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Inventory Model for Deteriorating Stocks Items with Time-Dependent Demand, Backlogged Partially When Late Payment is Allowed

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Abstract

In this article, we investigate a deterministic inventory model for deteriorating items with time dependent demand and shortages, delay in payments is permissible. Besides shortages are partially backlogged on the assumption that backlogging rate varies inversely as the waiting time for the next replenishment. Replenishment cycle length, the time at which shortage begins and the average total inventory cost are taken as decision variables. Numerical examples are given for illustrating the model.

1. INTRODUCTION

Almost every business must stock goods to ensure smooth and efficient running of its operation. The time factor may decide the demand rate. i.e. the demand rate may go up or down with time. The control and maintenance of inventories of deteriorating items with shortages have received much attention of several researchers in the recent years because most physical goods deteriorate over time. Some of the items either damaged or vaporized or decayed or affected by some other factors, are not in a perfect condition to satisfy the demand. Chung-Yuan Dye [5] developed a deterministic inventory model for deteriorating items with stock dependent demand and shortages and considered the condition of permissible delay in payments and partial backlogging.

Levin et al. [13] noted that inventory has a motivational effect on demand. Silver and Peterson [21] also noted that sales at the retail level tend to be proportional to the amount of inventory displayed. For these reasons, more researchers developed the EOQ models that focussed on stock dependent demand rate patterns. Gupta and Vrat [10] assumed that the demand rate was a function of initial stock level.

Key words: *Inventory control, Deteriorating Items, Time-Dependent Demand, Permissible Delay in payments.*

Mandal and Phaujdar [14] developed a production inventory model for deteriorating items with uniform rate of production and linearly stock dependent demand. Backer and Urban [2], Datta and pal [6] and Goh [8] concentrated on the polynomial function dependent on the instantaneous stock level. Some of the recent work in this area may refer to Padmanabhan and Vrat [16], Ray and Chaudhuri [17], Sarker et al. [18], Giri and Chaudhuri [7] and Mandal and Maiti [15].

Under the classical EOQ model, payment for the quantity ordered is made when the quantity is received. However this may not be in many cases. Often the suppliers offer a fixed credit period to demand. Goyal [9] first studied an EOQ model under conditions of permissible delay in payments. Chung [4] presented the DCF (Discounted Cash Flow) approach for the analysis of the optimal inventory policy in the presence of trade credit. Later Shinn et al. [20] extended Goyal's [9] model to consider the quantity discounts for freight cost. Nowadays to consider more practical features of the real inventory systems, Aggarwal and Jaggi [1] and Hwang and shinn [11] extended Goyal's [9] model to consider the deterministic inventory model with a constant deterioration rate. Shah and Shah [19] developed a probabilistic inventory model when delay in payments is permissible. They developed an EOQ model for deteriorating items in which time and deterioration of units are treated as continuous variables and demand is a random variable. After Aggarwal and Jaggi [1], Jamal et al. [12] extended their model to allow for shortages and make it more applicable in real world.

Contrary to that many researches assumed that the shortages are either completely backlogged or completely lost. Others have assumed that demand during stock out period is partially met. (Wee [22] and Yan and Cheng [23]). In 1995, Padmanabhan and Vrat [16] considered an EOQ model for perishable items developed with a stock dependent demand. Particularly they assumed that the demand during stock out period depends linearly on the inventory level. In some inventory systems, for fashionable goods the length of the waiting time for the next replenishment becomes main factor for determining whether the backlogging rate will be accepted or not. If the waiting time is longer, the backlogging rate is smaller. So the backlogging rate is depending on the waiting time for the next replenishment. A model in the field of deteriorating items with time varying demand and shortages has recently been developed by Chang and Dye [3] in which the backlogging rate is inversely proportional to the waiting time for the next replenishment.

In this paper, we develop a deteriorating inventory model with time dependent demand, including the conditions of allowable shortages and permissible delay in payments. Following Chang and Dye [3] the backlogging rate during the stock out period is assumed to be inversely proportional to the waiting time for the next replenishment. This article is presented as follows. In section 2, the notations and assumptions are given. In section 3, we present the mathematical model. In section 4, numerical examples are given to illustrate the model and we analyze the optimal solution with respect to some parameters. Finally we conclude this paper.

2. NOTATIONS AND ASSUMPTIONS

The mathematical model of the inventory problem considered here in is developed on the basis of the following notations and assumptions.

2.1 Notations

k	=	Ordering cost of inventory, \$ per order.
I(t)	=	The inventory level at time t.
θ	=	Deterioration rate, a fraction of the on-hand inventory.
P	=	Purchase cost, \$ per unit
h	=	Holding cost excluding interest charges, \$ per unit/ year.
s	=	Shortage cost, \$ per unit/year
π	=	Opportunity cost due to lost sales, \$ per unit
I_e	=	Interest which can be earned, \$/year
I_r	=	Interest charges which invested in inventory, \$/year $I_r \geq I_e$.
M	=	Permissible delay in settling the accounts and $0<M<1$.
T	=	The length of replenishment cycle.
T_1	=	Time at which shortage starts, $0 \leq T_1 \leq T$.
TVC (T_1, T)	=	The average total inventory cost per unit time.
$TVC_1(T_1, T)$	=	The average total inventory cost per unit time for $T_1 \geq M$ in case 1.
$TVC_2\,(T_1, T)$	=	The average total inventory cost per unit time for $T_1 < M$ in case 2.

