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Application of Fractional Derivative to functions System

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Abstract

Making use of fractional derivative and subordination a subclass of univalent functions with negative coefficients is introduced and some properties are proved e.g. coefficient bounds, Geometric property, inclusion property, extreme points, convex combination and integral representation.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in $\Delta = \{z : |z| < 1, z \in \mathbb{C}\}$ and $\mathcal{A}_0$ denotes the subclass of $\mathcal{A}$ consisting functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \tag{1}$$

which are analytic and univalent in Δ and normalized by

$$f(0) = f'(0) - 1 = 0.$$

Let T be the subclass of $\mathcal{A}_0$ consisting functions of the type

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \tag{2}$$

So we have $T \subset \mathcal{A}_0 \subset \mathcal{A}$.

Now suppose

$T(\alpha, \beta) = \{F(z) : F(z) = (1-\alpha) f(z) + \alpha z f'(z) + \beta z^2 f''(z) + (1-\beta) z^3 f'''(z), f(z) \in T\}$.

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So, if

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$$

$$F(z) = z - \sum_{n=2}^{\infty} A_n^{(\alpha,\beta)} z^n \in T\,(\alpha,\beta) \tag{3}$$

where

$$A_n^{(\alpha,\beta)} = (1 - \alpha + \alpha n + \beta n(n-1) + (1-\beta)\, n\,(n-1)\,(n-2))a_n \tag{4}$$

Univalent functions with negative coefficients were studied by many authors. See [3], [5].

Definition 1: The fractional derivative of order k is defined by

$$D_z^k F(z) = \frac{1}{\Gamma(1-k)} \frac{d}{dz} \int_0^z \frac{F(t)}{(z-t)^k}\, dt, \quad 0 \le k < 1. \tag{5}$$

$F(z)$ is an analytic function in simply-connected domain of the z-plane containing the origin and the multiplicity of $(z-t)^{-k}$ is removed by requiring $\log\,(z-t)$ to be real when $z - t > 0$.

By a simple calculation we obtain

$$\lim_{k\to 0} D_z^k F(z) = F(z), \quad \lim_{k\to 1} D_z^k F(z) = F'(z)$$

and

$$\Gamma(2-k) z^k\, D_z^k F(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-k)}{\Gamma(n+1-k)} A_n^{(\alpha,\beta)} z^n \tag{6}$$

See [6] and [7].

Definition 2: The generalization of the fractional derivative of order k for a function in a simply-connected region of z-plane containing the origin is denoted by $\mathcal{J}_{0,z}^{k,\eta,\gamma} F(z)$ and defined for the cases $0 \le k < 1$ and $n \le k < n+1$, $n \in \{1, 2, 3, \cdots\}$ as follows

$$\begin{cases} \dfrac{1}{\Gamma(1-k)} \dfrac{d}{dz} \{z^{k-\eta} \int_0^z (z-t)^{-k}\ {}_2F_1(-k+\eta, 1-\gamma, 1-k; 1-\dfrac{t}{z}) F(t)dt\} \\ \dfrac{d^n}{dz^n} \mathcal{J}_{0,z}^{k-n,\eta,\gamma} F(z) \end{cases} \tag{7}$$

where ${}_2F_1\,(a,\,b;\,c;\,z)$ is a Gaussian Hypergeometric function and defined by

$$ {}_2F_1\,(a,\,b;\,c;\,z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n!}z^n, \quad |\,z\,|<1 \tag{8}$$

$$\left((a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},\ \ c>b>0\ \ \text{and}\ \ \ c>a+b\right).$$

Also the multiplicity of $(z-t)^{k-1}$ is removed by requiring $\log\,(z-t)$ to be real when $z-t>0$, provided that

$$F(z) = O(|z|^{\theta})\ (z\to 0),\ \theta > \max\,\{0,\,\eta-\gamma\}-1.$$

From the above definition we obtain

$$\mathcal{J}_{0,z}^{k,k,\gamma}F(z) = D_z^k F(z),\ 0\le k<1 \tag{9}$$

where $D_z^k F(z)$ is the fractional derivative of order k introduced in Definition 1.

