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ON COMPLETELY RANDOM GAMES

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ABSTRACT. We consider completely random games with a finite number of players, that is, games in which the strategies are laws of probability and the players' moves are random variables with the distribution given by the strategies. The payoff of the game is given by a function of the conditional expectation of the move of a player versus the move of the others, which allows studying games in which the strategies may be probability distributions dependent on one another. In this context we first present a static game with two players and two strategies for which we show numerically the existence of an equilibrium point which is a fixed point either of the global best possible reply or best possible average reply of players in the game. We verify numerically that the strategies given by the equilibrium point are optimal. We also present a more general approach that encompasses the example presented. In this general approach the strategies are probability measures indexed by parameters belonging to a compact convex set of real vectors. Under mild hypothesis we show that the game admits equilibriums.

1. Introduction

The origin of contemporary game theory may be traced, firstly, to the monograph of von Neumann and Morgensten (see [19]) and then to the works of John Nash (see [10] and also the collection of essays [6]). Since then, game theory has had a myriad of important developments and found many applications in Economics and many other scientific areas. General presentations of different aspects of mathematical game theory are given in [12], [17], [4], [7] and [3].

John Nash's note *Equilibrium Points in n-Person games* (see [6, pp. 49–50]) already contains some of the features of the present work, namely, considering pure strategies given by the possible moves of a player and defining strategies as probability distributions over the set of pure strategies.

In contrast with previous approaches, we deal with a formalisation of games that dwell with uncertainty by means of strategies given by general probability distributions indexed by parameters, the moves of the players given by random variables with laws given by the correspondent strategies and payoffs which are functions of the conditional expectations of moves of one player given the moves of all other players.

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In the following we provide a review of some of the works that have approaches that may overlap, if only mildly, with our own. In [2] a solution for two-person zero-sum games with random payoffs is given by means of attaining a specified level of confidence. The payoff is given by a matrix with random entries with known distribution. The problem is formulated, firstly, as a linear programming problem having as coefficients the tail probabilities of the random matrix entries and secondly, as an non linear programming problem of the payoff level of a player subjected to the constraint that the player receives that payoff with at least a specified level of confidence. The problems are shown to admit solutions under hypothesis on the distributions of the matrix entries.

The work in [20] deals with finding equilibrium in stochastic optimisation problems related to supply and demand. Under reasonable assumptions the existence and unicity of equilibrium is proved.

In [11] the authors study, and we quote: "...n-player game with random payoffs and continuous strategy sets. The payoff function of each player is defined by its expected value and the strategy set of each player is defined by a joint chance constraint." The random constraint vectors defining the joint chance constraint used to define the strategy of players are dependent and follow elliptically symmetric distributions; the dependence among random constraint vectors is modelled by the Archimedean copula. The existence of a Nash equilibrium is proved under a set of hypothesis related to the particular types of distributions and copulas used.

According to [16], and we quote: "...The original Nash equilibrium theory was conceived for deterministic games, which makes it limited to handle real applications with random payoffs and strategy sets." In its excellent literature review the authors describe the evolution of game models with random payoffs that culminates in the work [11] where *Nguyen et alii* write, and we quote: "...extended the results in [14] and in [13] to the general case where the payoff function is random and the strategy profile set of each player is defined by elliptically distributed dependent joint chance constraints." *Riccardi et alii* in [16], propose and we quote: "...an n-player non-cooperative game where the payoff function of each player follows a multivariate distribution." The authors also signal the two main approaches to deal with random payoffs: the first one is via the expected values (see [11]) and the second approach, followed in this work (and also in [2]), amounts to require the players attaining a maximum of their payoff with a given level of confidence. Several models for a zonal electricity model are proposed that are then exploited via simulation in prescribed scenarios.

Let us detail the main content of what follows.

- In Section 2 we deal with a simple example of a completely random game, with two players, each one having as moves Bernoulli random variables taking two values—the pure strategies—and so, players moves are random variables taking values in the set of pure strategies. The laws of these random variables for the players—the correspondent game strategies—are given Bernoulli laws. The dependence structure between the laws of the players is given as well. Of course, in the case of more than two pure strategies we can consider multinomial laws. The joint law of the strategies of the players, that is the dependence structure of the game, is given by means of copulas in order to illustrate the possibility of fitting the model to observations. The payoffs are given by the image—by a continuous function—of the conditional expectation of the move of one player with respect to the move of the other player. This conditional expectation is computed by means of the joint law. Since the moves are discrete random variables, these conditional expectations may be explicitly computed once the parameters are known or estimated. in Section 3 we detail the numerical analysis of an instance of the game introduced in Section 2 explicitly computing the equilibrium points and analysing the results.

- In Section 4 we use Choquet's representation theorem to express the value of the game best possible reply for the example introduced in Section 2.
- In Section 5 we present a more general framework for completely random games that encompasses the example presented in Section 2.

2. A first simple example: 2 players, 2 pure strategies - $(2p, 2s)$

In this Section we present a simple example of a static game, with two players, each one having as a strategy a Bernoulli law. This example will serve a guide and motivation for a more general theory in Section 5. Let us state first the assumptions and notations for what follows.

(1) We denote the players by P_1 and P_2.

(2) The strategy of player P_1 is given by the Bernoulli law $\mathcal{B}(p)$; we can then say—following J. Nash in [6, pp. 49–50]—that $\Theta = \{\theta_a, \theta_b\}$ are the pure strategies of player P_1. The strategy of player P_2 is given by the Bernoulli law $\mathcal{B}(q)$; similarly we may say that $\Xi = \{\xi_a, \xi_b\}$ are the pure strategies of player P_2. The elements of Θ and Ξ and can be chosen according to some numerical return specific to each game; as a consequence we can identify $\Theta = \{\theta_a, \theta_b\}$ with $\widetilde{\pi}_1(\Theta) = \{\widetilde{\pi}_1(\theta_a), \widetilde{\pi}_1(\theta_b)\}$ and $\Xi = \{\xi_a, \xi_b\}$ with $\widetilde{\pi}_2(\Xi) = \{\widetilde{\pi}_2(\xi_a), \widetilde{\pi}_2(\xi_b)\}$ with $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ being the returns—or the pure strategies payoffs specific to the game—of the pure strategies.

(3) In the game, a move of player P_1 is given by X_1 a random variable taking values in Θ with law Bernoulli $\mathcal{B}(p)$. A move of player P_2 is given by X_2 a random variable taking values in Ξ with law Bernoulli $\mathcal{B}(q)$. We suppose that the joint law of (X_1, X_2) is also given.

(4) Let $\pi_1 : \mathbb{R} \mapsto [0,1]$ and $\pi_2 : \mathbb{R} \mapsto [0,1]$ be two payoff continuous functions. We define the payoff for player P_1 for a move with law given by $\mathcal{B}(p)$ whenever the player P_2 makes a move with law given by $\mathcal{B}(q)$ by:

$$\pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2]) ,$$

and the payoff for player P_2 for a move with law given by $\mathcal{B}(q)$ by whenever the player P_1 makes a move with law given by $\mathcal{B}(p)$ by:

$$\pi_2(\mathbb{E}_{(p,q)}[X_2 \mid X_1]) .$$

We observe that by taking, for instance $\pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2])$, the image by the function π_1 of the conditional expectation of X_1 with respect to X_2—instead of taking, as it is usually done, $\pi(X_1, X_2)$, that is the image by a payoff function π of the couple (X_1, X_2)—we are considering that the payoff for move of player P_1 depends on the information made available by the move of player P_2; this is a realistic assumption in the context of games that are played under external supervision or with the presence of a referee, a game in which the players place their bets and these bets are known.

(5) Then, the *best possible random reply* of player P_2 for the move made by player P_1—with law given by $\mathcal{B}(p)$—is given by:

$$r_{2\to 1}(p) = \arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2]) \ , \tag{2.1}$$

where, $\arg\sup_q$ is the value $\widehat{q}$ such that,

$$\sup_q u(\mathbb{E}_{(p,q)}[X_1 \mid X_2]) = u(\mathbb{E}_{(p,\widehat{q})}[X_1 \mid X_2]) \ .$$

Also, the *best possible random reply* of player P_1 for a move made by player P_2—with law given by $\mathcal{B}(q)$—is given by:

$$r_{1\to 2}(q) = \arg\sup_p \pi_2(\mathbb{E}_{(p,q)}[X_2 \mid X_1]) \ , \tag{2.2}$$

with a similar convention for $r_{1\to 2}(q)$ as the one given above for $r_{2\to 1}(p)$.

