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A STUDY OF FUZZY SEMI I-CONNECTEDNESS

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ABSTRACT
This paper presents the concept of fuzzy semi I-connectedness within fuzzy ideal topological spaces, and explores several properties and descriptions thereof.

1. INTRODUCTION
A specific instance of the relevant fuzzy notion is the basic definitions and theories of scientific investigations, particularly mathematical ones, with regard to ordinary sets. A theory of fuzzy topology is the logical outcome of applying the idea of point set topology to fuzzy sets. The foundational concept of a fuzzy set was introduced by Lotfi A. Zadeh in 1965 (Wikipedia)[1]). Building on this, Chang defined fuzzy topology in 1968 (De Gruyter Brill [2]). Since then, various aspects of general topology have been reinterpreted within the fuzzy framework by numerous researchers. Of particular interest is how local properties of a space—which, in some cases, can characterize the entire space—play a significant role in both classical and fuzzy topology. In classical topology, the introduction of ideal-based notions—most notably by Kuratowski and Vaidyanathaswamy—provided valuable tools for localization analysis (De Gruyter Brill [2]). These ideas have inspired analogous concepts in fuzzy topology, where notions like fuzzy points and the Q-neighborhoods of a fuzzy point serve as foundational structures for further study [3]. In 1997, Sarkar extended the concept of ideals to fuzzy settings and introduced the notion of a local function in fuzzy topology, laying the groundwork for further developments (De Gruyter Brill[2]). Since then, researchers have explored a variety of fuzzy topological concepts through the lens of fuzzy ideals.

In this paper, we extend that trajectory by introducing and investigating the concept of fuzzy semi I-connectedness within fuzzy ideal topological spaces.

2. PRELIMINARIES
Let X be a nonempty set. A family τ of fuzzy sets of X is called a fuzzy topology [1] on X if the null fuzzy set 0 and the whole fuzzy set 1 belongs to τ and τ is closed with respect to any union and finite intersection .If τ is a fuzzy topology on X, then the pair (X, τ) is called a fuzzy topological space and the members of τ are called fuzzy open sets and their complements are called fuzzy closed sets. The closure [1] of a fuzzy set A of X denoted by Cl(A) , is the intersection of all fuzzy closed sets which contains A . The interior [1] of a fuzzy set A of X denoted by Int(A) is the union of all fuzzy sets of X contained in A . A fuzzy set A in fuzzy topological space (X, τ) is said to be quasi-coincident [12] with a fuzzy set B denoted by AqB if there exists a point $x \in X$ such that $A(x) + B(x) > 1$. The negation of this statement is written as (AqB). A fuzzy set V in a fuzzy topological space (X, τ) is called a Q-neighborhood [12] of a fuzzy point x_β if there exists a fuzzy open set U of X such that $X_\beta qU \leq V$.

Lemma 2.1: Let A and B be two fuzzy sets of X. Then $A \leq B$ $(Aq(1-B))$[12].

A nonempty collection of fuzzy sets I of a set X satisfying the conditions (i) if $A \in I$ and $B \leq A$, then $B \in I$ (heredity) , (ii) if $A \in I$ and $B \in I$ then $A \cup B \in I$ (finite additivity) is called a fuzzy ideal [15] on X . The triplex (X, τ, I) denotes a fuzzy ideal topological space with a fuzzy ideal I and fuzzy topology τ . The local function [15] for a fuzzy set A of X with respect to τ and I denoted by

$A^*(\tau, I)$ (briefly A^*) in a fuzzy ideal topological space (X, τ, I) is the union of all fuzzy points X_β such that if U is a Q-neighborhood of x_β and $E \in I$ then for at least one point $y \in X$ for which $U(y) + A(y) - 1 > E(y)$ [15] . The $*$-closure [15] operator of a fuzzy set A denoted by $Cl^*(A)$ in (X, τ, I) defined as $Cl^*(A) = A \cup A^*$. In a fuzzy ideal topological space (X, τ, I) , the collection $\tau^*(I)$ means an extension of fuzzy topological space finer than τ via fuzzy ideal which is constructed by considering the Class $\beta=\{U-E : U \in \tau , E \in I\}$ as a base .