2.2. Assumptions

1. The inventory system involves only one item.
2. The replenishment rate is infinite.
3. There is no replacement or repair of deteriorated units.
4. The demand rate function R(t) is deterministic and is a known function of time and it is given by

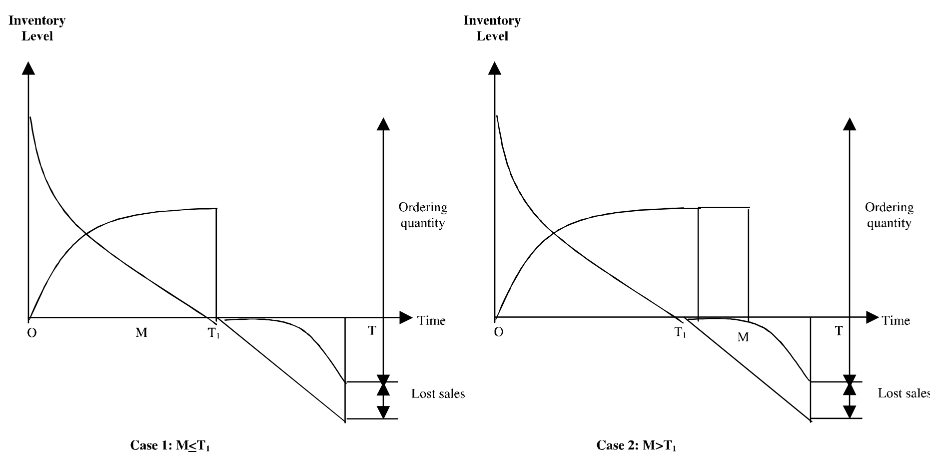
$$R(t) = \begin{cases} \alpha + \beta t & ;0 \le t < T_1 \\ \alpha & ;T_1 \le t < T \end{cases}$$

where $\alpha > 0$ & $0 < \beta < 1$.

5. Shortages are allowed and the backlogged rate is defined to be $\dfrac{1}{1+\delta(T-t)}$ when inventory is negative. The backlogging parameter δ is a positive constant and $T_1 \le t < T$.

3. MODEL FORMULATION

As in figure 1, the reduction of the inventory is due to the combined effects of the demand as well as the deterioration in the interval $[0,T_1)$ and the demand backlogged in the interval $[T_1,T)$. Hence the change in the inventory level I(t) with respect to time can be written as follows.



Graphical Representation of Inventory system

$$\frac{dI(t)}{dt} = \begin{cases} -\alpha - \beta t - \theta I(t) & ;0 \le t < T_1 \\ -\dfrac{\alpha}{1+\delta(T-t)} & ;T_1 \le t < T \end{cases} \qquad \ldots(1)$$

with boundary condition $I(T_1) = 0$.

Consider $0 \le t < T_1$

Solving the differential equation $\frac{dI(t)}{dt} = -\alpha - \beta t - \theta I(t)$ we get

$$I(t)e^{\theta t} = \int (-\alpha - \beta t)e^{\theta t}dt + c$$

where c is the constant of integration. Using the boundary condition $I(T_1) = 0$ we can find C and I(t) is given as

$$I(t) = \left[\frac{\alpha\theta - \beta}{\theta^2}\right]\left[e^{\theta(T_1-t)} - 1\right] + \left(\frac{\beta}{\theta}\right)\left[T_1 e^{\theta(T_1-t)} - t\right]$$

Now consider $T_1 \leq t < T$

$$\frac{dI(t)}{dt} = \frac{-\alpha}{1+\delta(T-t)}$$

By solving the above equation we get

$$I(t) = -\alpha \log\left[1+\delta(T-t)\right] \left(\frac{-1}{\delta}\right) + c$$

where c is the constant of integration.

Using the given boundary condition $I(T_1)=0$ we give I(t) as follows.

$$I(t) = -\frac{\alpha}{\delta}\left\{\log\left[1+\delta(T-T_1)\right] - \log\left[1+\delta(T-t)\right]\right\}$$

$$I(t) = \left(\frac{\alpha\theta - \beta}{\theta^2}\right)\left[e^{\theta(T_1-t)} - 1\right] + \frac{\beta}{\theta}\left[T_1 e^{\theta(T_1-t)} - t\right] \quad when \quad 0 \leq t < T_1$$

$$I(t) = -\left[\frac{\alpha}{\delta}\right]\left\{\log\left[1+\delta(T-T_1)\right] - \log\left[1+\delta(T-t)\right]\right\} \; when \;\; T_1 \leq t < T$$

…(2)

From I(t) we can find the holding cost in the interval $[0,T_1)$ denoted by HC.