(6) The *best possible average reply* of player P_2 for the move made by player P_1—with law given by $\mathcal{B}(p)$—is given by:

$$\overline{r}_{2\to 1}(p) = \mathbb{E}\left[\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2])\right] \ , \tag{2.3}$$

where, $\arg\sup_q$ is the value $\widehat{q}$ such as above. Also, the *best possible average reply* of player P_1 for a move made by player P_2—with law given by $\mathcal{B}(q)$—is given by:

$$\overline{r}_{1\to 2}(q) = \mathbb{E}\left[\arg\sup_p \pi_2(\mathbb{E}_{(p,q)}[X_2 \mid X_1])\right] \ , \tag{2.4}$$

with similar conventions as stated above for $\overline{r}_{2\to 1}(p)$.

Remark 2.1 (On the nature of the *best possible replies* of the players). Since $\pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2])$ in Formula (2.4) is a random variable we have to interpret $\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2])$, also, as a random variable taking values in sets, that is, a random set. So, for each $\omega \in \Omega$, in the probability space Ω,

$$\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2])(\omega) \ ,$$

may be a set with more than one element since it is a set of maximising points. Being so $\overline{r}_{2\to 1}(p)$ is the expected value of a random set and so it may be a non random set with more than one element; in any case, for each p, we have that $\overline{r}_{2\to 1}(p) \subset [0,1]$. We will see in the numerical instances of an example of this game that, in fact, $\overline{r}_{2\to 1}(p)$, can be considered as a function the variable p.

Definition 2.2 (Game global best possible **random** reply)**.** The global game best possible random reply is given by the function R (or the correspondence),

$$R : [0,1] \times [0,1] \mapsto [0,1] \times [0,1]$$

such that $R(p,q) = (r_{2\to 1}(p), r_{1\to 2}(q))$.

Definition 2.3 (Equilibrium of the game)**.** An equilibrium for the game is a fixed point for R; if R is a correspondence then the equilibrium of the set is a set.

Definition 2.4 (Game global best possible **average** reply)**.** The global game best possible average reply is given by the function R (or the correspondence),

$$\overline{R} : [0,1] \times [0,1] \mapsto [0,1] \times [0,1]$$

such that $\overline{R}(p,q) = (\overline{r}_{2\to 1}(p), \overline{r}_{1\to 2}(q))$.

Definition 2.5 (Equilibrium of the game)**.** An equilibrium for the game is a fixed point for $\overline{R}$; if $\overline{R}$ is a correspondence then the equilibrium of the set is a set.

Remark 2.6 (On the nature of the equilibrium for game global best possible average reply correspondence). A fixed point for R is a set of two strategies determined by the parameters (p,q) that give the best possible replies for the two players and so it is a set of strategies that does not admit improvement.

Remark 2.7 (Detailing the particular assumptions of the game). We will suppose that the joint law of X_1 and X_2 is given by the Clayton copula with parameter c and that the marginals are, respectively, $\mathcal{B}(p)$ and $\mathcal{B}(q)$. We have that, for the conditional expectation $\mathbb{E}_{(p,q)}[X_1 \mid X_2]$,

$$\begin{aligned}
&\mathbb{E}_{(p,q)}[X_1 \mid X_2 = \xi_a] = \theta_a \mathbb{E}_{(p,q)}[X_1 = \theta_a \mid X_2 = \xi_a] + \theta_b \mathbb{E}_{(p,q)}[X_1 = \theta_b \mid X_2 = \xi_a] = \\
&= \theta_a \frac{\mathbb{P}_{(p,q)}[X_1 = \theta_a, X_2 = \xi_a]}{\mathbb{P}_{(p,q)}[X_2 = \xi_a]} + \theta_b \frac{\mathbb{P}_{(p,q)}[X_1 = \theta_b, X_2 = \xi_a]}{\mathbb{P}_{(p,q)}[X_2 = \xi_a]} = \\
&= \theta_a \frac{\mathbb{P}_{(p,q)}[X_1 = \theta_a, X_2 = \xi_a]}{q} + \theta_b \frac{\mathbb{P}_{(p,q)}[X_1 = \theta_b, X_2 = \xi_a]}{q} .
\end{aligned}$$

The terms of the form $\mathbb{P}_{(p,q)}[X_1 = \theta_b, X_2 = \xi_a]$ may be computed by the bivariate probability function associated to the Clayton copula with parameter c and marginal laws $\mathcal{B}(p)$ and $\mathcal{B}(q)$ which is given by:

$$\begin{aligned}
&\mathbb{P}_{(p,q)}[X_1 = \theta_{a,b}, X_2 = \xi_{a,b}] = \\
&= \begin{cases}
-\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} - q + 1 & X_1 = \theta_a \wedge X_2 = \xi_b \\
-\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} - p + 1 & X_1 = \theta_b \wedge X_2 = \xi_a \\
\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} & X_1 = \theta_b \wedge X_2 = \xi_b \\
\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} + p + q - 1 & X_1 = \theta_a \wedge X_2 = \xi_a ,
\end{cases}
\end{aligned}$$

with $\theta_{a,b} \in \{\theta_a, \theta_b\}$ and $\xi_{a,b} \in \{\xi_a, \xi_b\}$. It is now obvious the all branches of the function $\mathbb{P}_{(p,q)}[X_1 = \theta_{a,b}, X_2 = \xi_{a,b}]$ in the variables p, q can be shown to be Lipschitz with the same constant and so it is an equicontinuous family of continuous functions and so the supremums are continuous functions.

In the next theorem we suppose that the joint law of the strategies is determined by a determined copula—the Clayton copula—chosen for concept illustration purposes (see [8, pp. 184-237] or [18] for relevant information on copulas).

Theorem 2.8 (On the existence of equilibrium points for the game)**.** *Suppose that $p, q \in [\epsilon, 1-\epsilon]$ for $0 < \epsilon \ll 1$ and that the joint law of (X_1, X_2) is given by the Clayton copula with parameter c any positive number. Define R_ϵ to be the restriction of R to the square $[\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$. Then there exists an equilibrium for the game with global best possible average reply correspondence R_ϵ.*

Proof. We observe that the parameter space $[\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$. The proof is a consequence of Theorem 5.5 in Section 5. ☐

In the following we will detail the study a first example of a $(2p, 2s)$ game under some particular assumptions.

3. Computing the details of a concrete example of a $(2p, 2s)$ game

Suppose that a coin is drawn with law $\mathcal{B}(r)$; the result will be either H (heads) with probability r or T (tails) with probability $1-r$. The player P_1 makes the move X_1 according to the law $\mathcal{B}(p)$ and the player P_2 makes the move X_2 according to the law $\mathcal{B}(q)$. We will consider that the returns—or the specific payoffs of this game—are given by:

$$\{H, T\} = \Xi = \Theta = \{\theta_a, \theta_b\} = \{\xi_a, \xi_b\} = \{1, -1\}$$

Since the dependence structure we are using is given by:

$$\mathbb{P}_{(p,q)}[X_1 = \theta_{a,b}, X_2 = \xi_{a,b}] = \\ = \begin{cases} -\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} - q + 1 & X_1 = \theta_a \wedge X_2 = \theta_b \\ -\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} - p + 1 & X_1 = \theta_b \wedge X_2 = \theta_a \\ \left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} & X_1 = \theta_b \wedge X_2 = \theta_b \\ \left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} + p + q - 1 & X_1 = \theta_a \wedge X_2 = \theta_a \end{cases}$$

we will have that,

$$\mathbb{E}_{(p,q)}[X_1 \mid X_2 = \theta_a] = \theta_a \frac{\mathbb{P}_{(p,q)}[X_1 = \theta_a, X_2 = \theta_a]}{q} + \theta_b \frac{\mathbb{P}_{(p,q)}[X_1 = \theta_b, X_2 = \theta_a]}{q}$$
$$= \theta_a \frac{\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} + p + q - 1}{q} +$$
$$+ \theta_b \frac{-\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} - p + 1}{q} =$$
$$= \frac{2\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} + 2p + q - 2}{q}$$

and that,

$$\mathbb{E}_{(p,q)}[X_1 \mid X_2 = \theta_b] = \theta_a \frac{\mathbb{P}_{(p,q)}[X_1 = \theta_a, X_2 = \theta_b]}{1-q} + \theta_b \frac{\mathbb{P}_{(p,q)}[X_1 = \theta_b, X_2 = \theta_b]}{1-q}$$

$$= \frac{-\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} - q + 1}{1-q} +$$

$$- \frac{\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c}}{1-q} =$$

$$= \frac{-2\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} - q + 1}{1-q}$$

We will consider, as an example, $u(x) = \pi_1(x) = \pi_2(x) = x^2$ and $c = 0.06$. With these specifications we have that we can compute the random variable:

$$\pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2)]$$

We observe that $u(\mathbb{E}_{(p,q)}[X_1 \mid X_2 = \theta_a])$ and $u(\mathbb{E}_{(p,q)}[X_1 \mid X_2 = \theta_b])$, both as functions of the variables p and q, are shown as a contour plots in the following Figure 1. The random variable $\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2)]$ is then given by:

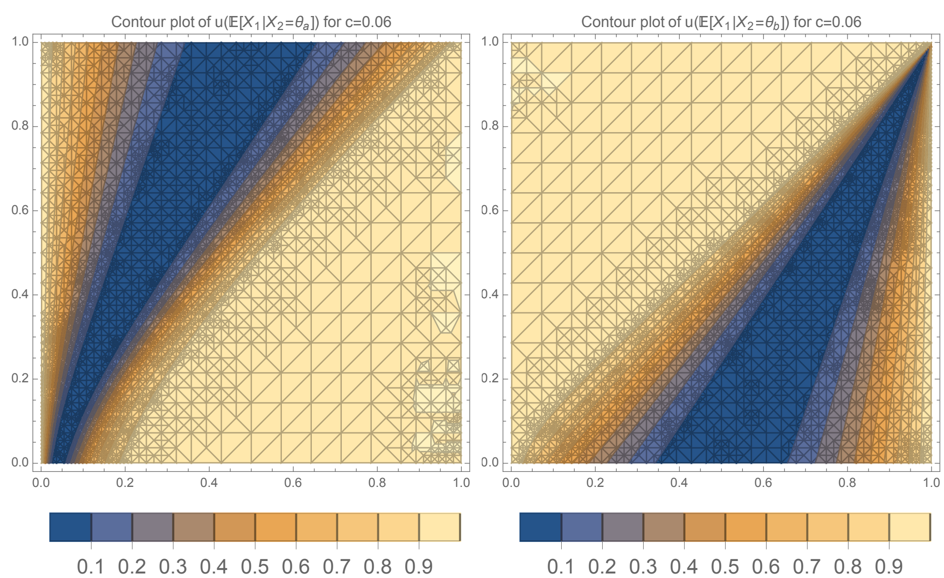

Figure 1. Contour plots of the images by u of the conditional expectations of the move of the first player with respect to the move of second player

$$
\begin{aligned}
&\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2)] = \\
&\qquad = \left(\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2 = \theta_a)]\right) I_{\{X_2=\theta_a\}} + \\
&\qquad + \left(\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2 = \theta_b)]\right) I_{\{X_2=\theta_b\}} \ ,
\end{aligned}
$$

and so we have that:

$$
\begin{aligned}
r_{2\to1}(p) &= \mathbb{E}\left[\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2)]\right] = \\
&= \left(\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2 = \theta_a)]\right) \mathbb{P}\left[X_2 = \theta_a\right] + \\
&+ \left(\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2 = \theta_b)]\right) \mathbb{P}\left[X_2 = \theta_b\right] \ .
\end{aligned}
$$

We observe that the terms of $r_{2\to1}(p)$, that is, $\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2 = \theta_a)]$ and $\arg\sup_q \pi_1(\mathbb{E}_{(p,q)}[X_1 \mid X_2 = \theta_b)]$ may be considered functions of the variable p since both are correspondences that take as values singular sets.

From the plot of Figure 3 depicting the function $r_{2\to1}(p)$ it is clear that there should exist a fixed point. Numerical calculations show that for $p_0 = 0.496$ we have that $r_{2\to1}(p_0) = 0.495363$ and we have thus obtained an approximation of the first component of a point of the equilibrium set.

Remark 3.1 (On the first component of the equilibrium point of the game). This result shows that for the first player the best option is to play with the law $\mathcal{B}(0.496)$.

We can perform the same calculations for the second player and the results are the following with only minor changes from the calculations for the first player.

$$
\begin{aligned}
&\mathbb{E}_{(p,q)}[X_2 \mid X_1 = \theta_a] = \theta_a \frac{\mathbb{P}_{(p,q)}[X_2 = \theta_a, X_1 = \theta_a]}{p} + \theta_b \frac{\mathbb{P}_{(p,q)}[X_1 = \theta_b, X_2 = \theta_a]}{p} \\
&= \frac{2\left((1-p)^{-1/c} + (1-q)^{-1/c} - 1\right)^{-c} + 2p + q - 2}{p}
\end{aligned}
$$

The graphics representation is given in the following Figure 2.

Remark 3.2 (On the contour plots). The non coloured regions in the contour plots of Figure 2 are due to very large values of the functions represented. These large values will also induce a particular result for the function $r_{1\to2}(q)$ shown below in Figure 3. The best possible average reply correspondence of player P_1 to the move of player P_2 is given by the following expression.

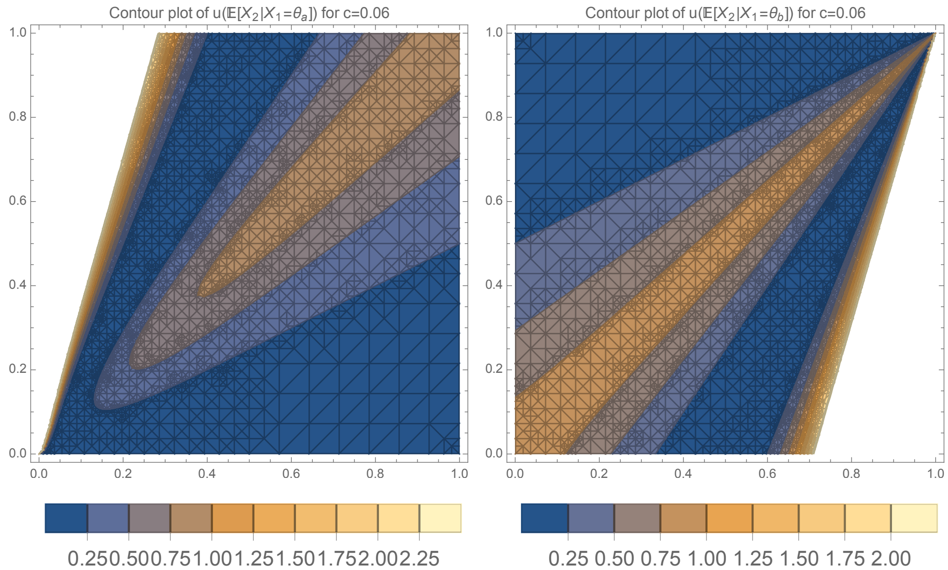

FIGURE 2. Contour plots of the images by u of the conditional expectations of the move of the second player with respect to the move of first player

$$
\begin{aligned}
r_{1\to 2}(q) = \mathbb{E}\left[\arg\sup_{p} \pi_2(\mathbb{E}_{(p,q)}[X_2 \mid X_1)]\right] &= \\
&= \left(\arg\sup_{p} \pi_2(\mathbb{E}_{(p,q)}[X_2 \mid X_1 = \theta_a)]\right) \mathbb{P}\left[X_1 = \theta_a\right] + \\
&+ \left(\arg\sup_{p} \pi_2(\mathbb{E}_{(p,q)}[X_2 \mid X_1 = \theta_b)]\right) \mathbb{P}\left[X_1 = \theta_b\right]
\end{aligned}
$$

We also have the terms of $r_{1\to 2}(q)$, that is, $\arg\sup_p \pi_2(\mathbb{E}_{(p,q)}[X_2 \mid X_1 = \theta_a)]$ and $\arg\sup_p \pi_2(\mathbb{E}_{(p,q)}[X_2 \mid X_1 = \theta_b)]$ may be considered functions of the variable q since both are correspondences that take as values singular sets.

The numerical computations corresponding to this correspondence—which in fact comes out as a function—give the graphic representation on the left of Figure 3, with the identity function of also plotted. From the plot of this Figure 3 (right) depicting the function $r_{1\to 2}(q)$ it is also clear that there should exist a fixed point. Numerical calculations show that for $p_0 = 0.5$ we have that $r_{1\to 2}(p_0) = 0.5$ and we have thus obtained an approximation of the second component of an equilibrium point.

3.1. The concrete example of a $(2p, 2s)$ game with given partial information. We now address the following question: what happens if some information on the initial draw is fed into the game? For instance, if we suppose that

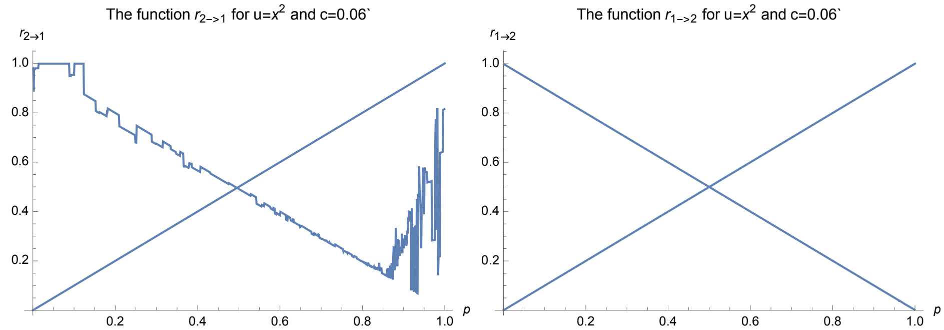

FIGURE 3. Functions $r_{2\to1}(p)$ (left) and $r_{1\to2}(q)$ (right) and the identity functions showing the approximate location of their respective fixed points.