The fractional derivative of a power function is given as follows:

$$\mathcal{J}_{0,z}^{k,\eta,\gamma}\,z^{\rho} = \frac{\Gamma(\rho+1)\Gamma(\rho-\eta+\gamma+1)}{\Gamma(\rho-\eta+1)\Gamma(\rho-k+\gamma+1)}z^{\rho-\eta},\ 0\le k<1 \tag{10}$$

$$\rho > \max\,\{0,\,\eta-\gamma\}-1.$$

By a simple calculation if we consider $F(z)\in T(\alpha,\,\beta)$ then

$$\frac{\Gamma(2-\eta)\Gamma(2-k+\gamma)}{\Gamma(2-\eta+\gamma)}\,z^{\eta}\,\mathcal{J}_{0,z}^{k,\eta,\gamma}F(z)=z-\sum_{n=2}^{\infty}\phi(n)\,A_n^{(\alpha,\beta)}z^n \tag{11}$$

where

$$\phi(n) = \frac{(1)_n\,(2-\eta+\gamma)_{n-1}}{(2-\eta)_{n-1}\,(2-k+\gamma)_{n-1}}, \quad n\in\{2,3,\ldots\} \tag{12}$$

is a non-increasing function of n, therefore

$$0<\phi(n)<\phi(2) = \frac{2(2-\eta+\gamma)}{(2-\eta)\,(2-k+\gamma)},\ \ n\in\{2,3,\ldots\}$$

See [4], [8].

Application of fractional calculus on univalent functions was carried out by several different authors in [1], [2].

Definition 3: A function $F(z) \in T(\alpha, \beta)$ is said to be in the class $\Omega(\lambda, \xi, \theta, \nu)$ if $F(z)$ satisfies the inequality

$$\left| \frac{H\,\mathcal{J}_{0,z}^{k,\eta,\gamma} F(z) - 1}{2\lambda H\,\mathcal{J}_{0,z}^{k,\eta,\gamma} F(z) - \xi(1+\theta)\lambda} \right| < \nu \tag{13}$$

where

$$\mathrm{H} = \frac{\Gamma(2-\eta)\Gamma(2-k+\gamma)}{\Gamma(2-\eta+\gamma)} z^{\eta-1}$$

Definition 4: Let $X(M, N, q)$ consist of all analytic functions $g(z)$ in Δ for which $g(0) = 1$ and

$$g(z) \prec \frac{1+[N+(M-N)(1-q)|z}{1+Nz}, \tag{14}$$

$-1 \leq M < N \leq 1,\ 0 < N \leq 1,\ 0 \leq q < 1.$

Definition 5 : Let $Y_{M,N}^{q}(k,\eta,\gamma)$ denote the class of all functions $F(z) \in \Omega(\lambda, \xi, \theta, \nu)$ for which

$$\frac{z(z^{\eta}\mathcal{J}_{0,z}^{k,\eta,\gamma} F(z))'}{(z^{\eta}\mathcal{J}_{0,z}^{k,\eta,\gamma} F(z))} \in X(M,N,q).$$

Lemma 1: Let $F(z) \in T(\alpha, \beta)$ then $F(z) \in \Omega(\lambda, \xi, \theta, \nu)$ if and only if

$$\sum_{n=2}^{\infty} \phi(n)\ (1+2\nu\lambda)\ A_n^{(\alpha,\beta)} \leq \lambda\nu(2-\xi(1+\theta)) \tag{16}$$

where $\phi(n)$ and $A_n^{(\alpha,\beta)}$ defined by (12), (4) respectively.

