k_1, \ldots, k_n, \ldots\}$ of $\{1, 2, \ldots, n, \ldots\}$ the process $Y = (Y_n, \mathcal{H}_n)_{n=0}^\infty$ admits only one martingale measure (it is just the initial measure P) then we say that P satisfies the special Haar uniqueness property (SHUP).

As for UHUP, the following proposition can be easily proved.

Proposition 7.2. *A martingale measure $P = (p_1, p_2, ...)$ satisfies* SHUP *if and only if $\forall i \in \mathbb{N}$ and for any set of indices $J \subset \mathbb{N}$ not containing i and such that its complement $\bar{J} = \mathbb{N} \setminus J$ is finite the following inequality is fulfilled:*

$$b_i \neq \frac{\sum\limits_{j\in J} b_j p_j}{\sum\limits_{j\in J} p_j}. \tag{7.1}$$

Put $J = \mathbb{N} \setminus \{i\}$ in the inequality (7.1). Then this inequality is equivalent to the following one:

$$b_i \neq \frac{a - b_i p_i}{1 - p_i} \Leftrightarrow b_i \neq a.$$

Thus a necessary condition for the the existence of a martingale measure satisfying SHUP is that the inequalities $b_i \neq a$ ($\forall i \in \mathbb{N}$) are fulfilled.

Definition 7.3. We say that a number $b \in \{b_k\}_{k=1}^{\infty}$ is of order m ($1 \leq m \leq \infty$) if this number appears in the sequence $\{b_k\}_{k=1}^{\infty}$ just m times.

The following two facts are evident:

- if the sequence $\{b_k\}_{k=1}^{\infty}$ contains a finite number of distinct values and only one of them is of order ∞, then there are not martingale measures satisfying SHUP;
- if the sequence $\{b_k\}_{k=1}^{\infty}$ contains only two different numbers, both of infinite order, then any martingale measure satisfies SHUP.

Now we will suppose that the sequence $\{b_k\}_{k=1}^{\infty}$ contains r different values ($3 \leq r < \infty$)

Without loss of generality we can assume that $b_1 < b_2 < \ldots < b_r$. Let these numbers be of oder m_1, m_2, ... , m_r respectively. Taking into account the above, it will be assumed that ***not less than two of numbers** m_1**,** m_2**, ... ,** m_r **are infinite***. Renumbering, we easily obtain the following system of notation:

$$\begin{gathered}
b_1 = b_{r+1} = b_{2r+1} = \ldots = b_{r(j-1)+1} = \ldots, 1 \leq j < m_1 + 1, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
b_k = b_{r+k} = b_{2r+k} = \ldots = b_{r(j-1)+k} = \ldots, 1 \leq j < m_k + 1, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
b_r = b_{2r} = b_{3r} = \ldots = b_{rj} = \ldots, 1 \leq j < m_r + 1.
\end{gathered}$$

Two next theorems were proved in [27].

Theorem 7.4. *Let $r = 3$ and $b_1 < a < b_2 < b_3$ or $b_1 < b_2 < a < b_3$. Then there are martingale measures satisfying SHUP.*

Theorem 7.5. *Let $3 < r < \infty$, the numbers $b_1 < b_2 < \ldots < b_r$ be rational, a be a real number, $b_1 < a < b_r$, $a \neq b_k$ ($k = 2, 3, \ldots, r-1$). Then there are martingale measures satisfying SHUP.*

Theorem 7.5 is not the final result in the description of (**B**,**S**)-markets, where there are martingale measures satisfying SHUP. According to the opinion of the authors of this paper, if $4 \leq r < \infty$, $b_1 < b_2 < \cdots < b_r$, $b_1 < a < b_r$ and

$a \neq b_k$ $(2 \leq k \leq r-1)$, then a martingale measure satisfying SHUP always exists. Currently, however, the authors have no proof of this fact even for the case of $r = 4$.

The main result of the paper [29] is the following theorem.

Theorem 7.6. *If $r = \infty$, all numbers b_k $(k \in \mathbb{N})$ are rational and number a is irrational, then there are martingale measures satisfying SHUP.*

Remark 7.7. Results that to some extent generalize the results of works [27] and [29] can be found in [31].