Definition 2.1: A fuzzy set A of a fuzzy ideal topological space (X,τ, I) is called :

(a) Fuzzy I-open if $A \leq Int(A^*)$ [11];

(b) Fuzzy semiopen if $A \leq Cl(Int(A))$ [4];

(c) Fuzzy semi-I-open if $A \leq Cl^*(Int(A))$ [4].

Definition 2.2: Complement of a Fuzzy I-open (resp. Fuzzy semiopen, Fuzzy semi-I-open) set in fuzzy ideal topological space (X,τ, I) is called Fuzzy I-closed (resp. fuzzy semiclosed, fuzzy semi-I-closed) [4].

Definition 2.3: The intersection of all fuzzy semi-I-closed sets containing a fuzzy set A of a fuzzy ideal topological space (X,τ, I) is called semi I-closure of A. It is denoted by sIcl(A) [4].

3. Fuzzy semi I-Separated Sets

Definition 3.1: Two non empty fuzzy sets A and B of a fuzzy ideal topological space (X,τ,I) are said to be fuzzy semi I-separated if (sIcl(A)qB) and (AqsIcl(B)).

Theorem 3.1: Let A and B be fuzzy semi I-separated sets in a fuzzy ideal topological space (X,τ, I) . If A_1 and B_1 are two non empty fuzzy sets such that $A_1 \leq A$ and $B_1 \leq B$, then A_1 and B_1 are fuzzy semi I-separated sets in X.

Proof: Since $A_1 \leq A$ and $B_1 \leq B$, we have sIcl $(A_1) \leq$ sIcl (A) and sIcl $(B_1) \leq$ sIcl (B). Therefore (sIcl(A)qB) $\Rightarrow$ (sIcl(A_1)qB_1) and (AqsIcl(B)) (A_1qsIcl(B_1)).

Theorem 3.2: Let A and B be fuzzy semi I-open in a fuzzy ideal topological space (X,τ, I) . Then A and B are fuzzy semi I-separated if and only if $\rceil$ (AqB).

Proof: Necessity. If AqB then there exists a point $x \in X$ such that $A(x) + B(x) > 1$. This implies that $(sIcl(A)(x)) + B(x) > 1$ and $A(x) + sIcl(B)(x)) > 1$.Hence sIcl(A)qB and AqsIcl(B), which is a contradiction. Hence (AqB).

Sufficiency. Let (AqB) , then by Lemma 2.1. $A \leq 1-B$.Therefore $sIcl(A) \leq sIcl(1-B) = 1-B$ because $1-B$ is fuzzy semi I-closed set in X .Hence by , Lemma 2.1 (sIcl(A)qB). Similarly (AqsIcl(B)).

Theorem 3.3: Let A and B be fuzzy semi I-closed set in a fuzzy ideal topological space (X,τ,I) .Then A and B are fuzzy semi I-separated in X if and only if (AqB).

Proof: The proof of this theorem follows from the Definition 3.1 and Lemma 2.1.

Theorem 3.4: Let A and B be a fuzzy semi I-open set in a fuzzy ideal topological space (X,τ,I). Then the fuzzy sets $C_AB = A \cap (1-B)$ and $C_BA = B \cap (1-A)$ are fuzzy semi I-separated in X.

Proof: Since $C_AB = A \cap (1-B)$, $sIcl(C_AB) \leq sIcl(1-B) = 1-B$ because B is fuzzy semi I-open in X . And so by Lemma 2.1., $sIcl(C_AB)qB$. Thus $(sIcl(C_AB)q(C_BA))$.Similarly $(sIcl(C_BA)q(C_AB))$.Hence C_AB and C_BA are fuzzy semi I-separated sets in X.

Theorem 3.5: Let A and B be fuzzy semi I-closed set in a fuzzy ideal topological space (X,τ,I) .Then the fuzzy sets $C_AB = A \cap (1-B)$ and $C_BA = B \cap (1-A)$ are fuzzy semi I-separated in X.

Proof: Since A and B are fuzzy semi I-closed in X , $A = sIcl(A)$ and $B = sIcl(B)$. Now $C_A(B) \leq (1-B)$ we have $(sIcl(B)qC_A(B))$ and hence $(sIcl(C_BA) q(C_AB))$ .Similarly $(sIcl(C_AB)q(C_BA))$. Hence C_AB and C_BA are fuzzy semi I-separated in X.