$$HC = h\int_0^{T_1} I(t)dt$$

$$= h\left\{\int_0^{T_1}\left(\frac{\alpha\theta-\beta}{\theta^2}\right)\left[e^{\theta(T_1-t)}-1\right]dt+\int_0^{T_1}\frac{\beta}{\theta}\left[T_1e^{\theta(T_1-t)}-t\right]dt\right\}$$

$$= h\left\{\left(\frac{\alpha\theta-\beta}{\theta^2}\right)\left[\frac{e^{\theta(T_1-t)}}{-\theta}-t\right]_0^{T_1}+\frac{\beta}{\theta}\left[\frac{e^{\theta(T_1-t)}}{-\theta}T_1-\frac{t^2}{2}\right]_0^{T_1}\right\}$$

$$HC = h\left\{\left[\frac{e^{\theta T_1}-1}{\theta^3}\right](\alpha\theta-\beta)-\frac{T_1}{\theta^2}\left[\alpha\theta+\beta\left[\frac{\theta T_1}{2}-e^{\theta T_1}\right]\right]\right\} \qquad ...(3)$$

Deterioration cost in the interval $[0,T_{1-})$ denoted by DC is given as

$$DC = P\theta\int_0^{T_1} I(t)dt$$

$$= P\theta\left\{\left(\frac{\alpha\theta-\beta}{\theta^2}\right)\left[e^{\theta(T_1-t)}-1\right]dt+\int_0^{T_1}\frac{\beta}{\theta}\left[T_1e^{\theta(T_1-t)}-t\right]dt\right\}$$

$$DC = P\theta\left\{\left(\frac{e^{\theta T_1}-1}{\theta^3}\right)(\alpha\theta-\beta)-\frac{T_1}{\theta^2}\left[\alpha\theta+\beta\left(\frac{\theta T_1}{2}-e^{\theta T_1}\right)\right]\right\} \qquad ...(4)$$

During the stock out period we have to consider two costs. First we have to derive the shortage cost for the backlogged items and then we have to obtain the opportunity cost due to lost sales. The shortage cost over the period $[T_1, T)$ denoted by SC is given by

$$SC = s\int_{T_1}^{T} I(t)dt = s\int_{T_1}^{T}\left(\frac{-\alpha}{\delta}\right)\left\{\log\left[1+\delta(T-T_1)\right]-\log\left[1+\delta(T-t)\right]\right\}dt$$

On simplification, we get

$$SC = -\frac{s\alpha}{\delta}\left\{(T-T_1)\log\left[1+\delta(T-T_1)\right]-(T-T_1)\log\left[1+\delta(T-T_1)\right]+(T-T_1)-\frac{1}{\delta}\log\left[1+\delta(T-T_1)\right]\right\}$$

$$SC = -\frac{s\alpha}{\delta^2}\left\{\delta(T-T_1)-\log\left[1+\delta(T-T_1)\right]\right\}$$

The cost cannot be negative. So the shortage cost is given by

$$SC = \frac{s\alpha}{\delta^2}\left\{\delta\left(T - T_1\right) - \log\ \left[1 + \delta\left(T - T_1\right)\right]\right\} \qquad \ldots(5)$$

Now, the opportunity cost due to lost sales during the replenishment cycle denoted by OC is given as

$$OC = \pi\int_{T_1}^{T}\alpha\left[1 - \frac{1}{1+\delta\ \left(T-t\right)}\right]dt$$

$$= \pi\alpha\left[t - \frac{\log\left[1+\delta(T-t)\right]}{-\delta}\right]_{T_1}^{T}$$

$$= \pi\alpha\left\{T - T_1 - \frac{1}{\delta}\log\left[1+\delta\ \left(T-T_1\right)\right]\right\}$$

$$OC = \frac{\pi\alpha}{\delta}\left\{\delta\left(T-T_1\right) - \log\left[1+\delta\left(T-T_1\right)\right]\right\} \qquad \ldots(6)$$

Now, we have to consider M which is the permissible delay in settling the accounts offered by the supplier. There are two possibilities Case 1: $M \leq T_1$ or Case 2: $M > T_1$.

We shall discuss first case 1 and then case 2.

3.1 case 1: $M \leq T_1$

Since in this case the length of the period with positive inventory of the items is larger than the credit period the buyer can earn the interest with an annual rate I_e in $[0, T_1)$. The interest earned denoted by IE_1 is

$$IE_1 = PI_e\int_0^{T_1}\left(T_1 - t\right)\ \left(\alpha + \beta\ t\right)dt$$

$$= PI_e\int_0^{T_1}\ \left[\alpha T_1 - \alpha t + \beta T_1 t - \beta t^2\right]\ dt$$

$$= PI_e\left[\alpha\, T_1 t - \frac{\alpha\, t^2}{2} + \beta\, T_1 \frac{t^2}{2} - \frac{\beta\, t^3}{3}\right]_0^{T_1}$$

Now,

$$IE_1 = \frac{pI_e T_1^2}{6}\left(3\alpha + \beta T_1\right) \qquad \text{...(7)}$$

After the fixed credit period, the buyer has to pay the interest on the product still in stock with an annual rate I_r. Hence the interest payable denoted by IP is given as

$$IP = PI_r \int_M^{T_1} I(t)dt$$

$$= PI_r \int_M^{T_1} \left(\left(\frac{\alpha\theta-\beta}{\theta^2}\right)\left[e^{\theta\,(T-t)} - 1\right] + \frac{\beta}{\theta}\left[T_1 e^{\theta\,(T_1-t)} - t\right]\right)dt$$

$$= PI_r\left\{\left(\frac{\alpha\theta-\beta}{\theta^2}\right)\left[-\ \frac{1}{\theta}\ - T_1 - \frac{e^{\theta\,(T_1-M)}}{-\theta} + M\right] + \frac{\beta\ T_1}{\theta}\left[\frac{T_1}{-\theta} - \frac{T_1^2}{2} - T_1\frac{e^{\theta\,(T_1-M)}}{-\theta} + \frac{M^{\ 2}}{2}\right]\right\}$$

