Stochastic Modelling and Computational Sciences
Vol. 2 No. 1 (June, 2022)

Received:30th January 2022 *Revised: 29th March 2022* *Accepted: 25th April 2022*

BOUNDED VARIATION FUNCTIONS AND THE RATE OF CONVERGENCE

Zhou Jin, School of Mathematical Sciences, University of Jihan, China

Abstract

In the present paper, we investigate the rate of approximation by Bezier variant of a new sequence of linear positive operators for functions of bounded variation. Here we extend and generalize the results of Gupta [2].

1. Introuction

Agrawal and Thamer [1] introduced a sequence of linear positive operators M_n and estimated some direct results in simultaneous approximation of unbounded functions. The operators introduced in [1] are defined by

$$M_n(f,x) = (n-1)\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-1}(t)f(t)dt + (1+x)^{-n} f(0) \tag{1}$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}.$$

Recently, Gupta [2] estimated the rate of pointwise approximation by the operators (1) for bounded variation functions. He introduced the Bezier variant of the operators (1), for each $\alpha \geq 1$, as

$$M_{n,\alpha}(f,x) = (n-1)\sum_{k=1}^{\infty} Q_{n,k}^{(\alpha)}(x)\int_0^{\infty} p_{n,k-1}(t)f(t)dt + Q_{n,0}^{(\alpha)}(x)f(0) \tag{2}$$

AMS Subject Classification: 41A25, 41A30.

Key words and phrase : Linear positive operators, Bézier type operators, Bounded variation, Rate of convergence

where $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$ and $J_{n,k}(x) = \sum_{j=k}^{\infty} p_{n,j}(x)$.

It is obvious that $M_{n,\alpha}$ are positive linear operators and $M_{n,\alpha}(1, x)=1$. In the special case $\alpha = 1$, the operators $M_{n,\alpha}$ reduce to the operators M_n defined by (1).

Some basis properties of $J_{n,k}(x)$ are as follows:

(i) $J_{n,k}(x) - J_{n,k+1}(x) = p_{n,k}(x),\ k = 0,1,2,\ldots;$

(ii) $J'_{n,k}(x) = np_{n+1,k-1}(x),\ k = 1,2,3,\ldots;$

(iii) $J_{n,k}(x) = n\int_0^x p_{n+1,k-1}(u)du,\ \ k = 1,2,3,\ldots;$

(iv) $J_{n,0}(x) > J_{n,1}(x) > \ldots > J_{n,k}(x) > J_{n,k+1}(x) > \ldots,$

and for every natural number $k,\ 0 \le J_{n,k}(x) < 1.$

Alternatively we may rewrite the operators (2) as

$$M_{n,\alpha}(f, x) = \int_0^{\infty} W_{n,\alpha}(t, x) f(t)dt$$

where

$$W_{n,\alpha}(t, x) = (n-1)\sum_{k=1}^{\infty} Q_{n,k}^{(\alpha)}(x) p_{n,k-1}(t) + Q_{n,0}^{(\alpha)}(x)\delta(t)$$

$\delta(t)$ being Dirac delta function.

Srivastava and Gupta [5] introduced a general sequence of linear positive operators which includes the operators defined in [6] and [3] as special cases. The authors investigated the rate of convergence of this operators by means of the decomposition technique for functions of bounded variation.

Zeng and Gupta [8] introduced Bezier variant of the Baskakov operators and the rate of convergence for locally bounded functions by using some inequalities and results probability theory. Zeng and Chen also studied rate of convergence for Durrmeyer- Bezier operators for functions of bounded variation [7].

This motivated us to extend the results of Gupta [2] and in the present paper, we obtain the rate of convergence of the operators $M_{n,\alpha}(f, x)$ defined by (2), for functions of bounded variation.

2. Auxiliary Results

In order to prove our main result we require following Lemmas.

Lemma 1 [1]. Let the m-th order moment be defined by

$$M_n((t-x)^m, x) = T_{n,m}(x) = (n-1)\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-1}(t)(t-x)^m dt + (1+x)^{-n}(-x)^m$$

then we have

$$T_{n,0}(x)=1,\ T_{n,1}(x) = \frac{2x}{n-2}, T_{n,2}(x) = \frac{x^2(2n+6)+2nx}{(n-2)(n-3)}, n>3$$

and

$$T_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

Lemma 2. For each $\lambda > 2$ and for all sufficiently n, we have for all $x \in (0,\infty)$

(i) $\beta_{n,\alpha}(y,x) = \int_0^y W_{n,\alpha}(t,x)dt \le \dfrac{\alpha\lambda x(1+x)}{n(x-y)^2}, \quad 0 \le y < x,$

(ii) $1-\beta_{n,\alpha}(z,x) = \int_z^{\infty} W_{n,\alpha}(t,x)dt \le \dfrac{\alpha\lambda x(1+x)}{n(z-x)^2}, \quad x < z < \infty.$

Proof**.** First, we prove (i). In view of Lemma 1 and using inequality $\left|a^{\alpha} - b^{\alpha}\right| \le \alpha|a-b|,\ 0 \le a,b \le 1$ and $\alpha \ge 1$, we have

$$\int_0^y W_{n,\alpha}(t,x)dt \le \int_0^y W_{n,\alpha}(t,x)\frac{(x-t)^2}{(x-y)^2}dt \le \alpha(x-y)^{-2}T_{n,2}(x) \le \frac{\alpha\lambda x(1+x)}{n(x-y)^2}.$$

The proof of (ii) is similar.

