Stochastic Modelling and Computational Sciences
Vol. 3 No. 2 (December, 2023)

STOCHASTIC EQUATION OF A POROUS MEDIUM WITH FRACTIONAL LAPLACIAN AND WHITE NOISE

MAMADSHO ILOLOV, JAMSHED SH. RAHMATOV, AND SUHROB LASHKARBEKOV

ABSTRACT. This work is devoted to the study of the existence and uniqueness of soft solutions of the Cauchy problem for stochastic equations of a porous medium with a fractional Laplacian and white noise. The required results are obtained using the technique of stochastic analysis, fractional calculus and the theory of non-linear semigroups. The regularity properties of solutions for the generalized stochastic equation of a porous medium are also established.

1. Introduction

The stochastic porous medium equation (SPME) as a filtration model is encountered in the modeling of various phenomena, such as fluid dynamics, stochastic underground hydrodynamics, astrophysics and statistical physics (see, for example, [1-4] and detailed bibliography therein). When the fractional Laplacian $(-\triangle)^{\alpha/2}$ is taken instead of the Laplace operator in SPME, a new mathematical model arises to describe diffusion processes in fractal and disordered media of fluid flows and the propagation of thermal waves in a porous medium.

In a number of works by A. de Pablo at all. [5-9] developed the theory of existence and uniqueness for the equation of a porous medium with a fractional Laplacian of the form

$$\begin{cases} \frac{\partial u}{\partial t} + (-\triangle)^{\alpha/2}(|u|^{m-1}u) = 0, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

for the initial function $u_0(x) \in L^1(\mathbb{R}^N)$, fractional order $\alpha \in (0,2)$, and porosity index $m > 0$. The existence and uniqueness theorem for a weak solution is proved for $m \le m_* = (N-\alpha)/N$ on the basis of the L^1-contracting semigroup theory.

Cycle of works V.Barbu at all. [10-12] is devoted to stochastic equations of a porous medium in $\mathbb{R}^N$ of the form

$$\begin{cases} du - \triangle\Psi(u)dt = udW(t), & x \in \mathbb{R}^N, t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

Date: Date of Submission September 01, 2023; Date of Acceptance September 25, 2023; Communicated by Igor V. Pavlov.

2010 *Mathematics Subject Classification.* Primary 60H10; Secondary 47H04 60H30.

Key words and phrases. Stochastic porous medium equation, fractional Laplacian, Brownian motion, regularity properties.

where Ψ is a monotone nondecreasing function on $\mathbb{R}$ (possibly multivalued) and $W(t)$ is a Wiener process

$$W(t) = \sum_{k=1}^{\infty} \mu_k e_k \beta_k, t \geq 0. \tag{1.3}$$

Here $\{\beta_k\}_{k=1}^{\infty}$ are independent Brownian motions given on the probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}, \mu_k \in \mathbb{R}$ and $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis in fractional Sobolev dual space $H^{\alpha/2}(\mathbb{R}^N)$. The existence and uniqueness theorem was proved in [12] under the additional Lipschitz property with respect to the function g. It is noted that despite the limitations of generality (the g-Lipschitz property requirement), there exist physical models described by such equations. For example, the two-phase transient Stefan problem perturbed by Gaussian white noise is such a model.

We also note the work of D. Conus, D. Khoshnevisan [13] devoted to the Cauchy problem for the nonlinear stochastic heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}n + \sigma(u)\dot{W}(x,t), & x \in \mathbb{R}^N, t \geq 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.4}$$

where $\mathcal{L}$ is the generator of the Levy process $\{X_t\}_{t\geq 0}$ with the Levy exponent Ψ normalized so that

$$Ee^{i\xi X_t} = e^{-t\Psi(\xi)}$$

for each $\xi \in \mathbb{R}$ and $t \geq 0; \sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function; $\dot{W}$ - white noise in time and space; the initial function $u_0(x)$ is the Borel measure on $\mathbb{R}$. It is well known [14] that the fractional Laplacian in the one-dimensional case is the infinitesimal generator of the stable Lévy process. The existence and uniqueness theorem for a weak solution to problem (1.4) was proved in [13]. More precisely, it was established that from the condition

$$\delta(\beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\beta + 2Re\Psi(\xi)} < \infty, \beta > 0$$

it follows that if $u_0 = \mu$ is a Borel measure on $\mathbb{R}$ that satisfies the integrability condition

$$\int_0^{\infty} e^{-\beta s} ds \sup_{z\in\mathbb{R}} (\int_{-\infty}^{\infty} |\eta(dx)|(P_s\mu)(x-z)^2) < 0, \tag{1.5}$$

where $P_s(\mu) \in B^k_{\beta,\eta}$ is the Banach space of predictable processes with norm

$$N^k_{\beta,\eta}(z) = (\int_0^{\infty} e^{-\beta s} dt \sup_{z\in\mathbb{R}} \int_{-\infty}^{\infty} \eta(dx) \|Z(x-z,t)\|_k^2)^{1/2},$$

and $\|X\|_K = \{E(|X|^k)\}^{1/k}$ for all $k \in [1,\infty), X \in L^k(p)$, then there is a unique element $u \in B^k_{\beta,\eta}$ such that $u(x,t)$ almost sure satisfies the relation

$$u(x,t) = (P_t u_0)(x) + (\tilde{p} * (\sigma \circ u))\dot{W}(x,t).$$

Here " $*$ " denotes the space-time type of "stochastic convolution" $\tilde{p}$ with martingale measure $(\sigma \circ u)\dot{W}$ and $\tilde{p}(x,t) = p(-x,t)$ for all $x \in \mathbb{R}$. Such a solution is not a random field. Most likely, it takes values in some space of generalized functions.

In this paper, we consider the generalized stochastic equation of a porous medium with a fractional Laplacian and multiplicative white noise

$$\frac{\partial u}{\partial t} + (-\triangle)^{\alpha/2}(|u|^{m-1}u) = g(u)\dot{W}(t), x \in \mathbb{R}^N, t > 0, \tag{1.6}$$

with initial condition

$$u(x,0) = u_0(x), x \in \mathbb{R}^N. \tag{1.7}$$

The term on the right side of equation (1.6), i.e.

$$g(u)\dot{W}(t) = g(u)\frac{dW(t)}{dt}, \tag{1.8}$$

describes random noise $W(t), t > 0$ depending on the state of the system and is $L^2(\mathbb{R}^N)$-valued $\mathcal{F}_t$-adapted Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with expectation $\mathbb{E}$ and normal filtering $\mathcal{F}_t = \sigma(W(s) : 0 \leq s \leq t)$. The operator $(-\triangle)^{\alpha/2}, \alpha \in (0,2)$ denotes the fractional power of the Laplacian given on $\mathbb{R}^N$ [6]. The number $m > 0$ is an indicator of the porosity of the medium.