By simplifying we have

$$IP = PI_r\left\{\left(\frac{\alpha\theta-\beta}{\theta^{\,3}}\right)\left[\left[e^{\theta\,(T_1-M)} + M\theta - (\theta T_1 + 1)\right] + \frac{\beta}{\theta^{\,2}}\left[T_1\left(e^{\theta\,(T_1-M)} - 1\right)\right] + \frac{\theta}{2}\left(M^{\ 2} - T_1^{\ 2}\right)\right]\right\}$$

$$\text{…(8)}$$

The total average cost in this case is

$$TVC_1 = \frac{K + HC + DC + SC + OC + IP - IE_1}{T}$$

$$= \frac{1}{T}\left\{\begin{array}{l} K + h\left[\left(\frac{e^{\theta T_1}-1}{\theta^3}\right)\ (\alpha\theta-\beta) - \frac{T_1}{\theta^2}\left[\alpha\theta + \beta\left(\frac{\theta T_1}{2} - e^{\theta T_1}\right)\right]\ \right] + P\theta\left[\left(\frac{e^{\theta T_1}-1}{\theta^3}\right)\ (\alpha\theta-\beta) - \frac{T_1}{\theta^2}\left[\alpha\theta + \beta T_1\left(\frac{\theta}{2} - e^{\theta\, T_1}\right)\ \right]\ \right] + \\ \frac{S\alpha}{\delta^2}\left[\delta\ (T-T_1) - \log\left[1 + \delta\ (T-T_1)\ \right]\ \right] + \frac{\pi\alpha}{\delta}\left[\delta(T-T_1) - \log\left[1 + \delta\ (T-T_1)\ \right]\ \right] + PI_r\left(\frac{\alpha\theta-\beta}{\theta^3}\right)\ \left[e^{\theta(T_1-M)} + M\theta - (\theta T_1 + 1)\right] + \\ \frac{\beta}{\theta^2}\left\{T_1\left[e^{\theta(T_1-M)} - 1\right] + \frac{\theta}{2}\left(M^2 - T_1^2\right)\right\} - \frac{PI_e T_1^2}{6}(3\alpha + \beta T_1) \end{array}\right\}$$

$$TVC_1=\frac{1}{T}\left\{\begin{array}{l}K+(h+P\theta)\left[\left(\frac{e^{\theta T_1}-1}{\theta^3}\right)(\alpha\theta-\beta)-\frac{T_1}{\theta^2}\left[\alpha\theta+\beta\left(\frac{\theta T_1}{2}-e^{\theta T_1}\right)\right]\right]+\\ \frac{\alpha(s+\pi\delta)}{\delta^2}\left[\delta(T-T_1)-\log\ \left[1+\delta\ (T-T_1)\right]\ \right]+PI_r\left(\frac{\alpha\theta-\beta}{\theta^3}\right)\left[e^{\theta(T_1-M)}+M\theta-(\theta T_1+1)\right]+\\ \frac{\beta}{\theta^2}\left[T_1\left(e^{\theta(T_1-M)}-1\right)+\frac{\theta}{2}\left(M^2-T_1^2\right)\right]-\frac{PI_eT_1^2}{6}(3\alpha+\beta T_1)\end{array}\right\}$$

…(9)

The total average cost per unit time can be minimized and the optimal values of T1 and T (Say T1* and T*) can be found by solving the following equations simultaneously.

$$\frac{\partial TVC_1(T_1,T)}{\partial T_1}=0 \text{ and } \frac{\partial TVC_1(T_1,T)}{\partial T}=0 \qquad ...(10)$$

provided they satisfy the sufficient conditions

$$\left[\frac{\partial^2 TVC_1(T_1,T)}{\partial T^2{}_1}\right]_{at(T_1^*,T^*)}>0,\left[\frac{\partial^2 TVC_1(T_1,T)}{\partial T^2}\right]_{at(T_1^*,T^*)}>0$$

and $\left[\left(\frac{\partial^2 TVC_1(T_1,T)}{\partial T^2{}_1}\right)\ \left(\frac{\partial^2 TVC_1(T_1,T)}{\partial T^2}\right)-\left[\frac{\partial^2 TVC_1(T_1,T)}{\partial T_1\partial T}\right]^2\right]_{at\ \ (T_1^*,T^*)}>0$

$$\frac{\partial TVC_1(T_1,T)}{\partial T_1}=0\Rightarrow$$

$$\frac{1}{T}\left\{\begin{array}{l}(h+P\theta)\left[\frac{e^{\theta T_1}}{\theta^2}(\alpha\theta-\beta)-\frac{1}{\theta^2}\left[\alpha\theta+\beta\left(\frac{\theta T_1}{2}-e^{\theta T_1}\right)\right]-\frac{T_1}{\theta^2}\beta\left(\frac{\theta}{2}-e^{\theta T_1}\theta\right)\right]-\frac{\alpha(s+\pi\delta)(T-T_1)}{1+\delta(T-T_1)}+\\ PI_r\left[\left(\frac{\alpha\theta-\beta}{\theta^3}\right)\left[e^{\theta(T_1-M)}\theta-\theta\right]+\frac{\beta}{\theta^2}\left[e^{\theta(T_1-M)}-1+T_1e^{\theta(T_1-M)}\theta-\theta T_1\right]\right]-\frac{PI_eT_1}{3}(3\alpha+\beta T_1)\\ -\frac{PI_eT_1^2\beta}{6}\end{array}\right\}=0$$