Lemma 3 [2]. For all $x \in (0,\infty)$, we have

$$(n-1)\int_x^{\infty} p_{n,k}(t)dt = \sum_{j=0}^{k} p_{n-1,j}(x).$$

Lemma 4 [2]. Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent random variable with the same geometric distribution

$$P(\xi_i = k) = (\frac{x}{1+x})^k \frac{1}{1+x}, \; k \in N, \mathrm{x} > 0 .$$

Then $E\xi_1 = x,\ \sigma = E(\xi_1 - E\xi_1)^2 = x^2 + x,\ E\left|\xi_1 - E\xi_1\right|^3 \le 3x(1+x)^2$.

Lemma 5 [4]. Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables with the expectation $E\xi_1 = a_1$, the variance $\sigma^2 = E(\xi_1 - a_1)^2 > 0,\ \ \rho = E\left|\xi_1 - a_1\right|^3 < \infty$, and let F_n standard distribution function of $\sum_{k=1}^{n} \frac{(\xi_k - a_1)}{\sigma\sqrt{n}}$. Then there exists an absolute constant $C,\ 1/\sqrt{2\pi} \le C < 0.8$, such that for all $n \in N$,

$$\sup_{x \in R}\left|F_n(x) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt\right| \le \frac{C\rho}{\sigma^3\sqrt{n}}.$$

The following Lemma can be prove by Lemma 4 and Lemma 5.

Lemma 6. For all $x \in (0,\infty)$ and $k \in N$, we have

(i) $\left|J_{n,k}^{\alpha}(x) - J_{n-1,k}^{\alpha}(x)\right| \le \frac{\alpha(9x+8)}{\sqrt{nx(1+x)}}$

and

(ii) $\left|J_{n,k}^{\alpha}(x) - J_{n-1,k-1}^{\alpha}(x)\right| \le \frac{\alpha(9x+8)}{\sqrt{nx(1+x)}}$.

Proof (i). $\left|J_{n,k}^{\alpha}(x) - J_{n-1,k}^{\alpha}(x)\right| \le \alpha\left|J_{n,k}(x) - J_{n-1,k}(x)\right| = \alpha I$, say.

Then

$$\begin{aligned} I &= \left|P(\eta_{n-1} \le k-1) - P(\eta_n \le k-1)\right| \\ &\le \left|P(\eta_{n-1} \le k-1) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{A_2} e^{-t^2/2}dt\right| + \left|P(\eta_n \le k-1) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{A_1} e^{-t^2/2}dt\right| + \left|\frac{1}{\sqrt{2\pi}}\int_{A_1}^{A_2} e^{-t^2/2}dt\right| \end{aligned}$$

where $A_1 = \dfrac{k-1-nx}{\sqrt{nx(1+x)}}$ and $A_2 = \dfrac{k-1-(n-1)x}{\sqrt{(n-1)x(1+x)}}$.

From Lemma 4 and Lemma 5

$$\left|P(\eta_{n-1} \le k-1) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{A_2} e^{-t^2/2}dt\right| \le (0.8)\frac{\rho}{\sigma^3\sqrt{n-1}} \le \frac{(3.6)(x+1)}{\sqrt{nx(1+x)}}$$

and

$$\left|P(\eta_{n} \le k-1) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{A_1} e^{-t^2/2}dt\right| \le (0.8)\frac{\rho}{\sigma^3\sqrt{n}} \le \frac{(2.4)(x+1)}{\sqrt{nx(1+x)}}.$$

For the last integral, we can write

$$\left|\frac{1}{\sqrt{2\pi}}\int_{A_1}^{A_2} e^{-t^2/2}dt\right| \le \frac{A_2 - A_1}{\sqrt{2\pi}(1+A_1^2/2)}.$$

If $1 \le k \le 3nx+2$, then

$$A_2 - A_1 = \frac{1}{\sqrt{x(1+x)}}\left(\frac{x}{\sqrt{n}+\sqrt{n-1}} + \frac{k-1}{\sqrt{n}\sqrt{n-1}(\sqrt{n}+\sqrt{n-1})}\right) \le \frac{(4.2)x+(1.44)}{\sqrt{nx(1+x)}}.$$