The main contribution of the work is the establishment of properties of existence, uniqueness and regularity of soft (strong) solutions of fractional stochastic equations of porous media. The obtained statements are generalizations of the results of works [6-9], [11-14]. The article has the following structure. In the next section, we introduce the Sobolev function spaces of integer and fractional orders needed below and give a definition of a soft solution to problem (1.6)–(1.7). In Section 3, fractional functional inequalities relevant for this work are presented. Section 4 is devoted to the main results of the existence, uniqueness, and regularity of solutions to problem (1.6)–(1.7).

2. Preliminaries

The fractional Laplace operator $(-\triangle)^{\alpha/2}, 0 < \alpha < 2$, is an infinitesimal generator of isotropic stable Lévy processes. Using the fractional diffusion operator instead of the normal (standard) Laplace operator, we significantly expand the theory of partial differential equations, taking into account the presence of interactions with long tails. Let $\mathbb{Z}_+ = \{n \in \mathbb{Z}, n \geq 0\}$ denote non-negative integers. Let the multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$ be composed of n non-negative integers $\alpha_i \geq 0$. For multi-indices $\alpha = (\alpha_1, ..., \alpha_n)$, define $|\alpha| = \sum_{i=1}^{n} d_i$. If $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$ then the partial derivative of order $|\alpha|$ is defined as follows

$$D^\alpha = \partial^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Let $\Omega \subset \mathbb{R}^N$ and $C^\infty(\Omega)$ be the space of infinitely differentiable functions in Ω, i.e., set of functions with continuous partial derivatives of any order and $C_0^\infty(\Omega)$ functions of class $C^\infty(\mathbb{R}^N)$ with compact support in Ω.

Let $u \in L^1(\Omega)$ be the space of integrable functions on $\Omega \subset \mathbb{R}^n$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$. We say that a function u has a weak derivative $D^\alpha u$ if there exists a function $v \in L^1(\Omega)$ such that

$$\int\limits_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int\limits_\Omega v \varphi dx, \varphi \in C_0^\infty(\Omega),$$

where $v = D^\alpha u$.

Schwartz spaces $S(\mathbb{R}^N)$ are function spaces $\psi \in C^\infty(\mathbb{R}^N)$ such that

$$\sup_{x \in \mathbb{R}^N} |x^\alpha \partial_x^\beta \psi(x)| < \infty \forall \alpha, \beta \in \mathbb{Z}_+^N.$$

If we take $\varphi \in S$, then for each $N \in \mathbb{N}$ and $\alpha \in \mathbb{Z}_+^N$ there exists a constant $C_{N,\alpha}$ such that

$$|D^\alpha \varphi(x)| \leq \frac{C_{N,\alpha}}{(1 + |x|^2)^{N/2}}, \forall x \in \mathbb{R}^N.$$

Next, denote by $S'(\mathbb{R}^N)$ the space of temporary growing distributions on $\mathbb{R}^N$ and by $\mathcal{H}$ the space

$$\mathcal{H} = \{\varphi \in S'(\mathbb{R}^N); \xi \to |\xi| \mathcal{F}(\varphi)(\xi) \in L^2(\mathbb{R}^N)\} \tag{2.1}$$

where $\mathcal{F}$ is the Fourier transform of the function φ. Let $L^2(\mathbb{R}^N)$ be the space of square-integrated functions on $\mathbb{R}$ with norm $|\cdot|$ and inner product $< \cdot, \cdot >_2$. In general, by $|\cdot|_p$ we denote the norm in $L^p(\mathbb{R}^N)$ or $L^p(\mathbb{R}^N, \mathbb{R}^N), 1 \leq p < \infty$. The dual space $\mathcal{H}^{-1}$ for $\mathcal{H}$ is given by the equality

$$\mathcal{H}^{-1} = \{\eta \in S'(\mathbb{R}^n); \xi \to \mathcal{F}(n)(\xi)|\xi|^{-1} \in L^2(\mathbb{R}^N)\}. \tag{2.2}$$

The duality relation between $\mathcal{H}^{-1}$ and $\mathcal{H}$ denoted by $< \cdot, \cdot >$ is given by

$$< \varphi, \eta > = \int\limits_{\mathbb{R}^N} \mathcal{F}(\varphi)(\xi) \overline{\mathcal{F}}(\eta)(\xi) d\xi, \tag{2.3}$$

and the norm $\mathcal{H}$ denoted as $\|\cdot\|_1$ is given as

$$\|\varphi\|_1 = (\int\limits_{\mathbb{R}^N} |\mathcal{F}(\varphi)(\xi)|^2 |\xi|^2 d\xi)^{1/2} = (\int\limits_{\mathbb{R}^N} |\nabla \varphi|^2 d\xi)^{1/2}. \tag{2.4}$$

The norm $\mathcal{H}^{-1}$ denoted as $\|\cdot\|_{-1}$ is given as

$$\|\eta\|_{-1} = (\int\limits_{\mathbb{R}^N} |\xi|^2 |\mathcal{F}(\eta)(\xi)|^2 d\xi)^{1/2} = (< (-\triangle)^{-1} \eta, \eta >)^{1/2}. \tag{2.5}$$

Note that the operator $-\triangle$ is an isomorphism from $\mathcal{H}$ to $\mathcal{H}^{-1}$. The scalar product on $\mathcal{H}^{-1}$ is defined as follows

$$(\eta_1, \eta_2)_{-1} = < (-\triangle)^{-1} \eta_1, \eta_2 > . \tag{2.6}$$

Regarding the relation between $\mathcal{H}$ and $L^p(\mathbb{R}^N)$, the space of p - summable functions on $\mathbb{R}^N$ we have a next statement.

Lemma 2.1. *Let $N \geq 3$. Then we have an inclusion*

$$\mathcal{H} \subset L^{\frac{2N}{N-2}}(\mathbb{R}^N) \tag{2.7}$$

in both algebraic and topological sense.

Indeed, from the Sobolev embedding theorem [15] we obtain

$$|\varphi|_{\frac{2N}{N-2}} \leq C|\nabla\varphi|_2, \forall\varphi \in C_0^\infty(\mathbb{R}^N),$$

and, based on the density properties, (2.7) follows from this.