Proof : Let (14) holds true and assume $|z| = 1$. By putting

$$\frac{\Gamma(2-\eta)\Gamma(2-k+\gamma)}{\Gamma(2-\eta+\gamma)z^{1-\eta}} = H$$

we obtain

$$\left|H\,\mathcal{J}_{0,z}^{k,\eta,\gamma} F(z) - 1\right| - \nu\left|2\lambda H\,\mathcal{J}_{0,z}^{k,\eta,\gamma} F(z) - \xi(1+\theta)\lambda\right|$$

$$= \left| -\sum_{n=2}^{\infty} \phi(n) A_n^{(\alpha,\beta)} z^{\eta-1} \right| - \nu \left| 2\lambda - 2\lambda \sum_{n=2}^{\infty} \phi(n) A_n^{(\alpha,\beta)} z^{\eta-1} - \xi(1+\theta)\lambda \right|$$

$$= \sum_{n=2}^{\infty} \phi(n)(1+2\nu\lambda)\ A_n^{\alpha,\beta} - \lambda\nu(2-\xi(1+\theta)) \le 0.$$

Hence by maximum modulus theorem, we conclude that $F(z) \in \Omega\ (\lambda, \xi, \theta, \nu)$.

Conversely, let $F(z)$ defined by (3) be in the class $\Omega(\lambda, \xi, \theta, \nu)$ so the condition (11) yields

$$\left| \frac{H\,\mathcal{J}_{0,z}^{k,\eta,\gamma} F(z) - 1}{2\lambda H\,\mathcal{J}_{0,z}^{k,\eta,\gamma} F(z) - \xi(1+\theta)\lambda} \right| = \left| \frac{\sum_{n=2}^{\infty} \phi(n)\, A_n^{(\alpha,\beta)} z^{n-1}}{2\lambda - \sum_{n=2}^{\infty} 2\lambda\phi(n)\, A_n^{(\alpha,\beta)} z^{n-1} - \xi(1+\theta)\lambda} \right| < \nu,\ \ z \in \Delta\,.$$

Since for any z, $|Re(z)| < |z|$ then

$$Re\left\{ \frac{\sum_{n=2}^{\infty} \phi(n)\, A_n^{(\alpha,\beta)} z^{n-1}}{\lambda(2-\xi(1+\theta)) - 2\lambda \sum_{n=2}^{\infty} \phi(n)\, A_n^{(\alpha,\beta)} z^{n-1}} \right\} < \nu$$

by letting $z \to 1$ through real values we conclude the required result.

For functions belonging to the class $Y_{M,N}^{q}(k,\eta,\gamma)$ we prove coefficient bounds and find geometry of this class

2. Coefficient Bounds

Theorem 2.1: $F(z) \in Y_{M,N}^{q}(k,\eta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} \left[1 + \frac{(N+1)(n-1)}{(N-M)(1-q)} \right] \phi(n)\, A_n^{(\alpha,\beta)} < 1. \tag{17}$$

Proof: Let $F(z) \in Y_{M,N}^{q}(k,\eta,\gamma)$ then by definitions 3, 4 and 5 we have

$$\left| \frac{z - \sum_{n=2}^{\infty} \phi(n)\, A_n^{(\alpha,\beta)} z^n - z + \sum_{n=2}^{\infty} n\phi(n)\, A_n^{(\alpha,\beta)} z^n}{Nz\left(1 - \sum_{n=2}^{\infty} n\phi(n)\, A_n^{(\alpha,\beta)} z^{n-1}\right) - [N + (M-N)(1-q)]\left(z - \sum_{n=2}^{\infty} \phi(n)\, A_n^{(\alpha,\beta)} z^n\right)} \right| < 1$$

which implies that

$$Re\left[\frac{\sum_{n=2}^{\infty}(n-1)\phi(n)\ A_n^{(\alpha,\beta)}z^{n-1}}{(N-M)(1-q)-\sum_{n=2}^{\infty}[N(n-1)+(N-M)(1-q)]\ A_n^{(\alpha,\beta)}\phi(n)z^{n-1}}\right]<1$$

Now we choose the values of z on the real axis and letting $z \to 1^-$, so we obtain

$$\frac{\sum_{n=2}^{\infty}(n-1)\phi(n)A_n^{(\alpha,\beta)}z^{n-1}}{(N-M)(1-q)-\sum_{n=2}^{\infty}[N(n-1)+(N-M)(1-q)]A_n^{(\alpha,\beta)}\phi(n)}<1$$

after a simple calculation we obtain the result.