$r \in [\rho_1, \rho_2]$? The natural result would be to have the laws of the players with parameters also in that interval. We will now suppose that $r \in [0.25, 0.45]$ all the other parameters being kept equal. The same method for the numerical computations was applied giving for the equilibrium point the numerical approximation $(p_0, q_0) = (0.311, 0.445)$ with:

$$R(p_0, q_0) = (r_{2\to1}(0.311), r_{1\to2}(0.445)) = (0.31081, 0.445174)$$

We will, secondly, suppose that $r \in [0.65, 0.85]$ all the other parameters being kept equal. The same method for the numerical computations was applied giving for the equilibrium point the numerical approximation $(p_0, q_0) = (0.311, 0.445)$ with:

$$R(p_0, q_0) = (r_{2\to1}(0.709), r_{1\to2}(0.792)) = (0.7084, 0.791998)$$

The main conclusion we get is that the asymmetry of the two strategies' players becomes more visible with more information being fed int the game.

3.2. Interpreting the equilibrium point obtained in the concrete example of a (2p,2s) game with given partial information. We simulated 10^5 runs of the game studied when we took $r \in [0.65, 0.85]$ using the previously determined equilibrium points as strategies. The Mathematica programming used is the following.

We obtained the following couple of success rates for the two players respectively: $(0.60355, 0.64599)$. It is clear that both players performed better than average with a small advantage for the second player, the player with the higher fixed point.

We also simulated 10^5 runs of the game studied when we took $r \in [0.25, 0.45]$. The correspondent results were $(0.55687, 0.5186)$ in this case showing a small advantage of the first player, the player with the smallest fixed point and a slightly better performance than average of both players.

For comparison we simulated 10^5 runs of the game with no prior information, that is when $r \in [0, 1]$ we have the result $(0.50342, 0.50245)$ in this case showing a

The meaning of the equilibrium point

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In[•]:= Players[c_, p_, q_] := CopulaDistribution[{"Clayton", c},
          {BernoulliDistribution[p], BernoulliDistribution[q]}]

In[•]:= c = 0.06; p0 = 0.709`;
        q0 = 0.792`; (* These are the parameters obtained and/or given *)

        ene = 100 000; U = {}; V = {};
        For[i = 1, i ≤ ene, i ++,
          aa = RandomVariate[UniformDistribution[{0.65, 0.85}], 1][[1]];
          (* We simulate a choice for the law of the first draw *)
          draw = RandomVariate[BernoulliDistribution[aa], 1][[1]];
          (* We simulate a choice for the first draw *)
          game = RandomVariate[Players[0.06, p0, q0], 1];
          (* We simulate the players mooves *)
          U = Append[U, Abs[game[[1, 1]] - draw]];
          V = Append[V, Abs[game[[1, 2]] - draw]];
          (* We determine the list of "fails" (the ones) for each player comparing
           the moove of the player to the draw *)
         ];
        {1 - N[Mean[U]], 1 - N[Mean[V]]}
        (* This gives the percentage of successes for player 1 and player 2
         respectively *)