Remark 7.8. Since UHUP $\Longrightarrow$ SHUP, then Theorem 6.2 also provides sufficient conditions for the existence of martingale measures satisfying SHUP.

We see that the theory of martingale measures satisfying SHUP is much more developed than for martingale measures satisfying UHUP. This is due to the fact that solving system of inequalities (6.1)–(7.1) which is a system of linear inequalities is easier than solving system of inequalities (6.1)–(6.2) which is a system of quadratic inequalities.

Remark 7.9. An algorithm for hedging (using Haar interpolations) of Markov type contingent claims in a market with a countable number of states is contained in [30].

8. Inverse Problems

The content of this chapter is based on the results of the articles [32] and [33].

In static financial markets defined on finite or countable probability spaces, the inverse problem is formulated as follows: for a predetermined probability measure P and an initial condition $a > 0$, prove the existence of a stock whose price at the initial moment coincides with a, and the measure P for the price process is an interpolating martingale measure (i.e. satisfying UHUP or SHUP). It is shown that this is true for finite probability spaces. For countable probability spaces, sufficient conditions are found for this statement to hold.

The financial meaning of the inverse problem is as follows. Let a hedger acting in the financial market believe that calculations of fair prices of contingent claims should be made using some probabilistic measure P. Suppose that he may describe all markets in which this measure is risk-neutral (martingale). Solving the inverse problem (that is, building the market according to such a predetermined measure) he can select those markets in which the martingale measure P allows to obtain the most fair prices. In view of the ability to interpolate these markets to complete ones, the hedger is able to form standard hedging portfolios of various contingent claims.

First, we formulate the result about the existence of the desired static market with a finite number of states (this result is complete).

Theorem 8.1. *Let Ω be a finite set, $P = (p_1, p_2, \ldots, p_r)$ be a non-degenerate probability measure on Ω and $a > 0$. Then there exists a stock with the price $Z = (Z_0, Z_1)$ such that $Z_0 = a$, the values $b_k \neq a$ $(1 \leq k \leq r)$ of Z_1 are strictly positive and strictly monotone, and in the obtained market the probability P is a martingale measure satisfying UHUP.*

Now let us pass to the question on the existence of the desired static market with a countable number of states, but when the value of a stock on which can take only a finite number of different values (this result also has a final form).

Theorem 8.2. *Let Ω be countable, $a > 0$, and P be a non-degenerate probability measure on Ω. Then for any integer $1 < r < \infty$ there exists a stock with the price $Z = (Z_0, Z_1)$ such that $Z_0 = a$ and:*

1) among the values of the r.v. Z_1 exactly r values (which we denote $b_1, \dots, b_r$) are different and strictly positive;

2) $b_k \neq a$ $(1 \leq k \leq r)$;

3) the numbers $b_1, \dots, b_s$, $2 \leq s \leq r$, have infinite order and the numbers $b_{s+1}, \dots, b_r$ have a finite order;

4) in the obtained market the probability P is a martingale measure satisfying SHUP.

Unfortunately, it was not possible to obtain any acceptable result about the existence of a static market with a countable number of states, where the value of a stock can take on an infinite number of different values.

9. Signed Interpolating Deflators

Currently, the development of the theory of Haar interpolations of financial markets with the use of martingale measures continues. The existence of martingale measures of discounted stock prices means that this kind of interpolation can only be used in arbitrage-free markets. However, real financial markets often contain elements of arbitrage opportunities. Therefore, it is important to develop techniques for interpolating processes that do not admit martingale measures. This work is devoted to just this problem. Here, signed deflators serve as the main interpolation tool. With their help, the Haar interpolation procedure is defined. In the case of the existence of martingale measures, this procedure leads to the interpolating process, which coincides with the martingale interpolation. The paper introduces the concept of an admissible deflator, defines (as when martingale measures exist) the universal Haar uniqueness property (UHUP) and its weakened (special) variant (SHUP). The main results we obtained up to the moment are related to SHUP, which leads to the uniqueness of the admissible deflator.