Conversely, assume that the condition (17) holds true. We must show that $F(z)$ $\in Y_{M,N}^{\alpha}(k,\eta,\gamma)$, or equivalently

$$|L| = \left|\frac{z^{\eta}\mathcal{J}_{0,z}^{k,\eta,\gamma}F(z)-z(z^{\eta}\mathcal{J}_{0,z}^{k,\eta,\gamma}F(z))'}{Nz(z^{\eta}\mathcal{J}_{0,z}^{k,\eta,\gamma}F(z))'-[N+(N-M)(1-q)]\,(z^{\eta}\mathcal{J}_{0,z}^{k,\eta,\gamma}F(z))}\right|<1.$$

But we have

$$|L| = \left|\frac{\sum_{n=2}^{\infty}(n-1)\phi(n)A_n^{(\alpha,\beta)}z^{n-1}}{(N-M)(1-q)-\sum_{n=2}^{\infty}[N(n-1)+(N-M)(1-q)]\phi(n)A_n^{(\alpha,\beta)}z^{n-1}}\right|$$

$$< \frac{\sum_{n=2}^{\infty}(n-1)\phi(n)A_n^{(\alpha,\beta)}}{(N-M)(1-q)-\sum_{n=2}^{\infty}[N(n-1)+(N-M)(1-q)]\phi(n)A_n^{(\alpha,\beta)}}$$

(by (17))
and the proof is complete.

Corollary : Let $F(z) \in Y_{M,N}^{q}(k,\eta,\gamma)$ then

$$A_n^{(\alpha,\beta)} < \frac{(N-M)(1-q)}{[(N-M)(1-q)+(N+1)(n-1)]\phi(n)}. \tag{18}$$

Theorem 2.2 : Let N ≠ 1 and $F(z) \in Y^{q}_{M,N}(k,\eta,\gamma)$ and

$$w = x + iy = \frac{z(z^{\eta}\mathcal{J}^{k,\eta,\gamma}_{0,z}F(z))'}{(z^{n}\mathcal{J}^{k,\eta,\gamma}_{0,z}F(z))}$$

Then the values of w lie in the circle.

Proof: By definitions 4 and 5 we have

$$w = x + iy = \frac{1+e\theta(z)}{1+N\theta(z)} \ (e = N + (M - N)(1 - q), \ |\theta(z)| < 1).$$

Then $(x + iy)(1 + N\theta(z)) = 1 + e\theta(z)$ or $x - 1 + iy = [(e - xN) - iyN]\theta(z)$.

So

$$(x - 1)^2 + y^2 < (e - xN)^2 + y^2 N^2.$$

After a simple calculation we obtain

$$\left[x - \frac{1-Ne}{1-N^2}\right]^2 + y^2 < \frac{(N-e)^2}{(1-N^2)^2}.$$

Hence the values of w lie in the circle with center at $\left(\frac{1-Ne}{1-N^2}, 0\right)$ and radius $\frac{N-e}{1-N^2}$.

Theorem 2.3 : If

$$\frac{N+1}{(N-M)(1-q)} \leq \frac{1+\lambda\nu\xi(1+\theta)}{\lambda\nu(2-\xi(1+\theta))} \tag{19}$$

then

$$\Omega(\lambda, \xi, \theta, \nu) = Y^{q}_{M,N}(k,\eta,\gamma)$$

Proof: By definition 5 we have $Y^{\alpha}_{M,N}(k,\eta,\gamma) \subseteq \Omega(\lambda,\xi,\theta,\nu)$. Now assume that $F(z) \in \Omega(\lambda, \xi, \theta, \nu)$ then by Lemma 1 we have

$$\sum_{n=2}^{\infty} \phi(n)(1+2\nu\lambda)A_n^{(\alpha,\beta)} \leq \lambda\nu(2-\xi(1+\theta))$$

By theorem 2.1 it is enough to show that (17) holds true, which is possible when

$$\left[1+\frac{(N+1)(n-1)}{(N-M)(1-q)}\right]\le\frac{1+2\nu\lambda}{\lambda\nu(2-\xi(1+\theta))}$$

or equivalently

$$\frac{(N+1)(n-1)}{(N-M)(1-q)}\le\frac{1+\lambda\nu\xi(1+\theta)}{\lambda\nu(2-\xi(1+\theta))}.$$

Since n starts from 2 then $n-1 \geq 1$ and hence from last inequality we obtain the result.