Out[•]= {0.60287, 0.64495}
```

FIGURE 4. Mathematica code for getting computational confirmation of the optimality of the equilibrium points

very small advantage of the first player, the player with the smallest fixed point and a slightly better performance than average of both players.

The interpretation of these results is that, using as strategies the determined equilibrium points, increases even if only slightly the gains of the players with respect of complete random strategy.

Remark 3.3 (On the difference between strategies). The differences observed should be related not only to the Clayton copula used but also to the function used for the payoff $\pi_1(x) = \pi_2(x) = x^2$. The derivative of this function for $x < 0.5$ is less than one and for $x > 0.5$ is greater than one; this may explain the quantitative differences observed between the first and the second case.

4. On the Choquet's representation of the game global best possible reply

Choquet's theorem allows the representation of the game global best possible reply by means of the extreme points of its convex compact domain. Recall that the game global best possible reply function considered is R_ϵ defined in $[\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$ which is a compact convex set of the locally convex space $\mathbb{R}$.

Theorem 4.1 (Choquet's theorem). *If C is a metrisable compact convex subset of a locally convex space and x_0 is an arbitrary element of C, then there is a* *probability measure which represents x_0 and is supported by the extreme points of C (see* [15, pp. 20-21]*).*

Let $\mu_{(p,q)}$ be the probability measure given by Choquet's theorem to $C := [\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$ supported by the set,

$$\mathcal{E} = \{(\epsilon, \epsilon), (\epsilon, 1-\epsilon), (1-\epsilon, \epsilon), (1-\epsilon, 1-\epsilon)\}$$

of extreme points of C that represent a given point $(p, q) \in C$. It is clear due to the discrete nature of $\mathcal{E}$ that we must have that $\mu_{(p,q)}$ is a convex combination of Dirac measures each supported by an extreme point of C, that is, for some constants $\alpha_1, \ldots, \alpha_4 \in [0, 1]$ such that $\alpha_1 + \cdots + \alpha_4 = 1$ we have that:

$$\mu_{(p,q)} = \alpha_1 \delta_{(\epsilon,\epsilon)} + \alpha_2 \delta_{(\epsilon,1-\epsilon)} + \alpha_3 \delta_{(1-\epsilon,\epsilon)} + \alpha_4 \delta_{(1-\epsilon,1-\epsilon)}$$

The representation property says considering $(p, q) \in C$ a fixed point of R_ϵ that for all linear function f we have that:

$$\begin{aligned} f(p, q) = \int_{\mathcal{E}} f(u, v) d\mu_{(p,q)}(u, v) = \\ = \alpha_1 f(\epsilon, \epsilon) + \alpha_2 f(\epsilon, 1-\epsilon) + \alpha_3 f(1-\epsilon, \epsilon) + \alpha_4 f(1-\epsilon, 1-\epsilon) \end{aligned}$$

And that means that since $(p, q) \in C$ a fixed point of R_ϵ, that is, $R_\epsilon(p, q) = (p, q)$, that we have:

$$R_\epsilon(p, q) = \alpha_1 \cdot (\epsilon, \epsilon) + \alpha_2 \cdot (\epsilon, 1-\epsilon) + \alpha_3 \cdot (1-\epsilon, \epsilon) + \alpha_4 \cdot (1-\epsilon, 1-\epsilon)$$

that is, the game global best possible reply function at the equilibrium point may be written as a linear combination of the extreme points of the set $[\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$.

Remark 4.2 (On the arbitrariness of ϵ)*.* : The coefficients $\alpha_1, \ldots, \alpha_4$ **depend on** and so on they also depend on ϵ. At the moment, it does not seem possible to use some procedure in order to pass to the limit and, as so, defining a game on $[0, 1] \times [0, 1]$.

Remark 4.3 (On the importance of Choquet's representation)*.* For parameter models with a finite number of parameters in compact intervals Choquet's representation provides an obvious result obtained simply by the fact that intervals are convex sets with two extreme points. The result will become interesting in the case of strategies belonging to some infinite dimensional subspace of probability measures.

5. Completely random games with arbitrary finite number of players and with strategies indexed by real vector parameters

In this section we present a general theory for random static games with a finite number of players. In the following there is a list of assumptions and notations for the approach we propose.

(1) Let $P_1, P_2, \ldots, P_r$ denote the players.
(2) Consider a set $\mathcal{M} = \{\mu_{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$ of probability measures defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\boldsymbol{\Theta} \Subset \mathbb{R}^d$ a convex compact space of parameters in an Euclidean space. In a given game, each player P_i, $i = 1, \ldots, r$, has a **strategy** given by a probability measure $\mu_{\boldsymbol{\theta}_i} \in \mathcal{M}$.

(3) In a given game, each player P_i, $i = 1, \dots, r$ has a **move**, that is, a random variable $X_i : \Omega \to \mathbb{R}$ having as law, or probability distribution, the probability measure $\mu_{\boldsymbol{\theta_i}}$.

(4) In a given game, $\mu_{(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_r})}$ the joint law of the vector $(X_1, X_2, \dots, X_r)$ is determined. We observe that a better approach to be thee notion of kernel such as it is developed in [5]. In an application this joint law may be given by an adequate copula.

(5) In a given game, for each player P_i, $i = 1, \dots, r$ there is a continuous function $\pi_i : \mathbb{R} \to [0, 1]$ such that the payoff of player P_i, $i = 1, \dots, r$ is given by the image by π_1 of the conditional expectation:

$$\mathbb{E}_{(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_r})}[X_i \mid X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_r] \ ,$$

that is,

$$\pi_i \left(\mathbb{E}_{(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_r})}[X_i \mid X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_r] \right) \ .$$

(6) We will use next the following notation; $\widehat{i} := \{1, 2, \dots, r\} \setminus \{i\}$, that is, $\widehat{i}$ is the set of all the players indexes $\{1, 2, \dots, r\}$ with the index i removed. With the notation:

$$\boldsymbol{\theta_{\widehat{i}}} = (\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_{i-1}}, \boldsymbol{\theta_{i+1}}, \dots, \boldsymbol{\theta_r}) \text{ and} (\boldsymbol{\theta_i}, \boldsymbol{\theta_{\widehat{i}}}) = (\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_{i-1}}, \boldsymbol{\theta_i}, \boldsymbol{\theta_{i+1}}, \dots, \boldsymbol{\theta_r}) \ ,$$

we define the (random) best possible reply by:

$$\begin{aligned} r_{\widehat{i} \to i}(\boldsymbol{\theta_i}) &= \arg \sup_{\boldsymbol{\theta_j},\, j \in \widehat{i}} \pi_i(\mathbb{E}_{(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_r})}[X_i \mid X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_r]) = \\ &= \{\boldsymbol{\theta_{\widehat{i}}} : \pi_i(\mathbb{E}_{(\boldsymbol{\theta_i}, \boldsymbol{\theta_{\widehat{i}}})}[X_i \mid X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_r]) = \\ &= \sup_{\boldsymbol{\theta_j},\, j \in \widehat{i}} \pi_i(\mathbb{E}_{(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_r})}[X_i \mid X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_r])\} \ , \end{aligned} \tag{5.1}$$

which a random variable taking as values sets in $\boldsymbol{\Theta}^{r-1}$ since it is defined by means of a conditional expectation.

(7) With the same notation the *best possible average reply* for a move made by the player P_i of players P_j for $j \in \widehat{i}$ is given by:

$$\overline{r}_{\widehat{i} \to i}(\boldsymbol{\theta_i}) = \mathbb{E} \left[\arg \sup_{\boldsymbol{\theta_j},\, j \in \widehat{i}} \pi_i(\mathbb{E}_{(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_r})}[X_i \mid X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_r]) \right] \ , \tag{5.2}$$

that is, the *best possible average reply*, for the player of index i given by the other players, is the expected value of the set of values of the parameters—defining the law of the moves of players P_j for $j \in \widehat{i}$—which maximizes the image, by the function π_i of the conditional expectation of the move of of player P_i given all the moves of the other players defined by their strategies. So, this *best possible average reply* may be considered as a correspondence from $\boldsymbol{\Theta}$ into $\boldsymbol{\Theta}^{r-1}$.

Remark 5.1 (The *best possible replies* depend on the laws of the players moves). We stress that the *best possible average reply* $r_i(\boldsymbol{\theta_i})$, of a player P_i, for $i = 1, \dots, r$, depends on the move X_i and also on X_j for $j \neq i$ and so it depends on $\boldsymbol{\theta_i}$, for $i = 1, \dots, r$, that is on the probability distribution of the moves of all players—the strategies—and so it may be used to define a correspondence that takes values in the parameter set $\boldsymbol{\Theta}^r$.

We have two notions of best possible reply for the game; one corresponding to best possible average reply in Formula (5.2) and the second one corresponding to best possible random reply in Formula (5.1).

Definition 5.2 (Game global random best possible reply)**.** The global *best possible random reply* of the game is given by the correspondence R defined, for almost all $\omega \in \Omega$, by:

$$R = R(\omega) : \boldsymbol{\Theta}^r = \boldsymbol{\Theta} \times \cdots \times \boldsymbol{\Theta} \twoheadrightarrow \boldsymbol{\Theta}^r \ ,$$

such that:

$$(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \ldots, \boldsymbol{\theta_r}) \mapsto \bigcup_{i \in \{1,\ldots,r\} \ ,\boldsymbol{\theta_{\widehat{i}}} \in r_{\ \widehat{i} \to i}(\boldsymbol{\theta_i})(\omega)} \left\{ \left(\boldsymbol{\theta_i}, \boldsymbol{\theta_{\widehat{i}}} \right) \right\} \ ,$$

for almost all $\omega \in \Omega$.

Naturally we will expect to be able to define similarly the next concept.

Definition 5.3 (Game global best possible average reply)**.** The global *best possible average reply* of the game is given by the correspondence $\overline{R}$ defined by:

$$\overline{R} : \boldsymbol{\Theta}^r = \boldsymbol{\Theta} \times \cdots \times \boldsymbol{\Theta} \twoheadrightarrow \boldsymbol{\Theta}^r \ ,$$

such that:

$$(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \ldots, \boldsymbol{\theta_r}) \mapsto \bigcup_{i \in \{1,\ldots,r\} \ ,\boldsymbol{\theta_{\widehat{i}}} \in \overline{r}_{\ \widehat{i} \to i}(\boldsymbol{\theta_i})} \left\{ \left(\boldsymbol{\theta_i}, \boldsymbol{\theta_{\widehat{i}}} \right) \right\}$$

We can now state the definition of a Nash like equilibrium point for the random game.

Definition 5.4 (Equilibrium points of the game)**.** A random (respectively, average) equilibrium for the game is a fixed point for the closure of the convex hull of the correspondence R (respectively,$\overline{R}$).

The question that subsides now is to determine the conditions under which a game has equilibrium points.

Theorem 5.5 (On the existence of—random—equilibrium points for the game)**.** *Let us suppose that:*

(1) *The parameter set* $\boldsymbol{\Theta}$ *is convex and compact.*
(2) *Defining, for all integers* $s \geq 1$, *the map* $\Phi_s : \boldsymbol{\Theta}^s \to \mathcal{M}^s$ *such that:*

$$\Phi_s(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \ldots, \boldsymbol{\theta_s}) = (\mu_{\boldsymbol{\theta_1}}, \mu_{\boldsymbol{\theta_2}}, \ldots, \mu_{\boldsymbol{\theta_s}}) \ ,$$

we suppose that this map is continuous from $\boldsymbol{\Theta}^s$ *equipped with any* s*-product distance of the Euclidean distance of the ambient space of* $\boldsymbol{\Theta}$ *into* $\mathcal{M}^s$ *equipped with the* s*-product of the total variation distance on* $\mathcal{M}$.
(3) *We suppose that for all integers* s *the joint law of* $(\mu_{\boldsymbol{\theta_1}}, \mu_{\boldsymbol{\theta_2}}, \ldots, \mu_{\boldsymbol{\theta_s}})$ *admits a density with respect to the Lebesgue measure given by* $f^{(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \ldots, \boldsymbol{\theta_s})}$.

Then there are equilibrium points for the game, namely the fixed points of the correspondence M *given by the closure of the convex hull of the game global random best possible reply correspondence* R.

Proof. We observe that if R is a function—and not a general correspondence—Brower's fixed point theorem [1] may allow us to conclude since R is a map from a compact convex set $\mathbf{\Theta}^r$ into itself. If R is, in general, a correspondence it will be necessary to use Berge maximum theorem and then Kakutani's fixed point theorem.

Let us show that $\mathbb{E}_{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})}\left[X_i \mid X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_r\right]$ is a continuous function of the parameter vector $(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})$. For that purpose we observe that if,

$$\begin{aligned}
&\mathbb{E}_{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})}\left[X_i \mid X_1=k_1,\ldots,X_{i-1}=k_{i-1},X_{i+1}=k_{i+1},\ldots,X_r=k_r\right] = \\
&= \int_{\mathbb{R}^d} x_i \frac{f_{(X_1,\ldots,X_r)}^{(\boldsymbol{\theta_1},\boldsymbol{\theta_2},\ldots,\boldsymbol{\theta_r})}(k_1,\ldots,k_{i-1},x_i,k_{i+1},\ldots,k_r)}{f_{(X_1,\ldots,X_{i-1},X_{i+1},\ldots X_r)}^{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_{i-1}},\boldsymbol{\theta_{i+1}},\ldots,\boldsymbol{\theta_r})}(k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_r)} d\lambda(x_i) = \\
&= \phi^{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_{i-1}},\boldsymbol{\theta_i},\boldsymbol{\theta_{i+1}},\ldots,\boldsymbol{\theta_r})}(k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_r)\ ,
\end{aligned} \tag{5.3}$$

then we have that:

$$\begin{aligned}
&\mathbb{E}_{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})}\left[X_i \mid X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_r\right] = \\
&= \phi^{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_{i-1}},\boldsymbol{\theta_i},\boldsymbol{\theta_{i+1}},\ldots,\boldsymbol{\theta_r})}(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_r)\ .
\end{aligned}$$

So it all amounts to prove the continuity of the function $\phi^{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_{i-1}},\boldsymbol{\theta_i},\boldsymbol{\theta_{i+1}},\ldots,\boldsymbol{\theta_r})}$ and by Formula 5.3 this amounts to prove that the densities are continuous functions of the vector parameters.

So consider two vectors of parameters $(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})$ and $(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})$ with the same indexes and the correspondent densities $f_{(X_{i_1},\ldots,X_{i_1})}^{(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})}$ and $f_{(X_{i_1},\ldots,X_{i_1})}^{(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})}$ of the joint laws (respectively) $\mu_{(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})}$ and $\mu_{(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})}$ with respect to the Lebesgue measure $d\lambda(x_{i_1},\ldots,x_{i_s})$ and observe that for any measurable set A,

$$\begin{aligned}
&\left|\int_A \left(f_{(X_{i_1},\ldots,X_{i_1})}^{(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})}(x_{i_1},\ldots,x_{i_s}) - f_{(X_{i_1},\ldots,X_{i_1})}^{(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})}(x_{i_1},\ldots,x_{i_s})\right) d\lambda(x_{i_1},\ldots,x_{i_s})\right| = \\
&= \left|\mu_{(\boldsymbol{\theta_{i_1}},\ldots,\boldsymbol{\theta_{i_s}})}(A) - \mu_{(\boldsymbol{\theta'_{i_1}},\ldots,\boldsymbol{\theta'_{i_s}})}(A)\right| \leq \sup_B \left|\mu_{(\boldsymbol{\theta_{i_1}},\ldots,\boldsymbol{\theta_{i_s}})}(B) - \mu_{(\boldsymbol{\theta'_{i_1}},\ldots,\boldsymbol{\theta'_{i_s}})}(B)\right| = \\
&= d_s\left(\mu_{(\boldsymbol{\theta'_{i_1}},\ldots,\boldsymbol{\theta'_{i_s}})}, \mu_{(\boldsymbol{\theta'_{i_1}},\ldots,\boldsymbol{\theta'_{i_s}})}\right)
\end{aligned}$$

Now choose $\epsilon > 0$ an then $(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})$ and $(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})$ close enough such that,

$$d_s\left(\mu_{(\boldsymbol{\theta'_{i_1}},\ldots,\boldsymbol{\theta'_{i_s}})}, \mu_{(\boldsymbol{\theta'_{i_1}},\ldots,\boldsymbol{\theta'_{i_s}})}\right) \leq \epsilon\ .$$

Then, for $(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})$ and $(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})$ close enough and for the set:

$$D^+(\epsilon) := \left\{(x_{i_1},\ldots,x_{i_s}) : f_{(X_{i_1},\ldots,X_{i_1})}^{(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})}(x_{i_1},\ldots,x_{i_s}) \geq f_{(X_{i_1},\ldots,X_{i_1})}^{(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})}(x_{i_1},\ldots,x_{i_s})\right\}$$

we will have that for all measurable sets A and for $(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})$ and $(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})$ close enough, the following positive integral verifies:

$$\begin{aligned}\int_{D^+(\epsilon)\cap A} \Big(f^{(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})}_{(X_{i_1},\ldots,X_{i_1})}(x_{i_1},\ldots,x_{i_s}) - \\ - f^{(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})}_{(X_{i_1},\ldots,X_{i_1})}(x_{i_1},\ldots,x_{i_s})\Big) d\lambda(x_{i_1},\ldots,x_{i_s}) \leq \epsilon \ ,\end{aligned} \tag{5.4}$$

and also for the set:

$$D^-(\epsilon) := \left\{ (x_{i_1},\ldots,x_{i_s}) : f^{(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})}_{(X_{i_1},\ldots,X_{i_1})}(x_{i_1},\ldots,x_{i_s}) \leq f^{(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})}_{(X_{i_1},\ldots,X_{i_1})}(x_{i_1},\ldots,x_{i_s}) \right\}$$

we will have that for all measurable sets A and for $(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})$ and $(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})$ close enough, the following positive integral verifies:

$$\begin{aligned}\int_{D^-(\epsilon)\cap A} \Big(f^{(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})}_{(X_{i_1},\ldots,X_{i_1})}(x_{i_1},\ldots,x_{i_s}) - \\ - f^{(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})}_{(X_{i_1},\ldots,X_{i_1})}(x_{i_1},\ldots,x_{i_s})\Big) d\lambda(x_{i_1},\ldots,x_{i_s}) \leq \epsilon \ .\end{aligned} \tag{5.5}$$

By the properties of the Lebesgue integral (see Lemma 6.1 in the Appendix) the bounds in Formulas (5.4) and (5.5) are sufficient to prove that $\lambda(x_{i_1},\ldots,x_{i_s})$ almost everywhere:

$$\lim_{(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})\to(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})} f^{(\boldsymbol{\theta}_{i_1},\ldots,\boldsymbol{\theta}_{i_s})}_{(X_{i_1},\ldots,X_{i_1})}(x_{i_1},\ldots,x_{i_s}) = f^{(\boldsymbol{\theta}'_{i_1},\ldots,\boldsymbol{\theta}'_{i_s})}_{(X_{i_1},\ldots,X_{i_1})}(x_{i_1},\ldots,x_{i_s}) \ ,$$

thus showing, as an application of Lebesgue's dominated convergence theorem that the functions $\phi^{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_{i-1}},\boldsymbol{\theta_i},\boldsymbol{\theta_{i+1}},\ldots,\boldsymbol{\theta_r})}$ are continuous as functions of the parameters $(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_{i-1}},\boldsymbol{\theta_i},\boldsymbol{\theta_{i+1}},\ldots,\boldsymbol{\theta_r})$.