We use in this chapter notations from Chapters 2 and 3 and results from [34]–[35].

Definition 9.1. Let $Z = (Z_k, \mathcal{F}_k)_{k=0}^N$ be an adabted process that can take any real values. A martingale $D = (D_k, \mathcal{F}_k, P)_{k=0}^N$ is said a signed deflator of the process Z if D_0=1 and the process $DZ = (D_k Z_k, \mathcal{F}_k, P)_{k=0}^N$ is a martingale.

Recall that Haar filtration $(\mathcal{H}_n)_{n=0}^L$ is said interpolating Haar filtration (IHF) of $(\mathcal{F}_k)_{k=0}^N$ if there exists an increasing sequence of integers $n_k, 0 \leq k < N+1$, such that $\mathcal{H}_{n_k} = \mathcal{F}_k$ (and hence $\mathcal{H}_L = \mathcal{F}_N$).

Let us fix an IHF $(\mathcal{H}_n)_{n=0}^L$ of $(\mathcal{F}_k)_{k=0}^N$ and let $D = (D_k, \mathcal{F}_k, P)_{k=0}^N$ be a signed deflator of the process $Z = (Z_k, \mathcal{F}_k)_{k=0}^N$. Denoting $X_{n_k} := D_k Z_k$ and $Y_{n_k} := D_k$, we obtain martingales $(X_{n_k}, \mathcal{H}_{n_k}, P)_{k=0}^N$ and $(Y_{n_k}, \mathcal{H}_{n_k}, P)_{k=0}^N$. Then we can define two martingales $X = (X_n, \mathcal{H}_n, P)_{n=0}^L$ and $Y = (Y_n, \mathcal{H}_n, P)_{n=0}^L$ in the following obvious way: for any $n < L+1$ find $n_k \geq n$ and put

$$X_n := E^P[X_{n_k}|\mathcal{H}_n],\ Y_n := E^P[Y_{n_k}|\mathcal{H}_n]. \tag{9.1}$$

It is clear that such definitions are correct.

Remark 9.2. From the properties of the mathematical expectation it follows the implication:

$$\{D_k = 0\} \subset \{D_k Z_k = 0\}\ (P-a.s.) \Rightarrow \{Y_n = 0\} \subset \{X_n = 0\}\ (P-a.s.). \tag{9.2}$$

Definition 9.3. The process $Z^{int} = (Z_n^{int}, \mathcal{H}_n)_{n=0}^L$ defined by the formula

$$Z_n^{int} = \begin{cases} Z_k,\ if\ n = n_k\ (0 \leq k < N+1), \\ \dfrac{X_n}{Y_n},\ if\ n \neq n_k, Y_n \neq 0, \\ 1, \quad if\ n \neq n_k, Y_n = 0, \end{cases} \tag{9.3}$$

will be called H-interpolation of the process Z with the help of the deflator D.

Remark 9.4. Let the process $Z = (Z_k, (\mathcal{F}_k)_{k=0}^N)$ admit a martingale measure Q, equivalent to the physical measure P, i.e. the process $(Z_k, \mathcal{F}_k, Q)_{k=0}^N$ be a martingale. Denote $h := \dfrac{dQ}{dP}$ and $D_k := E^P[h|\mathcal{F}_k]$. It is clear that the process $D = (D_k, \mathcal{F}_k)_{k=0}^N$ is a strictly positive deflator of the process Z. Hence for all $n \leq n_k$ $Y_n = E^P[Y_{n_k}|\mathcal{H}_n] = E^P[D_k|\mathcal{H}_n] > 0$ and $Z_n^{int} = \dfrac{X_n}{Y_n}$. Applying the generalized Bayes formula, it is easy to see that the process $(Z_n^{int}, \mathcal{H}_n, Q)_{n=0}^L$ is a martingale. From this fact it follows that H-interpolation of the process Z with the help of deflator D coincides with the Haar interpolation of Z with respect to the martingale measure Q (c.f. [2], [3]).