i.e. $\frac{1}{T}\left\{\begin{array}{l}\left(\frac{h+P\theta}{\theta}\right)(\alpha+\beta T_1)\left(e^{\theta T_1}-1\right)-\left(\frac{\alpha(s+\pi\delta)\ (T-T_1)}{1+\delta(T-T_1)}\right)+\frac{PI_r}{\theta}(\alpha+\beta T_1)\left[e^{\theta(T_1-M)}-1\right]\\ -\frac{PI_eT_1}{2}(2\alpha+\beta T_1)\end{array}\right\}=0$

…(11)

$$\frac{\partial\, TVC_1(T_1,T)}{\partial T}=0\Rightarrow$$

$$\frac{1}{T}\left[\frac{\alpha(s+\pi\delta)(T-T_1)}{1+\delta(T-T_1)}\right]-\frac{1}{T^2}\left\{\begin{array}{l} K+\left[(h+P\theta)\ \right]\left[\left(\frac{e^{\theta T_1}-1}{\theta^3}\right)(\alpha\theta-\beta)-\frac{T_1}{\theta^2}\left[\alpha\theta+\beta\left(\frac{\theta\, T_1}{2}-e^{\theta\, T_1}\right)\right]\right] \\ +\frac{\alpha(s+\pi\delta)}{\delta^2}\left\{\delta(T-T_1)-\log\left[1+\delta(T-T_1)\right]\right\}+PI_r\left\{\begin{array}{l}\left(\frac{\alpha\theta-\beta}{\theta^3}\right)\left[e^{\theta(T_1-M)}+M\theta-(\theta\, T_1+1)\right] \\ +\frac{\beta}{\theta^2}\left\{T_1\,[e^{\theta(T_1--M)}-1]+\frac{\theta}{2}\left(M^2-T_1^2\right)\right\}\end{array}\right\} \\ -\frac{PI_eT_1^2}{6}(3\alpha+\beta T_1)\end{array}\right\}=0$$

…(12)

To obtain the optimal values of T_1 and T which minimizes TVC_1 ($T_{1,}T$) we use the following algorithm.

3.1.1: Algorithm 1

Step 1. perform (i)-(iv)

(i) Start with $T_{1,(1)}$ = M

(ii) Substituting $T_{1,(1)}$ into equation (11) and find $T_{(1)}$

(iii) Using $T_{(1)}$ determine $T_{1,(2)}$ from equation (12)

(iv) Repeat (ii) and (iii) until no change occurs in the values of T_1 & T.

Step 2. compare T_1 and M.

(i) If $M \leq T_1$, T_1 is feasible go to step 3

(ii) If $M > T_1$, T_1 is not feasible. Set T_1=M and find the corresponding values of T from equation (12) then go to step 3.

Step 3. Calculate the corresponding TVC_1 (T_1^*,T^*).

3.2 Case 2: T_1<M

Since T_1<M the buyer earns the interest during the period [0, M) and pays no interest. The interest earned in this case denoted by IE_2 is given by

$$IE_2=PI_e\left\{\int_0^{T_1}(T_1-t)(\alpha+\beta\ t)\,dt+(M-T_1)\int_0^{T_1}(\alpha+\beta\ t)\,dt\right\}$$

$$=\frac{PI_eT_1^2}{6}\left(3\alpha+\beta T_1\right)+P\quad I_e\left(M-T_1\right)\left[\alpha t+\frac{\beta t^2}{2}\right]_0^{T_1}$$

$$IE_2=\frac{PI_eT_1^2}{6}\left(3\alpha+\beta\quad T_1\right)+\frac{PI_eT_1\left(M-T_1\right)}{2}\left(2\alpha+\beta\quad T_1\right) \qquad \ldots(13)$$

The total average cost in this case is

$$TVC_2=\frac{K+HC+DC+SC+OC-IE_2}{T}$$

$$TVC_2=\frac{1}{T}\left\{\begin{array}{l} K+\left(h+P\theta\right)\left[\left(\frac{e^{\theta T_1}-1}{\theta^3}\right)\left(\alpha\theta-\beta\right)-\frac{T_1}{\theta^2}\left[\alpha\theta+\beta\left(\frac{\theta T_1}{2}-e^{\theta T_1}\right)\right]\right]+ \\ \frac{\alpha\left(s+\pi\delta\right)}{\delta^2}\left[\delta\left(T-T_1\right)-\log\left[1+\delta\left(T-T_1\right)\right]\right]-\frac{PI_eT_1^2}{6}(3\alpha+\beta T_1) \\ -\frac{PI_eT_1\left(M-T_1\right)}{2}\left(2\alpha+\beta T_1\right) \end{array}\right\}$$

…(14)

For minimizing total average inventory cost per unit time, we have to find the optimal values of T_1 & T which are the solutions of the following equations.