In order to conclude we must apply the Berge maximum theorem and Kakutani fixed point theorem. The general reference for these results is [1, p. 583] for the Kakutani–Fan–Glicksberg theorem and [1, p. 570] for the Berge's maximum theorem. For the reader's commodity we also present the adequate versions of these results—respectively, Theorem 6.3 and Theorem 6.2—in the Appendix.

Let us chose a version of $\mathbb{E}_{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})}\left[X_i \mid X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_r\right]$ and for that version and $\omega \in \Omega_1$ with Ω_1 a set of probability one let us take:

$$\begin{aligned}&\mathbb{E}_{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})}\left[X_i \mid X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_r\right](\omega) = \\ &= \phi^{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_{i-1}},\boldsymbol{\theta_i},\boldsymbol{\theta_{i+1}},\ldots,\boldsymbol{\theta_r})}(X_1(\omega),\ldots,X_{i-1}(\omega),X_{i+1}(\omega),\ldots,X_r(\omega)) \ ,\end{aligned}$$

Now, considering the correspondence $\varphi_i : \boldsymbol{\Theta} \twoheadrightarrow \boldsymbol{\Theta}^{r-1}$ such that, for

$$(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_{i-1},\boldsymbol{\theta}_i,\boldsymbol{\theta}_{i+1},\ldots,\boldsymbol{\theta}_r) \in \boldsymbol{\Theta}^r$$

we have

$$\varphi_i(\boldsymbol{\theta}_i) = \left\{\boldsymbol{\theta}_{\widehat{i}} = (\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_{i-1},\boldsymbol{\theta}_{i+1},\ldots,\boldsymbol{\theta}_r)\right\} \subset \boldsymbol{\Theta}^{r-1} \ ,$$

define:

$$
\begin{aligned}
m_{\widehat{\imath}\to i}(\boldsymbol{\theta}_i) = m_{\widehat{\imath}\to i}(\boldsymbol{\theta}_i)(\omega) :=& \\
&= \sup_{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})} \mathbb{E}_{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})}\left[X_i \mid X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_r\right](\omega) = \\
&= \sup_{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})} \phi^{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})}(X_1(\omega),\ldots,X_{i-1}(\omega),X_{i+1}(\omega),\ldots,X_r(\omega)) = \\
&= \sup_{(\boldsymbol{\theta_i},\varphi_i(\boldsymbol{\theta}_i))\equiv\left(\boldsymbol{\theta}_i,\boldsymbol{\theta}_{\widehat{\imath}}\right)} \phi^{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})}(X_1(\omega),\ldots,X_{i-1}(\omega),X_{i+1}(\omega),\ldots,X_r(\omega)) \,.
\end{aligned}
$$

Then, considering the correspondence $r_{\widehat{\imath}\to i}(\boldsymbol{\theta_i})(\omega) : \boldsymbol{\Theta} \twoheadrightarrow \boldsymbol{\Theta}^{r-1}$ giving the maximisers,

$$
\begin{aligned}
&r_{\widehat{\imath}\to i}(\boldsymbol{\theta_i})(\omega) := \\
&= \left\{\boldsymbol{\theta}_{\widehat{\imath}} \in \varphi\left(\boldsymbol{\theta}_i\right) : \phi^{(\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_r})}(X_1(\omega),\ldots,X_{i-1}(\omega),X_{i+1}(\omega),\ldots,X_r(\omega)) = m_{\widehat{\imath}\to i}(\boldsymbol{\theta}_i)\right\} \,,
\end{aligned}
$$

we may conclude applying the Berge's maximum theorem, recalled ahead in the Appendix as Theorem 6.2, that for almost all $\omega \in \Omega$ we have that $m_{\widehat{\imath}\to i}(\omega)$ is continuous and the "argmax" correspondence $r_{\widehat{\imath}\to i}(\boldsymbol{\theta_i})(\omega)$ is upper hemicontinuous with compact values.

In order to conclude we aim to deal with the best possible random reply. So we first consider the correspondence $R(\omega) : \boldsymbol{\Theta}^r \twoheadrightarrow \boldsymbol{\Theta}^r$, proposed in Definition 5.2, defined for almost all $\omega \in \Omega$ by:

$$
R(\omega)\left(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_r\right) = \bigcup_{i\in\{1,\ldots,r\}\ ,\ \boldsymbol{\theta}_{\widehat{\imath}}\in r_{\widehat{\imath}\to i}(\boldsymbol{\theta_i})(\omega)} \left\{\left(\left(\boldsymbol{\theta}_1,\boldsymbol{\theta}_{\widehat{1}},\right),\ldots,\left(\boldsymbol{\theta}_r,\boldsymbol{\theta}_{\widehat{r}},\right)\right)\right\} \,, \tag{5.6}
$$

that is, the correspondence that to $(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_r)$ associates the set of all maximisers of the functions $m_{\widehat{\imath}\to i}(\boldsymbol{\theta}_i)$. Let us first observe that since $R(\omega)$ is the product of two correspondences—the first factor being the identity function in $\boldsymbol{\Theta}^r$ and the second factor being given by $r_{\widehat{\imath}\to i}(\boldsymbol{\theta_i})(\omega)$ which we already know is a upper hemicontinuous correspondence with compact values—is hemicontinuous with compact values (see [1, p. 568]). We can also observe that by the close graph theorem (see [1, p. 561]) $R(\omega)$ is a closed correspondence.

In order to prove the conditions for fixed point theorem to be applied we consider the convex hull of the correspondence $R(\omega)$, that is, we define:

$$
M(\omega)\left(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_r\right) = \overline{\mathbf{Conv}}\left(R(\omega)\left(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_r\right)\right)
$$

that is, the closure of the convex hull of $R(\omega)\left(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_r\right)$. Since $\boldsymbol{\Theta}^r$ is convex $M(\omega)$ is a correspondence defined in $\boldsymbol{\Theta}^r$ and taking convex set values in $\boldsymbol{\Theta}^r$. Now the closure of the convex hull of a hemicontinuous orrespondence taking compact values is hemicontinuous and takes compact values (see Theorem 17.35 in [1, p. 573]) and so again the closed graph theorem followed by the Kakutani-Fan-Glicksberg's fixed point theorem the correspondence $M(\omega)$ has a compact non empty set of fixed points and the proof is finished. □

We now study the best possible average reply fixed points detailed in the example studied in Sections 2 and 3. Let us state our starting point. By Formula (5.6) we have defined for almost all $\omega \in \Omega$ a correspondence $R(\omega)$ that to some vector $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r)$ in $\boldsymbol{\Theta}_r$ associates the random set on the righthand side of Formula (5.6). It is natural to define the average correspondence $\overline{R}$ to be a correspondence given by:

$$\overline{R}\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r\right) := \mathbb{E}\left[\bigcup_{i\in\{1,\ldots,r\}\ ,\ \boldsymbol{\theta}_{\widehat{i}}\in r_{\widehat{i}\to i}(\boldsymbol{\theta_i})} \left\{\left(\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_{\widehat{1}}\right), \ldots, \left(\boldsymbol{\theta}_r, \boldsymbol{\theta}_{\widehat{r}}\right)\right)\right\}\right] ,$$

where on the righthand side we have the expectation of a random set, since $r_{\widehat{i}\to i}(\boldsymbol{\theta_i}) = r_{\widehat{i}\to i}(\boldsymbol{\theta_i})(\omega)$. In order to clarify this definition we observe that after taking expectations we should obtain a correspondence. Let us consider a selection in the variables $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r)$; by definition it is a function S such that:

$$S\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r\right) \in \bigcup_{i\in\{1,\ldots,r\}\ ,\ \boldsymbol{\theta}_{\widehat{i}}\in r_{\widehat{i}\to i}(\boldsymbol{\theta_i})} \left\{\left(\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_{\widehat{1}}\right), \ldots, \left(\boldsymbol{\theta}_r, \boldsymbol{\theta}_{\widehat{r}}\right)\right)\right\}$$

that is, such that

$$\begin{aligned} S\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r\right) &= \left(\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_{\widehat{1}}\right), \ldots, \left(\boldsymbol{\theta}_r, \boldsymbol{\theta}_{\widehat{r}}\right)\right) |_{\boldsymbol{\theta}_{\widehat{i}}\in r_{\widehat{i}\to i}(\boldsymbol{\theta_i})\ ,\ i\in\{1,\ldots,r\}} = \\ &= \left(\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_{\widehat{1}}\right) \mathbb{1}_{r_{\widehat{1}\to 1}(\boldsymbol{\theta_1})}(\boldsymbol{\theta}_{\widehat{1}}), \ldots, \left(\boldsymbol{\theta}_r, \boldsymbol{\theta}_{\widehat{r}}\right) \mathbb{1}_{r_{\widehat{1}\to r}(\boldsymbol{\theta_r})}(\boldsymbol{\theta}_{\widehat{r}})\right) \end{aligned}$$

with, for $i \in \{1, \ldots, r\}$:

$$\mathbb{1}_{r_{\widehat{1}\to i}(\boldsymbol{\theta_i})}(\boldsymbol{\theta}_{\widehat{i}}) = \begin{cases} 1 & \text{if } \boldsymbol{\theta}_{\widehat{i}} \in r_{\widehat{i}\to i}(\boldsymbol{\theta_i}) \\ 0 & \text{if } \boldsymbol{\theta}_{\widehat{i}} \notin r_{\widehat{i}\to i}(\boldsymbol{\theta_i}) \ . \end{cases}$$

For such a selection, in case that,

$$\mathbb{E}\left[\mathbb{1}_{r_{\widehat{1}\to i}(\boldsymbol{\theta_i})}(\boldsymbol{\theta}_{\widehat{i}})\right] = \mathbb{1}_{\overline{r}_{\widehat{1}\to 1}(\boldsymbol{\theta_i})}(\boldsymbol{\theta}_{\widehat{i}}) \ , \tag{5.7}$$

which happens if $\# r_{\widehat{i}\to 1}(\boldsymbol{\theta_i}) = 1$, that is, if there is always an unique maximiser see [9, p.282], we should have:

$$\begin{aligned} \mathbb{E}\left[S\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r\right)\right] &= \mathbb{E}\left[\left(\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_{\widehat{1}}\right) \mathbb{1}_{r_{\widehat{1}\to 1}(\boldsymbol{\theta_1})}(\boldsymbol{\theta}_{\widehat{1}}), \ldots, \left(\boldsymbol{\theta}_r, \boldsymbol{\theta}_{\widehat{r}}\right) \mathbb{1}_{r_{\widehat{1}\to r}(\boldsymbol{\theta_r})}(\boldsymbol{\theta}_{\widehat{r}})\right)\right] = \\ &= \left(\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_{\widehat{1}}\right) \mathbb{E}\left[\mathbb{1}_{r_{\widehat{1}\to 1}(\boldsymbol{\theta_1})}(\boldsymbol{\theta}_{\widehat{1}})\right], \ldots, \left(\boldsymbol{\theta}_r, \boldsymbol{\theta}_{\widehat{r}}\right) \mathbb{E}\left[\mathbb{1}_{r_{\widehat{1}\to r}(\boldsymbol{\theta_r})}(\boldsymbol{\theta}_{\widehat{r}}\right]\right) \\ &= \left(\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_{\widehat{1}}\right) \mathbb{1}_{\overline{r}_{\widehat{1}\to 1}(\boldsymbol{\theta_1})}(\boldsymbol{\theta}_{\widehat{1}}), \ldots, \left(\boldsymbol{\theta}_r, \boldsymbol{\theta}_{\widehat{r}}\right) \mathbb{1}_{\overline{r}_{\widehat{1}\to r}(\boldsymbol{\theta_r})}(\boldsymbol{\theta}_{\widehat{r}})\right) \end{aligned}$$

As a consequence, we may write:

$$\mathbb{E}\left[S\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r\right)\right] \in \bigcup_{i\in\{1,\ldots,r\}\ ,\ \boldsymbol{\theta}_{\widehat{i}}\in \overline{r}_{\widehat{1}\to i}(\boldsymbol{\theta_i})} \left\{\left(\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_{\widehat{1}}\right), \ldots, \left(\boldsymbol{\theta}_r, \boldsymbol{\theta}_{\widehat{r}}\right)\right)\right\} ,$$

that is, $\mathbb{E}\left[S\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r\right)\right]$ is a selection of the non random correspondence R given by:

$$R\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r\right) = \bigcup_{i\in\{1,\ldots,r\}\ ,\ \boldsymbol{\theta}_{\widehat{i}}\in \overline{r}_{\widehat{1}\to i}(\boldsymbol{\theta_i})} \left\{\left(\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_{\widehat{1}}\right), \ldots, \left(\boldsymbol{\theta}_r, \boldsymbol{\theta}_{\widehat{r}}\right)\right)\right\} .$$

We can now formulate a result that guaranties the existence of average equilibrium points for the game.

Corollary 5.6 (On the best possible average reply)**.** *Under the same hypothesis of Theorem 5.5 suppose that for all $i \in \{1, \ldots, r\}$ the condition given by Formula (5.7) is verified. Then there are average equilibrium points for the game.*

Proof. The same arguments developed in the proof of the preceding theorem apply.

6. Conclusions and further work

There are some advantages of the approach we have taken in this work, namely, considering strategies given by probability laws, players's moves given by random variables with the respective laws given by the strategies and the payoffs given as functions of the conditional expectations of one player move with respect to the other, or others; the first advantage is that we can consider that the players do not play independently of each other, that is, we can introduce—in a simple and feasible way—a dependence structure on the strategies of the players; the second advantage is that with a dynamic game we have that the moves are stochastic processes; finally, a third advantage is that we contemplate the possibility of players not having deterministic moves, but preferably, random moves but with a given distribution corresponding to a defined strategy.

In what concerns for the example in Section 2, the main conclusion is that without information on the outcome of the initial draw of the game—or on its law—it seems natural that the best strategy for player one is to play with $\mathcal{B}(0.496)$ and for player two to play with $\mathcal{B}(0.5)$. The origin of this asymmetry comes perhaps from the value of the constant describing the dependence structure of the players laws. If some information is fed into the game, namely by stating that the law of the initial draw $\mathcal{B}(r)$ has a parameter that belongs to some interval, for instance, not containing $1/2$, then the asymmetry of the laws of the players is reinforced, the second player having to play a more extreme game, with its law parameter either close to zero or to one. A natural question is: what is the importance of considering that the pure strategies are given by $\{H, T\} = \{1, -1\}$? In principle it corresponds to H giving to the statute of success. But then, what happens if we use only positive values, or any other values? Another natural question is: what is the importance of the dependence structure of the players laws? The constant c in the Clayton's copula relates to lower tail dependence. The value $c = 0$ corresponds to independence and larger values of c to comonotonicity.

An interesting question relates to the determination of the optimality degree of the equilibrium points for the game. In the approach we are following the fixed points are obtained from functions that are averages; the consideration of optimal extreme equilibrium points requires a different and less classical approach.

One example of interesting game is to consider a finite number of players in some stock exchange. In the Black-Scholes model, if the moves of the players are given by buying call options the price of these call options will basically depend on a bet on the volatility, that is, on a probability law describing the choice of the volatility in some compact interval; such an example will be treated in future work.

Appendix

In this section we reference some important results that are needed in the main text.

The following result in a Euclidean space with the Lebesgue measure was used in the proof of Theorem 5.5; we state and prove it for the reader's commodity.

Lemma 6.1. *Suppose that, for $f_{\boldsymbol{\theta}} > 0$ a locally integrable function, we have:*

$$\forall \epsilon > 0 \ , \ \exists \boldsymbol{\theta}_\epsilon \ \forall \boldsymbol{\theta} \ \|\boldsymbol{\theta}\| \leq \|\boldsymbol{\theta}_\epsilon\| \ , \ \forall A \ \lambda(A) < +\infty \Rightarrow \int_A f_{\boldsymbol{\theta}} d\lambda \leq \epsilon \ . \tag{6.1}$$

Suppose furthermore that the whole space can be covered by a sequence of sets of finite measure (σ-finite space). Then, almost everywhere,

$$\lim_{\|\boldsymbol{\theta}\| \to 0} f_{\boldsymbol{\theta}} = 0 \ .$$

Proof. By Tchebycheff inequality we have that for A such that $\lambda(A) < +\infty$,

$$\lambda_A \left[f_{\boldsymbol{\theta}} \geq a \right] \leq \frac{1}{a} \int_A f_{\boldsymbol{\theta}} d\lambda \ ,$$

with λ_A denoting the restriction of λ to A. Choose $a = 1/n$ and, in Formula (6.1) $\epsilon = 1/n^3$ and we will now have:

$$\lambda_A \left[f_{\boldsymbol{\theta}} \geq \frac{1}{n} \right] \leq \frac{1}{n^2} \ .$$

By a Borel-Cantelli type result we now have that:

$$\lambda_A \left[\limsup_n \left\{ f_{\boldsymbol{\theta}_n} \geq \frac{1}{n} \right\} \right] = 0 \ ,$$

and so we have that almost everywhere over A that,

$$\exists m_{\boldsymbol{x}} \geq 1 \ \forall n \geq m_{\boldsymbol{x}} \forall \boldsymbol{\theta} \ , \ \|\boldsymbol{\theta}\| \leq \|\boldsymbol{\theta}_n\| \Rightarrow f_{\boldsymbol{\theta}}(\boldsymbol{x}) \leq \frac{1}{n} \ .$$

Since the space is σ-finite, that implies the result announced. □

The following well known results are crucial for the proof of the existence of equilibrium points in Theorem 5.5. A general reference for these results is [1]. We recall that we consider the following notation for $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r) \in \boldsymbol{\Theta}^r$:

$$\boldsymbol{\theta}_{\widehat{i}} := (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{i-1}, \boldsymbol{\theta}_{i+1}, \ldots, \boldsymbol{\theta}_r) \ .$$

In Formula (6.2) we will also use the convention:

$$\left(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{\widehat{i}}\right) \equiv (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r) \ .$$

Theorem 6.2 (Berge's maximum theorem)**.** *Let $\varphi : \boldsymbol{\Theta} \twoheadrightarrow \boldsymbol{\Theta}^{r-1}$ be a continuous correspondence with nonempty compact values. Let $\psi : \boldsymbol{\Theta} \times \boldsymbol{\Theta}^{r-1} \to \mathbb{R}$ be a continuous function. Consider the optimisation given by:*

$$m_{\widehat{i} \to i}(\boldsymbol{\theta}_i) = \sup_{\boldsymbol{\theta}_{\widehat{i}} \in \varphi \boldsymbol{\theta}_i)} \psi\left(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{\widehat{i}}\right) = \sup_{\boldsymbol{\theta}_{\widehat{i}} \in \varphi(\boldsymbol{\theta}_i)} \psi\left(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_r\right) \ , \tag{6.2}$$

Consider also the correspondence $\mu : \boldsymbol{\Theta} \twoheadrightarrow \boldsymbol{\Theta}^{r-1}$ of maximisers of the optimisation problem given by:

$$\mu(\boldsymbol{\theta}_i) := \left\{ \boldsymbol{\theta}_{\widehat{i}} \in \varphi\left(\boldsymbol{\theta}_i\right) : \psi\left(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{\widehat{i}}\right) = m_{\widehat{i} \to i}(\boldsymbol{\theta}_i) \right\} \ .$$

Then the value function $m_{\widehat{i} \to i}$ is continuous and the "argmax" correspondence μ is upper hemicontinuous with compact values.

Theorem 6.3 (Kakutani–Fan–Glicksberg)**.** *Let* $\boldsymbol{\Theta}$ *be a nonempty compact convex subspace of a locally convex Hausdorff space and let the correspondence* $R : \boldsymbol{\Theta}^r \twoheadrightarrow \boldsymbol{\Theta}^r$ *have a closed graph and nonempty convex values. Then the set of fixed points of* R *is compact and nonempty.*

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Notes

[1]Brower's fixed point theorem: every continuous function from a nonempty convex compact subset K of a Euclidean space to K itself has a fixed point.

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