It should be noted that (2.7) does not hold for $1 \leq N \leq 2$. However, due to the duality relation, we have

$$L^{\frac{2N}{N-2}}(\mathbb{R}^N) \subset \mathcal{H}^{-1}, \forall N \geq 3. \tag{2.8}$$

Denote by $H'(\mathbb{R}^N)$ the Sobolev space

$$H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N); \nabla u \in L^2(\mathbb{R}^N)\} =$$
$$= \{u \in L^2(\mathbb{R}^N); \xi \to \mathcal{F}(u)(\xi)(1+|\xi|^2)^{1/2} \in L^2(\mathbb{R}^N)\}$$

with the norm

$$|u|_{H^{-1}(\mathbb{R}^N)} = (\int\limits_{\mathbb{R}^N}(u^2+|\nabla u|^2)d\xi)^{1/2} = (\int\limits_{\mathbb{R}^N}|\mathcal{F}u(\xi)|^2(1+|\xi|^2)d\xi)^{1/2}$$

and $\mathcal{H}^{-1}(\mathbb{R}^N)$ denoted by $|\cdot|_{-1}$ and its inner product $< \cdot, \cdot >_{-1}$.

There are continuous and dense embeddings

$$H^1(\mathbb{R}^N) \subset \mathcal{H}, \mathcal{H}^{-1} \subset H^{-1}(\mathbb{R}^N).$$

It should be emphasized, however, that $\mathcal{H}$ is not a subspace of $L^2(\mathbb{R}^N)$ and thus $L^2(\mathbb{R}^N)$ is the corresponding space with respect to the duality $< \cdot, \cdot >$ given by (2.3).

Taking the Banach space Y, denote by $L^2(0,T,Y)$ the space of all Y - valued Bochner measurable p-integrable functions on $(0,T)$ and by $C([0,T],Y)$ is the space of all continuous Y-valued functions on $[0,T]$.

For two Hilbert spaces H_1, H_2 let $L(H_1,H_2)$ and $L_2(H_1,H_2)$ denote sets of all linear bounded operators and Hilbert-Schmidt operators, respectively.

3. Fractional equation of a porous medium (FEPM). Deterministic case

In this section, we present the necessary information about the existence and uniqueness of solutions to nonlinear diffusion equations for a porous medium with a fractional Laplacian.

Consider the Cauchy problem of the form

$$\begin{aligned} \frac{\partial u}{\partial t} + (-\triangle)^{\alpha/2}(|u|^{m-1}u) &= f, x \in \mathbb{R}^N, t > 0 \\ u(x,0) &= u_0(x), x \in \mathbb{R}^N \end{aligned} \tag{3.1}$$

where $f \in \mathcal{H}^{-1}, u_0(x) \in L^1(\mathbb{R}^N)$ and the unknown function u can be any sign. We will assume that the fractional degree is $\alpha \in (0, 2)$, the degree of porosity is $m > 1$. In the limiting case $\alpha \to 1$ we obtain the standard FEPM

$$\frac{\partial u}{\partial t} - \triangle(|u|^{m-1}u) = f$$

which is the main model for nonlinear and degenerate diffusion (see, for example, [5] and the detailed bibliography therein).

The nonlocal operator $(-\triangle)^{\alpha/2}$ known as the Laplacian of order α is defined for any function g from the Schwartz class via the Fourier transform.

If

$$(-\triangle)^{\alpha/2} g = h,$$

then

$$\hat{h}(\xi) = |\xi|^{\alpha} \hat{g}(\xi). \tag{3.2}$$

If $0 < \alpha < 2$, then we can also use the representation in the sense of the hypersingular kernel

$$(-\triangle)^{\alpha/2} g(x) = C_{N,\alpha} P \cdot V \cdot \int\limits_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N+\alpha}} dz,$$

where $C_{N,\alpha} = \frac{2^{\alpha-1}\alpha\Gamma((N+\alpha)/2)}{\pi^{N/2}\Gamma(1-\alpha/2)}$ [7].

An equation of the form (3.1) can be considered as a fractional-diffusion version of the FEPM. At the same time, equations of this kind can be considered as a non-linear version of linear fractional diffusion equations obtained for $m = 1$, which have the following solution representation

$$u(x, t) = \int\limits_{\mathbb{R}^N} K_{\alpha}(x - z, t) f(z) dz, \tag{3.3}$$

where K_{α} has the Fourier transform $\hat{K}_{\alpha}(\xi, t) = e^{-|\xi|^{\alpha} t}$. This means that the K_{α} kernel has the form

$$K_{\alpha}(x, t) = t^{N/\alpha} F(|x| t^{-1/\alpha})$$

for some profile positive and decreasing function F that behaves at infinity according to the rule

$$F(r) \sim r^{-(N+\alpha)}.$$

When $\alpha = 1$, F is defined explicitly, if $\alpha = 2$ then the function K will be the Gaussian heat kernel. The linear model has been well studied from the standpoint of probability theory, since the fractional Laplacian is an infinitesimal generator of stable Lévy processes [7]. However, integral representations of the type (3.3) do not exist in the nonlinear case. This is the main motivation for our work.

Note that in the case of FEPM, a theory of the existence and uniqueness of a weak solution is developed in the case when the degree of porosity m is greater than the critical value $m_* = (N - \alpha)/N, 0 < \alpha < 2$. The linear case $m = 1$ also fits into the framework of this theory.

Theorem 3.1. *Let $m > m_*$ and $\alpha \in (0,2)$. For each $f \in \mathcal{H}^{-1}$ and each $u_0 \in L^1(\mathbb{R}^N)$ there exists a unique weak solution to problem (3.1).*

The exact definition of a unique weak solution will be given below. The construction of the solution indicated in Theorem 3.1 will be based on the double limit procedure. First, an initial function from $L_2(\mathbb{R}^N)$ is approximated by a sequence of bounded functions, and then $\mathbb{R}^N$ is approximated by bounded domains with zero boundary data. In this connection, we will show the existence of a weak solution to the associated Cauchy-Dirichlet problem.

In the case of the homogeneous problem (3.1) (that is, $f = 0$), the layer solution has some good qualitative properties, which are stated in the following statement.