Definition 9.5. We say that a signed deflator $D = (D_k, \mathcal{F}_k, P)_{k=0}^N$ satisfies the universal Haar uniqueness property — UHUP (resp., the special Haar uniqueness property — SHUP) if for every interpolating (resp., special interpolating) Haar filtration $\mathbf{H} = (\mathcal{H}_n)_{n=0}^L$ of the initial filtration $\mathbf{F}$ the process (9.3) admits only one deflator, namely the deflator $Y = (Y_n, \mathcal{H}_n, P)_{n=0}^L$, defined by (9.1).

We use in the sequel the notation (2.2) and we put $p_i := P(B_i), d_i := D_{k+1}|_{B_i}$.

Definition 9.6. A signed deflator D of the process Z is said admissible if $\forall 0 \leq k < N+1$, for all atom $A \in \mathcal{F}_k$ and for all non-empty subset $I \subset \{1, 2, \ldots, m\}$

$$\sum_{i \in I} p_i d_i \neq 0.$$

Proposition 9.7. *If the deflator D is admissible then the process* $Y = (Y_n, \mathcal{H}_n, P)_{n=0}^L$ *is a signed deflator of the process* $Z^{int} = (Z_n^{int}, \mathcal{H}_n)_{n=0}^L$.

The following theorems are the main ones in this chapter.

Theorem 9.8. *Let* $\forall k : 0 \leq k < N+1$ *and for all atom* $A \in \mathcal{F}_k$ *we have* $m \geq 3$. *If there exists an admissible signed deflator D satisfying SHUP, then the numbers* $a, b_1, \ldots, b_m$ *are different.*

Theorem 9.9. *Let $\forall k : 0 \leq k < N+1$ and for all atom $A \in \mathcal{F}_k$ we have $m \geq 4$ and the numbers $a, b_1, \ldots, b_m$ be different. Then there exists an admissible signed deflater D satisfying SHUP.*

Unfortunately, we have not identified significant applications of Theorems 9.8 and 9.8 to the study of financial markets.

Remark 9.10. Application of the technique of Haar interpolation to arbitrage markets without the use of deflators can be found in [36]–[38].

10. Conclusion

The problem of interpolating stochastic systems in order to improve their properties is very important for carrying out various kinds of calculations within these systems, as well as for making optimal decisions. One of such problems, namely the task of transforming incomplete financial markets into complete ones, was considered in 1987 in the work of M. Takku and W. Willinger [39]. In this work the transition from incomplete markets to complete markets was carried out by replacing the original martingale measure with a nonequivalent martingale measure. Another technique was proposed by A.V. Melnikov and K.M. Feoktistov [40]. They completed the financial market by adding additional risky assets to the shares of this original market, functionally dependent on the original ones. In the works of I.V. Pavlov and M.N. Bogacheva [2]–[3] the foundation was laid for a principally different method of transition from incomplete markets to complete ones. To solve the problem of transforming incomplete arbitrage-free markets into complete arbitrage-free financial markets, the method of interpolation of financial markets was used, associated with the use of Haar filtrations and martingale interpolation. Further, in this direction, research continued in the articles of I.V. Pavlov, V.V. Gorgorova, A.G. Danekyants, T.A. Volosatova, V.V. Shamraeva, I.V. Tsvetkova, N.V. Neumerzhitskaia etc. The results of these works are presented in this review. However, this topic is very far from being completed. Let us note the following important (in our opinion) tasks that remain unsolved to date.

- The existence of martingale measures satisfying SHUP in the case of static markets with a countable number of states has not been fully explored.
- In the case of static markets with a countable number of states, questions of the density of the set SHUP in the set of martingale measures have not been studied.
- No satisfactory sufficient conditions have been obtained (except for one result) for the existence of martingale measures satisfying UHUP in the case of a static market with a countable number of states.
- Apart from one paper (c.f. [41]), there are no results on Haar interpolations for a market in which multiple types of shares are traded.
- The inverse problems of constructing financial markets in which predetermined probabilities satisfy UHUP have not been solved.
- The study of interpolating signed deflators is in its infancy.

The author of this review hopes that the listed problems will attract the attention of researchers.

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