$$\frac{\partial TVC_2\left(T_1,T\right)}{\partial T_1}=0 \quad \text{and} \quad \frac{\partial TVC_2\left(T_1,T\right)}{\partial T}=0 \qquad ...(15)$$

provided they satisfy the sufficient conditions

$$\left[\frac{\partial^2 TVC_2\left(T_1,T\right)}{\partial T_1^2}\right]_{\left(T_1^*,T^*\right)}>0\left[\frac{\partial^2 TVC_2\left(T_1,T\right)}{\partial T^2}\right]_{\left(T_1^*,T^*\right)}>0$$

$$\left[\left(\frac{\partial^2 TVC_2\left(T_1,T\right)}{\partial T_1^2}\right)\left(\frac{\partial^2 TVC_2\left(T_1,T\right)}{\partial T^2}\right)-\left[\frac{\partial^2 TVC_2\left(T_1,T\right)}{\partial T_1\partial T}\right]^2\right]_{\left(T_1^*,T^*\right)}>0$$

Now

$$\frac{\partial TV\,C_2\left(T_1,T\right)}{\partial T_1}=0 \Rightarrow$$

$$\frac{1}{T}\left\{\left(\frac{h+P\theta}{\theta}\right)(\alpha+\beta T_1)[e^{\theta\,T_1}-1]-\alpha\frac{(s+\pi\delta)(T-T_1)}{1+\delta(T-T_1)}+PI_e(\alpha+\beta T_1)(T_1-M)\right\}=0 \quad \ldots(16)$$

$$\frac{\partial TVC_2\left(T_1,T\right)}{\partial T}=0 \;=>$$

$$-\frac{1}{T^2}\left\{\begin{array}{l} K+\left(h+P\theta\right)\left\{\left[\dfrac{e^{\theta T_1}-1}{\theta^3}\right]\left(\alpha\theta-\beta\right)-\dfrac{T_1}{\theta^2}\left[\alpha\theta+\beta\left(\dfrac{\theta T_1}{2}-e^{\theta T_1}\right)\right]\right\}+ \\ \dfrac{\alpha\left(s+\pi\delta\right)}{\delta^2}\left[\delta(T-T_1)-\log\left[1+\delta(T-T_1)\right]\right]-\dfrac{PI_eT_1^2}{6}(3\alpha+\beta T_1)- \\ \dfrac{PI_eT_1\left(M-T_1\right)}{2}\left(2\alpha+\beta T_1\right) \end{array}\right\}+\frac{1}{T}\left[\frac{\alpha\left(s+\pi\delta\right)(T-T_1)}{1+\delta\,(T-T_1)}\right]=0$$

…(17)

To find the optimal values of T_1 and T we use the following algorithm.

3.2.1: Algorithm 2

Step 1. perform (i)-(iv)

(i) start with $T_{1,(1)}$=M

(ii) substituting $T_{1,(1)}$ into equation (16) and find $T_{(1)}$

(iii) using $_{T(1)}$ find $T_{1,(2)}$ form equation (17)

(iv) Repeat (ii) and (iii) until no change occurs in the values of T_1 and T

Step 2. Compare T_1 and M

(i) If T_1 <M, T_1 is feasible, then go to step 3.

(ii) If $T_1 \geq$ M, T_1 is not feasible. Set T_1=M and find the corresponding values of T from equation (17), then go to step3.

Step 3. Calculate the corresponding $TVC_2(T_1^*,T^*)$

Our aim in this problem is to find the optimal values of T_1,T such that TVC (T_1,T) is minimum.

Now, $\mathrm{TVC}(\mathrm{T}_1^*, T^*) = Min\ \left\{\mathrm{TVC}_1(\mathrm{T}_1^*, T^*), \mathrm{TVC}_2(\mathrm{T}_1^*, T^*)\right\}$

4. NUMERICAL EXAMPLES

To illustrate the preceding theory the following examples are presented.

Example-1

Let K=200, α=1000, β=0.3, P=20, h=1.2, s =30, π=15, I_e=0.13, I_r=0.15, θ = 0.08, M is taken as 5/365, 15/365 and 20/365 for δ = 2,3,4. In this example three different values of δ are adopted. For each value of δ three different values of M are tested. The results are given in Table 1. From Table 1, we observe that for each value of M the average total inventory cost increases as δ (the backlogging parameter) increases. Hence to minimize the average inventory cost the retailer should control the backlogging parameter.

Example-2

Let K=400, α=900, β=0.2, P=300, h=1.8, s=35, π=14, I_e=0.12, I_r=0.14, θ=0.07, M varies as 5/365, 20/365, 25/365 and 30/365, δ also varies as 1,2,3 and 5. Here four distinct values of δ are considered. Four different values of M are examined with each δ. The results are exhibited in Table 2. From this table, we get an observation that for each value of M the average total inventory cost increases as δ increases.

Example-3

Let K=100, α=800, β=0.4, P=150, h=1.4, s=40, π=20, I_e=0.14, I_r=0.17, θ=0.09. Here we take only two values of δ as 1 and 2.But we take many values for M as 5/365, 15/365, 25/365, 30/365, 35/365, 40/365, 45/365 and 55/365. Here also we get the same observation as in examples 1 and 2. The results are shown in Table 3.