Theorem 3.6: Let (X,τ,I) be a fuzzy ideal topological space.Let A and B be two fuzzy semi I-separated in X if and only if there exist fuzzy semi I-open sets U and V such that $A \leq U$, $B \leq V$, (AqV) and (BqU).

Proof: Necessity. Let A and B are fuzzy semi I-separated fuzzy sets in X. Now put $V = 1-sIcl(A)$ and $U = 1-sIcl(B)$.Then U and V are fuzzy semi I-open set in X such that $A \leq U$, $B \leq V$, (AqV) and (BqU).

Sufficiency. Let U and V are fuzzy semi I-open and sets in X such that $A \leq U$, $B \leq V$, (AqV) and (BqU). Now $1-U$ and $1-V$ are fuzzy semi I-closed sets in X, $sIcl(A) \leq (1-V) \leq (1-B)$ and $sIcl(B) \leq (1-U) \leq (1-A)$.Therefore by Lemma 2.1., $(sIcl(A)qB)$ and $(sIcl(B)qA)$. Hence A and B are fuzzy semi I-separated fuzzy sets in X.

4. Fuzzy semi I-Connectedness

Definition 4.1: A fuzzy set E in a fuzzy ideal topological space (X,τ,I) is said to be fuzzy semi I-connected if it cannot be expressed as the union of two fuzzy semi I-separated sets.

Theorem 4.1: Let A and B be fuzzy semi I-separated sets in a fuzzy ideal topological space (X,τ,I) and E be a fuzzy semi I-connected set in X such that $E \leq A \cup B$.Then exactly one of the following conditions holds:

(a) $E \leq A$ and $E \cap B = 0$.

(b) $E \leq B$ and $E \cap A = 0$.

Proof: We first note that when $E \cap B = 0$ then $E \leq A$, since $E \leq A \cup B$,Similarly , when $E \cap A = 0$ we have $E \leq B$ .Now since $E \leq A \cup B$ both $E \cap A = 0$ and $E \cap B = 0$ cannot hold simultaneously. Again if $E \cap B \neq 0$ and $E \cap A \neq 0$. Then $E \cap A$ and $E \cap B$ are fuzzy semi I-separated sets in X such that $E = (E \cap A) \cup (E \cap B)$ contradicting the fuzzy semi I-connectedness of E. Hence exactly one of the conditions (a) and (b) must holds.

Theorem 4.2: Let E , F be two fuzzy sets of a fuzzy ideal topological space (X,τ,I). If E is fuzzy semi I-connected and $E \leq F \leq sIcl(E)$. Then E is fuzzy semi I-connected.

Proof: If $E = 0$ then the result is true . Let $E \neq 0$ and suppose F is not fuzzy semi I-connected. Then there exist two fuzzy semi I-separated sets A and B in X such that $F = A \cup B$.Since E is fuzzy semi I -connected and $E \leq F = E \cup F$, so by Theorem 4.1 ; $E \leq A$ and $E \cap B = 0$ or $E \leq B$ and $E \cap A = 0$. Let $E \leq A$ and $E \cap$

B = 0. Then B = B ∩ F ≤ B ∩ sIcl(E) ≤ E ∩ sIcl(A) ≤ B ∩ (1−B) ≤ B. It follows that B = B ∩ (1−B) and since B ≠ 0 , B(x) = 1/2 for all x∈X. Thus B_0 = X where B_0 denotes the support of B. Now E ∩ B = 0 implies E_0 ∩ B_0 = ∅ ⇒ E_0 = ∅. Hence E = 0 which is contradiction. Similarly if E ≤ B and E ∩ A = 0 then we get E = 0 a contradiction. Hence F is fuzzy semi I-connected.

Theorem 4.3: Let { Y_α: α ∈ Λ} be a collection of fuzzy semi I-connected sets in a fuzzy ideal topological space (X,τ,I) .Then Y = ∪{ Y_α : α ∈ Λ} is fuzzy semi I-connected provided there exists β ∈ Λ such that either (i) Y_α and Y_β are not fuzzy semi I-separated for each α ∈ Λ , or (ii) Y_α ∩ Y_β ≠ 0 for each α ∈ Λ.