In the next section we prove the inclusion property and convex combination property.

3. Inclusion Property and Convex Combination

Theorem 3.1: Let $0 \le q_2 < q_1 < 1$ then $Y^{q_1}_{M,N}(k,\eta,\gamma) \subseteq Y^{q_2}_{M,N}(k,\eta,\gamma)$.

Proof: Suppose that $F(z) \in Y^{q_1}_{M,N}(k,\eta,\gamma)$ then

$$\sum_{n=2}^{\infty}\left[1+\frac{(N+1)(n-1)}{(N-M)(1-q_1)}\right]\phi(n)\, A_n^{(\alpha,\beta)} < 1.$$

We have to prove

$$\sum_{n=2}^{\infty}\left[1+\frac{(N+1)(n-1)}{(N-M)(1-q_2)}\right]\phi(n)\, A_n^{(\alpha,\beta)} < 1$$

but the last inequality holds true if

$$1+\frac{(N+1)(n-1)}{(N-M)(1-q_2)}\le 1+\frac{(N+1)(n-1)}{(N-M)(1-q_1)}$$

and this by hypothesis definitely holds true.

Theorem 3.2: Let $F_j(z) = z - \sum_{n=2}^{\infty} A_{n,j}^{(\alpha,\beta)} z^n (j = 1, 2, \cdots, m)$ be in $Y^{q}_{M,N}(k,\eta,\gamma)$ then the function $F^*(z) = \sum_{j=1}^{m} c_j F_j(z)$ where $\sum_{j=1}^{m} c_j = 1$ is also in $Y^{q}_{M,N}(k,\eta,\gamma)$.

Proof: We have

$$F^*(z) = \sum_{j=1}^{m} c_j\left(z - \sum_{n=2}^{\infty} A_{n,j}^{(\alpha,\beta)} z^n\right) = z - \sum_{j=1}^{n} c_j\left(\sum_{n=2}^{\infty} A_{n,j}^{(\alpha,\beta)} z^n\right)$$

$$= z - \sum_{n=2}^{\infty}\left(\sum_{j=1}^{n} c_j A_{n,j}^{(\alpha,\beta)}\right) z^n.$$

But we have

$$\sum_{n=2}^{\infty}\left[1+\frac{(N+1)(n-1)}{(N-M)(1-q)}\right]\left(\sum_{j=1}^{m} c_j A_{n,j}^{(\alpha,\beta)}\right)\phi(n)$$

$$= \sum_{j=1}^{m}\left(\sum_{n=2}^{\infty}\left[1+\frac{(N+1)(n-1)}{(N-M)(1-q)}\right] A_{n,j}^{(\alpha,\beta)}\,\phi(n)\right)c_j$$

$$< \sum_{j=1}^{m} c_j = 1.$$

This completes the proof.

Remark: The class $Y_{M,N}^{q}(k,\eta,\gamma)$ is a convex set.

4. Extreme Points and Integral Representation

Now we investigate about extreme points of the class $Y_{M,N}^{q}(k,\eta,\gamma)$ and find integral representation of this class.