Theorem 3.2. *Let the conditions of Theorem 3.1 be satisfied and let u be a weak solution of Problem 3.1 for $f = 0$. Then*

(i) $\partial_t u \in L^\infty C(T,\infty), L^1(\mathbb{R}^N)$ for each $T > 0$;

(ii) Mass is conserved:

$$\int\limits_{\mathbb{R}^N} u(x,t)dx = \int\limits_{\mathbb{R}^N} u_0(x)dx \text{ for all } t \geq 0;$$

(iii) Let u_1, u_1 be weak solutions to problem (3.1) with initial data $u_{0_1}, u_{0_2} \in L^1(\mathbb{R}^N)$. Then

$$\int\limits_{\mathbb{R}^N} (u_1 - u_1)_+(x,t)dx \leq \int\limits_{\mathbb{R}^N} (u_{0_1} - u_{0_2})_+(x)dx;$$

(iv) Any L^p is the solution norm, $1 \leq p \leq \infty$ non-increasing in time;

(v) The solution is limited to $\mathbb{R}^N \times [\tau,\infty)$ for every $\tau > 0$. Moreover, for all $p \geq 1$

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq ct^{\gamma p}\|u_0\|^{\delta_p}_{L^p(\mathbb{R}^N)},$$

where $\gamma_p = (m - 1 + \alpha p/N)^{-1}$, $\delta_p = \alpha p\gamma_p|\mathbb{N}$ and $C = C(m,p,\mathbb{N},\alpha)$;

(vi) If $u_0 \geq 0$ then the solution is positive for all x and $t > 0$;

(vii) If either $m \geq 1$ or $u_0 \geq 0$ then $u \in C^\beta(\mathbb{R}^N \times (0,\infty))$ for some $0 < \beta < 1$;

(viii) The solution depends continuously on the parameters $\alpha \in (0,2), m > m_*$ and $u_0 \in L^1(\mathbb{R}^N)$ in the norm of the space $C([0,\infty), L^1(\mathbb{R}^N))$.

We note an alternative approach to the question of the existence and uniqueness of a strong or soft solution. Using the results of [10], one can find conditions for the existence and uniqueness of a soft solution for $u_0 \in L^1(\mathbb{R}^N)$ for arbitrary m and α thanks to abstract theory of accretive operators.

Next, we present an equivalent problem for weak solutions of problem (3.1).

If ψ and φ belong to the Schwartz class, then the formulae (3.2) of the fractional Laplacian together with the Plancherel theorem implies

$$\int\limits_{\mathbb{R}^N} (-\triangle)^{\alpha/2}\psi\varphi = \int\limits_{\mathbb{R}^N} |\xi|^\alpha \hat{\psi}\hat{\varphi} =$$

$$= \int\limits_{\mathbb{R}^N} |\xi|^{\alpha/2}\hat{\psi}|\xi|^{\alpha/2}\hat{\varphi} = \int\limits_{\mathbb{R}^N} (-\triangle)^{\alpha/4}\psi(-\triangle)^{\alpha/4}\varphi. \tag{3.4}$$

After that, if we multiply the equation in (3.1) by the test function φ and intervene by parts, we get

$$\int_0^\infty \int_{\mathbb{R}^N} u\frac{\partial\varphi}{\partial t}dxdx - \int_0^\infty \int_{\mathbb{R}^N} (-\triangle)^{\alpha/4}(|u|^{m-1}u)(-\triangle)^{\alpha/4}\varphi dxdx = 0. \quad (3.5)$$

Identity (3.5) underlies our definition of a weak solution.

The integrals in (3.5) make sense if u and $|u|^{m-1}u$ belong to the corresponding spaces. The correct space for $|u|^{m-1}u$ is the fractional Sobolev space $\dot{H}^{\alpha/2}(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with norm

$$\|\psi\|_{\dot{H}^{\alpha/2}} = (\int_{\mathbb{R}^N} |\xi|^\alpha |\hat{\psi}|^2 d\xi)^{1/2} = \|(-\triangle)^{\alpha/4}\psi\|_2.$$

Let's move on to the local problem

$$\begin{cases} L_\alpha \omega = 0, & (x,y) \in \mathbb{R}_+^{N+1}, t > 0, \\ \frac{\partial\omega}{\partial y^\alpha} - \frac{\partial |\omega|^{\frac{1}{m}-1}\omega}{\partial t} = 0, & x \in \mathbb{R}^N, y = 0, t > 0, \\ \omega = |f|^{m-1}|f|, & x \in \mathbb{R}^N, y = 0, t > 0. \end{cases} \quad (3.6)$$

To determine a weak solution to this problem, we formally multiply the equation in (3.6) by the test function φ and integrate by parts to obtain

$$\int_0^\infty \int_{\mathbb{R}^N} u\frac{\partial\varphi}{\partial t}dxds - \mu_\alpha \int_0^\infty \int_{\mathbb{R}_+^{N-1}} y^{1-\alpha} < \nabla\omega, \nabla\varphi > dxdyds = 0, \quad (3.7)$$

where $u = |T_r(\omega)|^{1/m-1}T_r(\omega)$. This is true provided that φ vanishes at $t = 0$ and also for large $|x|, y, t$. Then we introduce the energy space $X^\alpha(\mathbb{R}_+^{N+1})$ as a completion of $C_0^\infty(\mathbb{R}_+^{N+1})$ with norm

$$\|v\|_{X^\alpha} = (\mu_\alpha \int_{\mathbb{R}_+^{N+1}} y^{1-\alpha}|\nabla v|^2 dxdy)^{1/2}.$$

Definition 3.3. A pair of functions (u, ω) will be a weak solution to problem (3.6) if:

$$u = |Tr(\omega)|^{1/2-1}Tr(\omega) \in C([0,\infty); L^1(\mathbb{R}^N)),$$
$$\omega \in L^2_{loc}((0,\infty); X^\alpha(\mathbb{R}_+^{N+1}));$$

Definition 3.4. The function u is a weak solution to Problem 3.1 if:

1) $u \in C|[0,\infty), L^1(\mathbb{R}^N), |u|^{n-1}u \in L^2_{loc}|(0,\infty), \dot{H}^{\alpha/2}_{(\mathbb{R}^N)}$;
2) identity (3.5) is true for every $\varphi \in C_0^1(\mathbb{R}^N \times (0,\infty))$;
3) $u(\cdot, 0) = u_0$ almost everywhere.

The main disadvantage of applying this definition is that there is no formula for the fractional Laplacian of the product or composition of functions. Moreover, there is no benefit in using test functions with compact supports, since their fractional Laplacians do not preserve this property. To overcome these difficulties, we will use the fact that our solution u is the trace solution of the local problem found by extending $|u|^{m-1}u$ to half-spaces whose boundary is our original space.

Here is the corresponding extension method.