Proof: Suppose Y is not fuzzy semi I-connected. Then Y = A ∪ B ,where A and B are fuzzy semi I-separated sets in X. For an arbitrary α ∈ Λ , either (a) Y_α ≤ A with Y_α ∩ B = 0 or (b) Y_α ≤ B with Y_α ∩ A = 0 ,by Theorem 4.1. Similarly , (c) Y_β = A with Y_β ∩ B = 0 or (d) Y_β ≤ B with Y_β ∩ A = 0 .Without loss of generality we can assume that each { Y_α : α ∈ Λ} to be non-null , and hence exactly one of the possibilities (a) and (b) , and exactly (c) and (d) will hold.

For case (ii) ,the possibilities (a) and (b) cannot happen and similarly (b) and (c) cannot hold simultaneously. For case (i), if (a) and (b) hold, then Y_α= Y_α ∩ A and Y_β = Y_β ∩ B are fuzzy semi I-separated , A and B are being so. This is a contradiction. Similarly for case (ii) the possibilities (b) and (c) together are to be ruled out.

Thus in any case, either Y_α ≤ A with Y_α ∩ B = 0 or Y_α ≤ B with Y_α ∩ A = 0 (but not both simultaneously) for each α ∈ Λ. Now, Y_α ≤ A and Y_α ≤ B = 0 and thus B = 0, a contradiction. Similarly, Y_α ≤ B and Y_α ≤ A = 0 for all α ∈ Λ implies A = 0 , again a contradiction.

Theorem 4.4: Let Y be a fuzzy set of a fuzzy ideal topological space (X,τ,I) such that there exists at least one point x ∈ X ,with Y(x) > 1/2. Then Y is fuzzy semi I-connected if and only if two fuzzy points of Y are contained in a fuzzy semi I-connected set contained in Y.

Proof: Necessity. Let Y be fuzzy semi I-connected, then the condition is clearly true, irrespective of whether Y(x) > 1/2 at some point x ∈ X.

Sufficiency. Let x ∈ X such that Y(x) > 1/2. For each y ∈ Y_0 −{x} (where Y_0 denotes the support of Y) there exists a fuzzy semi I-connected set A_y ≤ Y such that $X_{Y(x)}$, $Y_{Y(y)}$ ∈ A_y. Clearly ∪{ A_y : y ∈ Y_0−{x}}= Y and ∪{ A_y : y ∈ Y_0 −{x}} is fuzzy semi I-connected by Theorem 4.2.

Corollary 4.1: A fuzzy ideal topological space (X, τ ,I) is fuzzy semi I-connected if and only if every pair of fuzzy points is contained in a fuzzy semi I-connected set.

Definition 4.1: A mapping f from a fuzzy ideal topological space (X, τ , I) to a fuzzy topological space (Y,σ) is said to be fuzzy semi I-continuous if the inverse image of every fuzzy open set in X is fuzzy semi I-open in X.

Theorem 4.5: Let f:(X, τ ,I)→(Y,σ) be a fuzzy semi I-continuous surjection. If E be fuzzy semi I-connected set in X . Then f(E) is fuzzy semi I-connected set of Y.

Proof: Suppose that f(E) is not fuzzy semi I-connected set of Y. Then there exist fuzzy semi I-separated sets A and B in Y such that f(E) = A ∪ B. Therefore there exist fuzzy open sets U and V such that A ≤ U, B ≤ V, (AqU) and (Bq V). Now $E = f^{-1}(f(E)) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and it can be easily verified that $f^{-1}(A) \leq f^{-1}(U)$; $f^{-1}B) \leq$ $(f^{-1}(A)\ q\ f^{-1}(U))$ and $(f^{-1}(V)\ q\ f^{-1}(B))$. Since f is fuzzy semi I-continuous $f^{-1}(U)$ and $f^{-1}(V)$ are fuzzy semi I-open set in X. Thus by Theorem 3.6, $f^{-1}(A)$ and $f^{-1}(B)$ are fuzzy semi I-separated in X. Hence E is not fuzzy semi I-connected, which is a contradiction.

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