If $k > 3nx+2$, then

$$\frac{A_2 - A_1}{(1+A_1^2/2)} \le \frac{1}{\sqrt{x(1+x)}}\left(\frac{x}{\sqrt{n}} + \frac{2nx(1+x)}{\sqrt{n}(k-1-2nx+(nx)^2/(k-1))}\right) \le \frac{3x+2}{\sqrt{nx(1+x)}}.$$

Thus

$$\left|\frac{1}{\sqrt{2\pi}}\int_{A_1}^{A_2} e^{-t^2/2}dt\right| \le \frac{(1.44)x+(0.68)}{\sqrt{nx(1+x)}} \text{ and } I \le \frac{(7.44)x+(6.68)}{\sqrt{nx(1+x)}}.$$

The proof of (ii) is similar.

3. Main Theorem

As main result we derive the following estimation the rate of convergence.

Theorem. Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$. If $\alpha \geq 1$, $x \in (0,\infty)$, $r \in N$ and $\lambda > 2$ are given, then for $f(t) = O((1+t)^r)$, $t \to \infty$, there exists a constant $K(f,\alpha,r,x)$ such that for sufficiently large n

$$\left| M_{n,\alpha}(f,x) - \left[\frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) \right] \right| \leq \frac{2\alpha(9x+8)}{\sqrt{nx(1+x)}} \left| f(x+) - f(x-) \right|$$
$$+ \frac{6\alpha\lambda(1+x)}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + K(f,\alpha,r,x) n^{-r}$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-), & 0 \leq t < x \\ 0, & t = x \\ f(t) - f(x+), & t > x \end{cases},$$

$V_a^b(g_x)$ is the total variation of g_x on $[a,b]$.

Proof. For any f bounded variation function, it is known that

$$f(t) = \frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) + g_x(t) + \frac{f(x+) - f(x-)}{2} \left(sign_x(t) + \frac{\alpha-1}{\alpha+1} \right)$$
$$+ \delta_x(t) \left[f(x) - \frac{f(x+)}{2} - \frac{f(x-)}{2} \right]$$

where

$$sign_x(t) = \begin{cases} -1, & t < x \\ 0, & t = x \\ 1, & t > x \end{cases} \quad \text{and} \quad \delta_x(t) = \begin{cases} 1, & t = x \\ 0, & t \neq x \end{cases}.$$

Obviously $M_{n,\alpha}(\delta_x, x) = 0$. Hence we have

$$\left| M_{n,\alpha}(f,x) - \left[\frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) \right] \right| \leq \left| M_{n,\alpha}(g_x, x) \right|$$
$$+ \left| \frac{f(x+) - f(x-)}{2} \right| \left| M_{n,\alpha}(sign_x, x) + \frac{\alpha-1}{\alpha+1} \right|$$

In order to prove the theorem we need the estimates for $M_{n,\alpha}(g_x, x)$ and $M_{n,\alpha}(sign_x, x)$. Considering Lemma 3, we first estimate $M_{n,\alpha}(sign_x, x)$ as follows,

$$M_{n,\alpha}(sign_x, x) = -1 + 2(n-1)\sum_{k=1}^{\infty} Q_{n,k}^{(\alpha)} \int_x^{\infty} p_{n,k-1}(t)dt$$
$$= -1 + 2\sum_{j=0}^{\infty} p_{n-1,j}(x) \sum_{k=j+1}^{\infty} Q_{n,k}^{(\alpha)}(x)$$
$$= -1 + 2\sum_{j=0}^{\infty} p_{n-1,j}(x) J_{n,j+1}^{\alpha}(x).$$

Therefore, we obtain

$$M_{n,\alpha}(sign_x, x) + \frac{\alpha-1}{\alpha+1} = 2\sum_{j=1}^{\infty} p_{n-1,j}(x) J_{n,j+1}^{\alpha}(x) - \frac{2}{\alpha+1}\sum_{j=0}^{\infty} Q_{n-1,j}^{(\alpha+1)}(x)$$

since $\sum_{j=0}^{\infty} Q_{n-1,j}^{(\alpha+1)}(x) = 1$.