Example-4

Let K=200, α = 1000, β = 0.3, P=20, h=1.2, s = 30, π = 15, I_e=0.13, I_r=0.15, θ = 0.08, M is taken as 5/365 for δ = 1, 2, 3, 4, 10, 25 & 50. Here M is fixed and it is examined with different values of δ. The results are given in Table 4. From Table 4, we can say that if δ is considered as large (δ = 10, 25 & 50) then the average total inventory cost also increases to a greater extent. For fixed M, the larger value of δ is the smaller proportion of customers who would like to accept backlogging at time t would be. For a fixed M, the average total inventory cost increases with increasing value of δ. Hence in order to minimize average total inventory cost, the retailer should add the fraction of each cycle in which there is no shortage.

Example-5

Let K=200, α = 1000, β = 0.3, P = 20, h = 1.2, s = 30, π = 15, I_e = 0.13, I_r = 0.15, θ = 0.08, M is taken as 10/365, 20/365, 50/365 & 55/365 for δ = 5. Here we fix δ and check it with distinct value of M. The results are given in Table 5. Table 5 informs us that for a fixed δ, the optimal average inventory cost decreases as M increases. It is natural in our daily life.

Table 1

M		δ		
		2	*3*	*4*
5	TVC	3093.83	4502.02	4800.38
	T_1*	0.0633	0.0275	0.0145
	T*	0.0660	0.0290	0.1494
15	TVC	2628.25	3169.74	4204.11
	T_1*	0.0751	0.0461	0.0473
	T*	0.0770	0.0479	0.0476
20	TVC	1059.26	1285.34	1538.05
	T_1*	0.2316	0.1688	0.1348
	T*	0.2458	0.1739	0.1378

Table 2

M		δ			
		1	*2*	*3*	*5*
5	TVC	3286.84	3393.85	6350.39	7050.99
	T_1*	0.1341	0.1467	0.0028	0.0847
	T*	0.2068	0.2131	0.0035	0.1392
20	TVC	2541.70	2562.45	2587.59	4627.65
	T_1*	0.1407	0.1669	0.1532	0.1093
	T*	0.1783	0.2110	0.1836	0.1634
25	TVC	2165.39	2220.02	2295.94	4025.56
	T_1*	0.1787	0.1765	0.2093	0.1043
	T*	0.2282	0.2150	0.2571	0.1468
30	TVC	1964.87	1992.38	2190.83	3500.92
	T_1*	0.1713	0.1749	0.2513	0.0705
	T*	0.2032	0.2020	0.3106	0.0806

Table 3

M		δ	
		1	*2*
5	TVC	1364.83	2820.96
	T_1*	0.0844	0.0207
	T*	0.1066	0.0265
15	TVC	996.53	1093.20
	T_1*	0.1063	0.1580
	T*	0.1238	0.1850
25	TVC	718.34	726.59
	T_1*	0.1462	0.1254
	T*	0.1650	0.1343
30	TVC	601.00	639.41
	T_1*	0.1572	0.1226
	T*	0.1737	0.1263
35	TVC	493.00	760.98
	T_1*	0.1459	0.0932
	T*	0.1527	0.1105
40	TVC	411.50	606.75
	T_1*	0.1122	0.1858
	T*	0.1419	0.1965
45	TVC	321.49	301.14
	T_1*	0.16165	0.1777
	T*	0.16185	0.1818
55	TVC	144.25	187.20
	T_1*	0.1979	0.2394
	T*	0.1983	0.2503

Table 4

M		$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 10$	$\delta = 25$	$\delta = 50$
5	TVC	1410.85	3093.83	4502.02	4800.38	6046.31	7659.41	8699.09
	T_1*	0.1567	0.0633	0.0275	0.0145	0.0221	0.0240	0.0199
	T*	0.1672	0.0660	0.0290	0.1494	0.0296	0.0265	0.0209

Table 5

M		$\delta = 5$
10	TVC	1282.78
	T_1*	0.1832
	T*	0.1882
	T_1*/ T*	0.9734
20	TVC	1181.88
	T_1*	0.1935
	T*	0.1980
	T_1*/ T*	0.9772
50	TVC	815.11
	T_1*	0.3107
	T*	0.3165
	T_1*/ T*	0.9817
55	TVC	779.46
	T_1*	0.3342
	T*	0.3403
	T_1*/ T*	0.9821

4.1. Managerial Implications

Based on the numerical examples considered above we now study the effects of change in M and δ on the optimal values of T_1,T and TVC(T_1,T).

1. In example 1,2 and 3 distinct values of δ are considered and each δ is tested with distinct values of M. From the computed results in Table 1,2,3 it is implied that the retailer should restrict the backlogging parameter with the aim of reducing the average total inventory cost.

2. In example 4, M is fixed as 5 and it is checked with various values of δ. From the results of Table 4,. we can give a suggestion that the acceptance of backlogging decrease when there is an increase in δ. The retailer bring down the average inventory cost if he considers the fraction of each cycle which has no shortage.

3. In example 5, δ is considered to be a constant. We analyse δ with varied values of M. Table 5 portrays a clear picture that supplier's permissible delay makes the retailer very lucrative, ie., the retailer is the most beneficiary if he gets longer permissible delay period from the supplier.