Theorem 4.1: Let $F_1(z) = z$ and

$$F_n(z) = z - \frac{(N-M)(1-q)}{[(N-M)(1-q)+(N+1)(n-1)]\phi(n)}, \quad n \geq 2.$$

Then the function $F(z) \in Y_{M,N}^{q}(k,\eta,\gamma)$ if and only if

$$F(z) = \sum_{n=1}^{\infty} d_n f_n(z), \tag{20}$$

where $d_n \geq 0$, $(n \geq 1)$ and $\sum_{n=1}^{\infty} d_n = 1$

Proof: Let $F(z) \in Y_{M,N}^{q}(k,\eta,\gamma)$, by (18) if we set

$$d_n = \frac{[(N-M)(1-q)+(N+1)(n-1)]\phi(n)}{(N-M)(1-q)} A_n^{(\alpha,\beta)}, n \geq 2$$

we have $d_n \geq 0$ and if we put $d_1 = 1 - \sum_{n=2}^{\infty} d_n$, then we obtain

$$F(z) = z - \sum_{n=2}^{\infty} A_n^{(d,\beta)} z^n = z - \sum_{n=2}^{\infty} \frac{(N-M)(1-q)}{[(N-M)(1-q)+(N+1)(n-1)]\phi(n)} d_n z^n$$

$$= z - \sum_{n=2}^{\infty} d_n (z - F_n(z)) = (1 - \sum_{n=2}^{\infty} d_n) z + \sum_{n=2}^{\infty} d_n F_n(z) = \sum_{n=1}^{\infty} d_n F_n(z).$$

Conversely suppose

$$F(z) = \sum_{n=1}^{\infty} d_n F_n(z)$$

then we have

$$F(z) = d_1 F_1(z) + \sum_{n=2}^{\infty} d_n F_n(z)$$

$$= d_1 z + \sum_{n=2}^{\infty} \left[z - \frac{(N-M)(1-q)}{[(N-M)(1-q)+(N+1)(n-1)]\phi(n)} z^n \right]$$

$$= z - \sum_{n=2}^{\infty} \frac{(N-M)(1-q)d_n}{[(N-M)(1-q)+(N+1)(n-1)]\phi(n)} z^n$$

Since

$$\sum_{n=2}^{\infty} d_n \left[1 + \frac{(N+1)(n-1)}{(N-M)(1-q)} \right] \phi(n) \frac{(N-M)(1-q)}{[(N-M)(1-q)+(N+1)(n-1)]\phi(n)}$$

$$= \sum_{n=2}^{\infty} d_n = 1 - d_1 < 1.$$

Therefore by Theorem 2.1 we conclude the result.

Remark: The extreme points of the class $Y^{q}_{M,N}(k,\eta,\gamma)$ are the functions $F_1(z)$ and $F_n(z)$, $n \geq 2$ as in the theorem 4.1.

Theorem 4.2: Let $F(z) \in Y^{q}_{M,N}(k,\eta,\gamma)$ then

$$\mathcal{J}^{k,\eta,\gamma}_{0,z} F(z) = z^{-\eta} e^{\int_0^z \frac{(1-[N+(M-N)(1-q)]Q(t)}{t(1-NQ(t))}dt}$$

Proof: Set $w = \dfrac{z(z^{\eta}\mathcal{J}^{k,\eta,\gamma}_{0,z}F(z))'}{z^{\eta}\mathcal{J}^{k,\eta,\gamma}_{0,z}F(z)}$, since $F(z) \in Y^{q}_{M,N}(k,\eta,\gamma)$ so

$$\left|\frac{w-1}{wN-[N+(M-N)(1-q)]}\right| < 1,$$

therefore

$$\frac{w-1}{wN-[N+(M-N)(1-q)]} = Q(z), |Q(z)| < 1$$

Hence we can write

$$\frac{(z^{\eta}\mathcal{J}^{k,\eta,\gamma}_{0,z}F(z))'}{z^{\eta}\mathcal{J}^{k,\eta,\gamma}_{0,z}F(z)} = \frac{1-[N+(M-N)(1-q)]Q(z)}{z(1-NQ(z))}$$

and

$$z^{\eta}\mathcal{J}^{k,\eta,\gamma}_{0,z}F(z) = \exp\left(\int_0^z \frac{1-[N+(M-N)(1-q)]Q(t)}{t(1-NQ(t))}\right)$$

and

$$\mathcal{J}^{k,\eta,\gamma}_{0,z}F(z) = z^{-\eta}\exp\left(\int_0^z \frac{1-[N+(M-N)(1-q)]Q(t)}{t(1-NQ(t))}\right).$$

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