Let $g = g(x)$ be a smooth bounded function defined on $\mathbb{R}^N$ and let its α-harmonic extension to the upper half-space $v = E(g)$ be the only smooth bounded solution $v = v(x, y)$ of the next problem

$$\begin{cases} \nabla \cdot (y^{1-\alpha}\nabla\theta) = 0, & \text{in } \mathbb{R}_+^{N+1} \equiv \{(x,y) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N, y > 0\}, \\ v(x,0) = g(x), & x \in \mathbb{R}^N. \end{cases} \tag{3.8}$$

Then, due to the results of [13], we get

$$-\mu_\alpha \lim_{y\to 0+} y^{1-\alpha}\frac{\partial v}{\partial y} = (-\triangle)^{\alpha/2}g(x), \mu_\alpha = 2^{\alpha-1}\Gamma(\alpha/2)\Gamma(1-\alpha/2). \tag{3.9}$$

In (3.6) the operator ∇ on all (x, y) variables, although in (3.7) $(-\triangle)^{\alpha/2}$ only on $x = (x_1, ..., x_N)$ variables. Next, we introduce the notation

$$L_\alpha v \equiv \nabla \cdot (y^{1-\alpha}\nabla v), \frac{\partial v}{\partial y^\alpha} \equiv \mu_\alpha \lim_{y\to 0+} y^{1-\alpha}\frac{\partial v}{\partial y}.$$

Operators of the form L_α with coefficients $y^{1-\alpha}$ belonging to the Mackenhoupt space with weights A_2 if $0 < \alpha < 2$ were studied in [6]. Taking this into account, we rewrite problem (3.1) as a quasistationary problem for $\omega = E(|u|^{m-1}u)$ with a dynamic boundary condition of the form

$$\begin{cases} L_\alpha\omega = 0, & (x,y) \in \mathbb{R}_+^{N+1}, t > 0, \\ \frac{\partial\omega}{\partial y^\alpha} - \frac{\partial|\omega|^{\frac{1}{m}-1}\omega}{\partial t} = 0, & x \in \mathbb{R}^N, y = 0, t > 0, \\ \omega = |u_0|^{m-1}u_0, & x \in \mathbb{R}^N, y = 0, t > 0. \end{cases} \tag{3.10}$$

To determine a weak solution to problem (3.8), we formally multiply the equation in (3.10) by a test function and integrate by parts. Then

$$\int_0^\infty \int_{\mathbb{R}^N} u\frac{\partial\varphi}{\partial t}dxds - \mu_\alpha \int_0^\infty \int_{\mathbb{R}_+^{N+1}} y^{1-\alpha} < \nabla\omega, \nabla\varphi > dxdyds = 0, \tag{3.11}$$

where $u = |T_r(\omega)|^{1/m-1}T_r(\omega)$.

Equality (3.9) is satisfied under the condition that φ vanishes at $t = 0$ and at large $|x|, y$ and t.

Then we introduce the energy space $X^\alpha(\mathbb{R}_+^{N+1})$, which is the completion of $C_0^\infty(\mathbb{R}_+^{N+1})$ in norm

$$\|v\|_X^\alpha = (\mu_\alpha \int_{\mathbb{R}_+^{N+1}} y^{1-\alpha}|\nabla v|^2 dxdy)^{1/2}.$$

Definition 3.5. A pair of functions (u, ω) is a weak solution to problem (3.7) if:

1) $u = |T_r(\omega)|^{\frac{1}{m}-1}T_r(\omega) \in C[0,\infty), L^1(\mathbb{R}^N), \omega \in L^2_{loc}C(0,\infty), X^\alpha(\mathbb{R}_+^{N+1})$;

2) identity (3.5) holds for each $\varphi \in C_0^1(\overline{\mathbb{R}_+^{N+1}} \times (0,\infty))$;

3) $(Cu(\cdot,0)) = u$ almost everywhere.

The extension operator is well defined in $\dot{H}^{\alpha/2}(\mathbb{R}^N)$. It will get an explicit expression when using the Poisson kernel. It was proved in [13] that the mapping

$$E : \dot{H}^{\alpha/2}(\mathbb{R}^N) \to X^\alpha(\mathbb{R}^{N+1}_+)$$

is an isometry. Trace operator

$$Tr : X^\alpha(\mathbb{R}^{N+1}_+) \to \dot{H}^{\alpha/2}(\mathbb{R}^N)$$

is surrective and continuous.

Indeed, for every $\Phi \in X^\alpha(\mathbb{R}^N)$ there is a trace embedding

$$\|Tr(\Phi)\|_{\dot{H}^{\alpha/2}} = \|E(Tr\Phi)\|_{X^\alpha} \leq \|\Phi\|_{X^\alpha}.$$

The main result of the above reasoning is the following statement about the equivalence of the two definitions of a weak solution.

Theorem 3.6. *The function u is a weak solution to problem (3.1) if and only if the pair $(u, E(|u|^{m-1}u))$ is a solution to problem (3.8).*

Proof. Since the mapping

$$E : \dot{H}^{\alpha/2}(\mathbb{R}^N) \to X^\alpha(\mathbb{R}^{N+1}_+)$$

is an isometry, then we have

$$\mu_\alpha \int\limits_{\mathbb{R}^{N+1}_+} y^{1-\alpha} < \nabla E(\Psi), \nabla E(\varphi) >= \int\limits_{\mathbb{R}^N} (-\triangle)^{\alpha/4}\Psi(-\triangle)^{\alpha/4}\varphi$$

for each $\Psi, \varphi \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$. Further, the result immediately follows from the following lemma, in which it is established that any α-harmonic function is orthogonal to every function with trace 0 on $\mathbb{R}^N$ in $X^\alpha(\mathbb{R}^{N+1}_+)$. □

Lemma 3.7. *Let $\Psi \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$ and $\Phi_1, \Phi_2 \in X^\alpha(\mathbb{R}^{N+1}_+)$ such that $Tr(\Phi_1) = Tr(\Phi_2)$. Then*

$$\mu_\alpha \int\limits_{\mathbb{R}^{N+1}_+} y^{1-\alpha} < E(\Psi), \Phi_1(\varphi) >= \mu_\alpha \int\limits_{\mathbb{R}^{N+1}_+} y^{1-\alpha} < \nabla E(\Psi), \Phi_2 > .$$

Proof. Let $h = \Phi_1 - \Phi_2$. Since $E(\Psi)$ is smooth for $y > 0$, taking $\varepsilon > 0$ we obtain after integration by parts

$$\mu_\alpha \int\limits_\varepsilon^\infty \int\limits_{\mathbb{R}^N} y^{1-\alpha} < \nabla E(\Psi), \nabla h > dxdy = \mu_\alpha \int\limits_{\mathbb{R}^N} \varepsilon^{1-\alpha} \frac{\partial E(\Psi)}{\partial y}(x,\varepsilon)h(x,\varepsilon)dx.$$

□

Left part of the last equality converges to $\mu_\alpha \int\limits_{\mathbb{R}^{N+1}_+} y^{1-\alpha} < E(\Psi), \nabla h >$, while the right side tends to 0 since identity (3.5) holds in the weak sense in $H^{-\alpha/2}(\mathbb{R}^N)$, and $Tr(h) = 0$

The lemma is proven.