5. CONCLUSION

In this paper we developed an inventory model with time- dependent demand and shortages under the condition of permissible delay in payments. The backlogging rate is considered to be a decreasing function of the waiting time for the next replenishment. This assessment is more realistic. For the conditions $\beta=0$ and $\delta=0$ the model reduces to the model by Aggarwal S.P. and Jaggi C.K.[1] and moreover they did not allow shortage.

Numerical Examples are presented to illustrate the model. The results of the sensitivity analysis are also consistent. From the tables 1, 2 & 3 we conclude that, for the proportion of customers would like to accept backlogging at time t decreases as δ increases and for fixedd, the increasing value of M will result in a significant decrease in the optimal average inventory cost.

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REFERENCES

[1] Aggarwal S.P. and Jaggi C.K. (1995), Ordering Policies of Deteriorating Items Under Permissible Delay in Payments *Journal of the Operational Research Society,* **46**, 658-662.

[2] Backer, R.C. and Urban, L.A. (1988), Deterministic Inventory System with an Inventory Level-Dependent Demand Rate, *Journal of the Operational Research Society,* **39**, 823-831.

[3] Chang, H.J. and Dye, C.Y. (1999), An EOQ Model for Deteriorating Items with Time Varying Demand and Partial Backlogging, *Journal of the Operational Research Society*, **50**, 1176-1182.

[4] Chung, K.H. (1989), Inventory Control and Trade Credit Revisited, *Journal of the Operational Research Society,* **40**, 495-498.

[5] Chung-Yuan Dye (2002), A Deteriorating Inventory Model with Stock-Dependent Demand and Partial Backlogging under conditions of Permissible Delay in Payments, Opsearch, **39**, 189-201.

[6] Datta T.K. and Pal, A.K. (1990), A Note on an Inventory Model with Inventory-Level-Dependent Demand Rate, *Journal of the Operational Research Society,* **41**, 971-975.

[7] Giri, B.C and Chaudhuri, K.S. (1998), Deterministic Models of Perishable Inventory with Stock-Dependent Rate and Non-linear Holding Cost, *European Journal of Operational Research,* **105**, 467-477.

[8] Goh M. (1994), EOQ Models with General Demand and Holding cost Functions European *Journal of Operational Research,* **73**, 50-54.

[9] Goyal, S.K. (1985), Economic Order Quantity under Conditions of Permissible Delay in Payments, *Journal of the Operational Research Society*, **36**, 335-338.

[10] Gupta, R. and Vrat, P. (1986), Inventory Model for Stock-Dependent Consumption Rate, Opsearch, **23**, (1986) 19-24.

[11] Hwang, H. and Shrinn, S.W. (1997), Retailers Pricing and Lot Sizing Policy for Exponentially Deteriorating Product Under the Condition of Permissible Delay in Payments, Computers and Operations Research, **24**, 539-547.

[12] Jamal, A.M., Sarker, B.R. and Wang S. (1997), An Ordering Policy for Deteriorating Items with Allowable Shortage and Permissible Delay in Payment, *Journal of the Operational Research Society*, **48**, 826-833.

[13] Levin, R.I., McLaughlin, C.P., Lamone, R.P. and Kottas, J.F., (1972), Productions Operations Management Contemporary Policy for Managing Operating Systems, McGraw–Hill, New York, 373.

[14] Mandal B.N and Phaujdar S. (1989), An inventory Model for Deteriorating Items and Stock-Dependent Consumption Rate, *Journal of the Operational Research Society*, **40**, 483-488.

[15] Mandal, M. and Maiti, M. (1999), Inventory of Damageable Items with Variable Replenishment Rate, Stock-Dependent Demand and Some Units in Hand, Applied Mathematical Modeling, **23**, 799-807.

[16] Padmanabhan G. and Vrat, P. (1995), EOQ Models for Perishable Items Under Stock Dependent Selling Rate, *European Journal of the Operational Research*, **86**, 281-292.

[17] Ray, J. and Chaudhuri, K.S. (1997), An EOQ Model with Stock-Dependent Demand, Shortage, Inflation and Time Discounting, *International Journal of Production Economics*, 53, 171-180.

[18] Sarker, B.R., Mukherjee, S. and Balan, C.V. (1997), An order–level Lot Size Inventory Model with Inventory–Level Dependent Demand and Deterioration, *International Journal of Production Economics*, **48**, 227-236.

[19] Shah, N.H. and Shah Y.K. (1998), A Discrete-In-Time Probabilistic Inventory Model for Deteriorating Items Under Conditions of Permissible Delay in Payments, International *Journal of Systems Science*, **29**, 121-126.

[20] Shinn, S.W., Hwang, H.P. and Sung S. (1996), Joint Price and Lot Size Determination Under Conditions of Permissible Delay in Payments and Quantity Discounts For Freight Cost, *European Journal of Operational Research*, **91**, 528-542.

[21] Silver, E.A. and Peterson, R. (1982), Decision Systems for Inventory Management and Production Planning, 2nd edition, Wiley, New York.

[22] Wee, H.M. (1995), Joint Pricing and Replenishment Policy for Deteriorating Inventory with Declining Market, *International Journal of Production Economics*, **40**, 163-171.

[23] Yan, H. and Cheng, T. (1998), Optimal Production Stopping and Restarting Times for an EOQ Model with Deteriorating Items, *Journal of the Operational Research Society,* 49, 1288-1295.