4. Some functional inequalities

In this section, we present some functional inequalities related to the fractional Laplacian defined in $\mathbb{R}^N$. The first inequality, called the Struk-Varopoulos inequality, was established in [9]. For the sake of completeness, we give a brief proof of it.

Lemma 4.1. *Let $0 < \gamma < 2, q > 1$. Then*

$$\int\limits_{\mathbb{R}^N} (|v|^{q-2}v)(-\triangle)^{\gamma/2}v \geq \frac{4(q-1)}{q^2} \int\limits_{\mathbb{R}^N} |(-\triangle)^{\gamma/4}|v|^{q/2}|^2 \tag{4.1}$$

for all $v \in L^q(\mathbb{R}^N)$ such that $(-\triangle)^{\gamma/2}v \in L^q(\mathbb{R}^N)$.

Proof. Using equalities (3.5) and Lemma 3.7 we obtain

$$\int\limits_{\mathbb{R}^N} (|v|^{q-2}v)(-\triangle)^{\gamma/2}v = \int\limits_{\mathbb{R}^N} |(-\triangle)^{\gamma/4}(|v|^{q-2}v)(-\triangle)^{\gamma/4}v =$$

$$= \mu_\alpha \int\limits_{\mathbb{R}^{N+1}_+} y^{1-\alpha} < \nabla(|E(\theta)|^{q-2}E(v)), \nabla E(v) >=$$

$$= \mu_\alpha \frac{y(q-1)}{q^2} \int\limits_{\mathbb{R}^{N+1}_+} y^{1-\alpha}|\nabla(|E(v)^{q/2}|)|^2 \geq$$

$$\geq \frac{y(q-1)}{q^2} \int\limits_{\mathbb{R}^N} |(-\nabla)^{\gamma/4}|v|^{q/2}|^2.$$

□

At the last step, the inequality is taken because the function $|E(\theta)|^{q/2}$ is not γ harmonic.

Using the same technique, one can prove the generalization of (4.1).

Lemma 4.2. *Let $0 < \gamma < 2$. Then*

$$\int\limits_{\mathbb{R}^N} (-\triangle)^{\gamma/2}v \geq \int\limits_{\mathbb{R}^N} |(-\triangle)|^{\gamma/4}\Psi(v)|^2 \tag{4.2}$$

where $\Psi' = (\Psi')^2$.

The proof is based on the extension method and the following property

$$< \nabla\Psi(\omega), \nabla\omega >= |\nabla\Psi(\omega)|^2.$$

To prove the second functional inequality, which we need later, we use the well-known Hardy-Littlewood-Sobolev inequality [9]: for each v such that $(-\triangle)^{\gamma/2}v \in L^r(\mathbb{R}^N), 1 < r < N/\gamma, 0 < \gamma < 2$

$$\|v\|_{r_1} \leq c(N, r, \gamma)\|(-\triangle)^{\gamma/2}v\|_{r_1}, r_1 = \frac{N_r}{N - \gamma r}. \tag{4.3}$$

Taking $r = 2, \gamma = \alpha/2$ for example, we find the inclusion $\dot{H}^{\alpha/2}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ when $N > \alpha$. What will happen for $N = 1 \leq \alpha < 2$? Or more general, for $r \geq N/\gamma$? The answers to these questions are in the next lemma.

Lemma 4.3. *(Nash-Gagliardo-Nirenberg inequality type). Let* $p \geq 1, r > 1, 0 < \gamma < min\{N, r\}$. *Then there is a constant* $C = C(p, r, \gamma, N) > 0$ *such that for any* $v \in L^p(\mathbb{R}^N)$ *with* $(-\triangle)^{\gamma/2} v \in L^p(\mathbb{R}^N)$ *we have*

$$\|v\|_{r_2}^{\beta+1} \leq C\|(-\triangle)^{\gamma/2} v\|_r \|v\|_p^{\beta}, \tag{4.4}$$

where $r_2 = \frac{N(rp+r-p)}{r(N-\gamma)}, \beta = \frac{p(r-1)}{r}$.

Proof. First, we use inequalities (4.1) and (4.3) to estimate the left side of inequality (5.3). Further, to estimate the right side of (5.3), Hölder's inequality is used. □

5. Main results

On the space $S'(\mathbb{R}^N)$, the Brownian list W is a zero-mean quadratically integrable Lévy process on a suitably chosen Hilbert space. Denote its covariance operator by K. We define $\mathcal{H}$ as the set of all $W \in S'(\mathbb{R}^N)$ in such a way that

$$|(W, \varphi)| \leq L\sqrt{K(\varphi, \varphi)}, \forall \varphi \in S'(\mathbb{R}^N) \tag{5.1}$$

with constant $L < \infty$ independent of φ. i.e. $(H_N, < \cdot, \cdot >)_{H_N}, N \in \mathbb{N}$ is a decreasing sequence of separable Hilbert spaces.

Then the next statement is true.

Lemma 5.1. *There are* N *and* C *such that*

$$|K(\varphi, \varphi)| \leq c|\varphi|_{H_N} |\psi|_{H_N} \text{ for all } \varphi, \psi \in S(\mathbb{R}^N).$$

The next assertion establishes that W can be represented as a spatially stationary random field on $\mathbb{R}^N \times [0, \infty)$.

Theorem 5.2. *If the spectral measure* μ *of the process* W *is finite, then* W *can be identified as a random field* $W(x,t), x \in \mathbb{R}^N, t \geq 0$ *as follows:*

$$(W(t), \varphi(t)) = \int_{\mathbb{R}^N} W(x,t)\Psi(x,t)dx, t \geq 0, \Psi \in S(\mathbb{R}^N \times [0, \infty)).$$

Let us now return to equation (1.6) with $g = 1$, i.e.

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = -(-\triangle)^{\alpha/2} u^m(x,t) + \dot{W}(x,t), & x \in \mathbb{R}^N, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{5.2}$$

We multiply equation (5.2) by the test function Ψ and integrate u by parts, and then we obtain

$$\int_0^T \int_{\mathbb{R}^N} u(x,t)\Psi_t(x,t)dxdt =$$

$$\int_0^T \int_{\mathbb{R}^N} (-\triangle)^{\alpha/2} u^m(x,t)(-\triangle)^{\alpha/4}\Psi dxdt + \int_0^T < W(t), \Psi_t(x) > dt. \qquad (5.3)$$

The integrals in (5.3) make sense if u and u^m lie in well-defined spaces, and a possible convenient space for u^m is the fractional function space $H^{\alpha/2}(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with the following norm

$$\|\varphi\|_{H^{\alpha/2}(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |\xi|^\alpha |\hat{\varphi}|^2 d\xi)^{1/2} = \|(-\triangle)^{\alpha/2}\varphi\|_{L^2(\mathbb{R}^N)}.$$

Definition 5.3. The function u is a weak solution of equation (5.2) if:
1. $u \in L^1(\mathbb{R}^N)$ for all $T > 0, u^m L^2_{loc}(H^{\alpha/2}(\mathbb{R}^N)(0,\infty))$;
2. W satisfies inequalities (5.1) and the conditions of the lemma 5.1;
3. Equation (5.2) is true for every $\varphi \in C_0^1(\mathbb{R}^N \times (0,T))$;
4. $u(\cdot,t) \in L^1(\mathbb{R}^N)$ for all $t > 0, \lim\limits_{t\to 0+} u(\cdot,t) = u_0 \in L^1(\mathbb{R}^N)$.

Considering the above results and definitions, we present the main result of the paper.

Theorem 5.4. *Suppose $u_0 \in L^1(\mathbb{R}^N)$, and let $u \in L^1(\mathbb{R}^N \times [0,T])$ for all $T > 0$ is a weak solution to problem (1.1), $u^m \in L^1_{loc}(H^{\alpha/2}(\mathbb{R}^N),(0,\infty))$ and W satisfies inequality (5.1) and the conditions of Lemma 5.1. Then*

$$\int_{\mathbb{R}^N} u(x,t)dx \in C(R_+), u \in C(L^1(\mathbb{R}^N,[0,T]))$$

and

$$\int_{\mathbb{R}^N} u(x,t)dx = \int_0^T \int_{\mathbb{R}^N} \sigma(u(x,s)\dot{W}(x,s))dxds + \int_{\mathbb{R}^N} u_0(x)dx =$$

$$= \int_0^T \int_{\mathbb{R}^N} \sigma(u(x,x)W(dx,ds)) + \int_{\mathbb{R}^N} u_0(x)dx.$$

Proof. Let a test function Ψ be defined in terms of $\Psi(x,t) = \varphi(x)\varphi(t)$ where $\varphi_R(x) = \varphi(\frac{x}{R})$ is defined as follows:

$$\varphi \in C_0^\infty(\mathbb{R}^N), 0 \le \varphi(\cdot) \le 1,$$
$$\varphi(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2. \end{cases}$$

For any test function $\varphi \in C_0^\infty(\mathbb{R}_+)$ we have

$$-\int_0^\infty \int_{\mathbb{R}^N} u(t,x)\varphi_R(x)\varphi'(t)dtdx =$$

$$= \int_0^\infty \int_{\mathbb{R}^N} [-u^m(x,t)(-\triangle)^{\alpha/2}\varphi_R(x) + \dot{W}(x,t)\varphi_R(x)] +$$

$$+ \int_{\mathbb{R}^N} u_0(x)\varphi_R(x)\varphi(0)x = \int_0^\infty \int_{\mathbb{R}^N} [-u^m(x,t)\frac{1}{R^\alpha}(-\triangle)^{\alpha/2}\varphi(\frac{x}{R}) +$$

$$+ \dot{W}(x,t)\varphi_R(x)]\varphi(t)dxdt + \int_{\mathbb{R}^N} u_0(x)\varphi_R(x)\varphi(0)dx.$$

Taking the limit as $R \to \infty$ and using the Lebesgue convergence theorem, we obtain

$$-\int_0^\infty \int_{\mathbb{R}^N} u(t,x)\varphi'(t)dtdx =$$

$$= \int_0^\infty \int_{\mathbb{R}^N} \dot{W}(x,t)\varphi(t)dxdt + \int_{\mathbb{R}^N} u_0(x)\varphi(0)dx.$$

Let $F_u(t) = \int_{\mathbb{R}^N} u(x,t)dx$ and $F_{\dot{W}}(t) = \dot{W}(x,t)dx$, where $F_u(t) \in L^1_{loc}(\mathbb{R}^N)$ and $F_W(t) \in L^2_{loc}(P)$. Let $T > 0$ and consider the test function $\varphi(t) = I_{(0 \le t \le T)}$; we also choose a sequence of test functions $\varphi_n \in C_0^\infty(\mathbb{R}^N)$ with decreasing φ_n so that $\varphi_n(t) \le \varphi(t)$ and $\varphi_n(t) = 1$ on the interval $[0, T - 1/n]$ for sufficiently large n. Then we apply Lebesgue's theorem to F_u and obtain

$$-\int_0^\infty F_u(t)\Psi_n'(t)dt =$$

$$= \int_0^\infty F_{\dot{W}}(t)\varphi_n(t)dt + \int_{\mathbb{R}^N} u_0(x)\varphi_n(0)dx \to \int_0^T F_{\dot{W}}(t)dt + \int_{\mathbb{R}^N} u_0(x)dx$$

for $n \to \infty$. This implies that $F_u \in C(\mathbb{R}^+)$ or otherwise F_u has a continuous representation in its Lebesgue class. Now choosing φ_n better, at each Lebesgue point T for F_u and passing to the limit on the left side, we get

$$F_u = \int_0^T F_{\dot{W}}(t)dt + \int_{\mathbb{R}^N} u_0(x)dx.$$

We have given proofs of the theorem for $g = 1$, for simplicity, however, with these steps we can establish this result for $g \neq 1$ when σ is continuous and satisfies the Lipschitz conditions. For the condition of existence and uniqueness, we need the following condition on σ.

Condition 5.5. *There is a finite positive constant Lip_σ such that for all $x, y \in \mathbb{R}$ we have $\sigma(0) = 0$, and*

$$|\sigma(x) - \sigma(y)| \leq Lip_\sigma |x - y|.$$

Theorem 5.6. *Under conditions 5.5, there exists a unique solution to problem (5.2).*

Proof. From Lemma 5.1, equation (5.1), and Hölder's inequality, we obtain

$$|\int_0^T \int_{\mathbb{R}^N} u(x,t)\Psi_t(x,t)dxdt| \leq$$

$$\|(-\triangle)^{\alpha/4}u^m\|_{L^2(\mathbb{R}^N\times(0,T))} \cdot \|(-\triangle)^{\alpha/4}\varphi\|_{L^2(\mathbb{R}^N\times(0,T))} + TL\sqrt{C}\sup_{0\leq t\leq T}|\Psi_t|_{H_n} =$$

$$= \|u^m\|_{H^{\alpha/2}(\mathbb{R}^N\times(0,T))} \cdot \|\Psi\|_{H^{\alpha/2}(\mathbb{R}^N\times(0,T))} + TL\sqrt{C}\sup_{0\leq t\leq T}|\Psi_t|_{H_n} < \infty$$

Returning to equation (1.5) we have

$$\mathbb{E}|\int_{\mathbb{R}^N} |u_1(x,t) - u_2(x,t)||dx|^2 = \mathbb{E}|\int_0^T \int_{\mathbb{R}^N} |g(u_1(x,t)) - g(u_2(x,t))||W(dx,dt)|^2 \leq$$

$$\leq Lip_g^2 \mathbb{E}\int_0^T \int_{\mathbb{R}^N} |u_1(x,t) - u_2(x,t)|dxdt.$$

Let there exist $C > 0$ such that

$$C\int_{\mathbb{R}^N} \mathbb{E}|u_1(x,t) - u_2(x,t)|^2 dx \leq \mathbb{E}|\int_{\mathbb{R}^N} |u_1(x,t) - u_2(x,t)|dx|^2$$

then it follows that

$$\|u_1 - u_2\|_{L^2(P)} \leq Lip_g\sqrt{T/C}\|u_1 - u_2\|_{L^2(P)}.$$

The last inequality proves the uniqueness of the solution since $1 - Lip_g\sqrt{T/C} > 0$. □

The next result concerns the growth property of the solution, taking into account Theorem 5.4.

Theorem 5.7. *There is a constant $L_\sigma > 0$ and $C > 0$ such that*

$$\|u(t)\|_{L^2(P)} \leq \frac{1}{\sqrt{C}}\|u_0\|_{L^1(\mathbb{R}^N)} exp(\frac{Lip_\sigma}{\sqrt{C}}t). \tag{5.4}$$

Proof. Taking into account Theorem 5.4, we obtain the inequality

$$\|u\|_{L^2(P)} \leq \frac{Lip_\sigma}{\sqrt{C}} \int\limits_0^t \|u(s)\|_{L^2(P)} ds + \frac{1}{\sqrt{C}} \|u_0\|_{L^1(\mathbb{R}^N)}. \tag{5.5}$$

Next, applying the Gronwall lemma to inequality (5.5), we obtain (5.4).

Note that the solution concept used here was proposed in [15]. Some other aspects of the theory of nonlinear stochastic equations are presented in [16-18.]

References

1. Benilan, Ph., Grandall, M.: The continuous dependence on φ of solutions of $u_t - \triangle\varphi(u) = 0$, *Indiana Univ. Mathematics Journal* **30 №2** (1981) 161–177.
2. Belaribi, N., Cuvelier, F., Russo, F: A probabulistic algorithm approximating solutions of a singular PDE of porous media type, *Monte Carlo Methods and Applications* **17** (2011) 317–369.
3. Belaribi, N., Cuvelier, F., Russo, F: *Probablistyc and deterministic lgrithms for Space multidimensional irregular porous media equation*, Stochastic Differential Equations: Analysis and Computations, vol. 1, issue 1, 2013, 3-62.
4. Krilov, N. V.: Itô's formula for the L_p-norm of stochastic W_p^1-value processes, *Probab. Thory Rel. Fields* **147** (2010) 583–605.
5. De Pablo, A., Qurros, F., Rodriguez, A., Vazquez, J.L.: A fractional porous medium equation, *Advances in Mathematics* **226, no.2** (2011) 1378–1409.
6. Stinga, P.R., Torrea J.L.: Extension problem and Harnack's inequality for some fractional operators, *Preprint* **arXiv:0910.2569v2**.
7. De Pablo, A., Qurros, F., Rodriguez, A., Vazquez, J.L.: A general fractional porous medium equation, *Preprint* **arXiv:1104.0306v1[math.Ap]2 Apr. 2011**.
8. Barenblatt, G. I.: *Scaling, Self-Similarity, and Intermediate Asymptotis*, Cambridge Univ. Press, Cambridge, 1996. Updated version of Similarity, Self-Similarity, and Intermediate Asymptotis, Consulting Bureau, New York, 1999.
9. Vazquez, J.L.: The fractional p-Laplacian evolution equation in $\mathbb{R}^N$ in the sublinear case, *Preprint* **arXiv:2011.01521v2[math.Ap]6 Jan. 2021**.
10. Barbu, V., Da Prato, G., Rockner, M.: Existence and uniqueness of nonnegative solutions to stochastic porous media equation, *Indiana University Math. Journal* **57** (2008) 187–212.
11. Barbu, V., Da Prato, G., Rockner, M.: Existence strong solutions for stochastic porous media equation under general monotonicity conditions, *An. Probab.* **37 (2)** (2009) 428–452.
12. Barbu, V., Rockner, M., Russo, F.: The stochastic porous media equations in $\mathbb{R}^N$, *Preprint* **arXiv:1312.6234v1[math.PR] 21 Dec. 2013**.
13. Conus, D., Khoshnevisan, D.: Weak nonmild solutions to some SPDES, *Illinois Journal of Mathematics* **54 (4)** (2010) 1321–1349.
14. Conus, D., Khoshnevisan, D.: On the existence and position of the farthest peaks of a family of stochastic heat and wave equations, *Probab. Theory Relat. Fields* **152** (2012) 681–701.
15. McSylvester Ejighikeme Omaba.: Weak Nonmild Solution of Stochastic Fractional Porous Medium Equation, *Preprint* **arXiv:1703.06115 [math.PR] (or arXiv:1703.06115v1) (2016)**
16. Ilolov, M., Kuchakshoev, K.S., Rahmatov, J.S.: Fractional stochastic evolution equations: Whitenoise model *Communications on Stochastic Analysis* **14(3-4)** (2020) 55-69
17. Ilolov, M., Kuchakshoev, K.S., Rahmatov, J.S.: Lyapunov function and stability of solutions of stochastic differential equations with fractional-like derivatives *Global and Stochastic Analysis* **Vol. 8 No. 2** (July-December, 2021) 87-99.
18. Ilolov, M., Lashkarbekov, S., Rahmatov, J.Sh.: Fractional stochastic evolution equations with Balakrishnan's white noise *Global and Stochastic Analysis* **Vol. 9 No. 3** (2022) 53-70

Mamadsho Ilolov: Center of Innovative Development of Science and New Technologies of National Academy of Sciences of Tajikistan, Abduhamid Juraev Institute of Mathematics of National Academy of Sciences of Tajikistan, Department of Mechanics and Mathematics of the Tajik National University, Dushanbe, 734025, Tajikistan

Email address: ilolov.mamadsho@gmail.com

Jamshed Sh. Rahmatov: Center of Innovative Development of Science and New Technologies of National Academy of Sciences of Tajikistan,Dushanbe, 734025, Tajikistan

Email address: jamesd007@rambler.ru

Suhrob Lashkarbekov: Sadriddin Ayni Tajik State Pedagogical University, Dushanbe, 734025, Tajikistan

Email address: Lashkarbekov89@bk.ru