By mean value theorem, it follows

$$Q_{n-1,j}^{(\alpha+1)} = J_{n-1,j}^{\alpha+1}(x) - J_{n-1,j+1}^{\alpha+1}(x) = (\alpha+1) p_{n-1,j}(x) \gamma_{n-1,j}^{\alpha}(x)$$

where $$J_{n-1,j+1}(x) < \gamma_{n-1,j}(x) < J_{n-1,j}(x).$$

From Lemma 6 and by inequality

$$J_{n,j+1}^{\alpha}(x) - J_{n-1,j}^{\alpha}(x) < J_{n,j+1}^{\alpha}(x) - \gamma_{n-1,j}^{\alpha}(x) < J_{n,j+1}^{\alpha}(x) - J_{n-1,j+1}^{\alpha}(x)$$

we obtain

$$\left| M_{n,\alpha}(sign_x, x) + \frac{\alpha-1}{\alpha+1} \right| \le 2\sum_{j=0}^{\infty} p_{n-1,j}(x) \left| J_{n,j+1}^{\alpha}(x) - \gamma_{n-1,j}^{\alpha}(x) \right| \le \frac{\alpha(9x+8)}{\sqrt{nx(1+x)}}.$$

We now estimate $M_{n,\alpha}(g_x, x)$. By Lebesgue- Stieltjes integral representation, we have

$$M_{n,\alpha}(g_x, x) = \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} + \int_{x+x/\sqrt{n}}^{\infty} \right) W_{n,\alpha}(t,x) g_x(t) dt$$
$$= E_1 + E_2 + E_3, say. \qquad (3)$$

We first estimate E_2. For $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$ we have

$$|E_2| \le \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} |g_x(t)| W_{n,\alpha}(t,x)dt .$$

Since $|g_x(t)| \le V_t^x(g_x) \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)$ and $\int_a^b W_{n,\alpha}(t,x)dt \le 1$, for $(a,b) \subset [0,\infty)$, we conclude

$$|E_2| \le \frac{1}{n}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x).$$

Next we estimate E_1. Writing $y = x - x/\sqrt{n}$ and using Lebesgue- Stieltjes integration by parts, we have

$$E_1 = \int_0^y g_x(t)d_t(\beta_{n,\alpha}(t,x)) = g_x(y)\beta_{n,\alpha}(y,x) - \int_0^y \beta_{n,\alpha}(t,x)d_t(g_x(t)).$$

Since $|g_x(y)| \le V_y^x(g_x)$ we conclude that

$$|E_1| \le V_y^x(g_x)\beta_{n,\alpha}(y,x) + \int_0^y \beta_{n,\alpha}(t,x)d_t(-V_t^x(g_x)).$$

Lemma 2 implies that

$$|E_1| \le V_y^x(g_x)\frac{\alpha\lambda x(1+x)}{n(x-y)^2} + \frac{\alpha\lambda x(1+x)}{n}\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_x)).$$

Integrating the last term by parts, we get

$$|E_1| \le \frac{\alpha\lambda x(1+x)}{n}\left[x^{-2}V_0^x(g_x) + 2\int_0^y \frac{V_t^x(g_x)}{(x-t)^3}dt\right].$$

Now replacing the variable y in the last integral by $x - x/\sqrt{u}$ we obtain

$$|E_1| \le \frac{\alpha\lambda x(1+x)}{n}\left[x^{-2}V_0^x(g_x) + x^{-2}\frac{1}{n}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^x(g_x)\right].$$

Hence

$$\left|E_1\right| \le \frac{2\alpha\lambda(1+x)}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x).$$

Finally we estimate E_3. We put

$$q_{n,\alpha}(t,x) = \begin{cases} 1-\beta_{n,\alpha}(t,x), & 0 \le t < 2x \\ 0, & t = 2x \end{cases}$$

and $z = x + x/\sqrt{n}$, then

$$\begin{aligned} E_3 &= \int_z^{2x} g_x(t)d_t(q_{n,\alpha}(t,x)) - g_x(2x)\int_{2x}^{\infty} W_{n,\alpha}(t,x)dt + \int_{2x}^{\infty} g_x(t)d_t(\beta_{n,\alpha}(t,x)) \\ &= E_{31} + E_{32} + E_{33},\ \ say. \end{aligned}$$

We estimate E_{31}.

$$E_{31} = g_x(z)q_{n,\alpha}(z,x) + \int_z^{2x} \tilde{q}_{n,\alpha}(t,x)d_t(g_x(t))$$

where $\tilde{q}_{n,\alpha}(z,x)$ is normalized form of $q_{n,\alpha}(z,x)$.

Since $q_{n,\alpha}(z-,x) = \tilde{q}_{n,\alpha}(z,x)$ and $g_x(z-) \le V_x^{z-}(g_x)$, we have

$$E_{31} \le V_x^{z-}(g_x)q_{n,\alpha}(z,x) + \int_z^{2x} \tilde{q}_{n,\alpha}(t,x)d_t(-V_x^t(g_x)).$$

Applying Lemma 2

$$\begin{aligned} \left|E_{31}\right| &\le V_x^{z-}(g_x)\frac{\alpha\lambda x(1+x)}{n(z-x)^2} + \frac{\alpha\lambda x(1+x)}{n}\int_z^{2x-}\frac{1}{(x-t)^2}d_t(-V_x^t(g_x)) \\ &\quad + \frac{1}{2}\left[V_x^{2x-}(g_x)\int_{2x}^{\infty} W_{n,\alpha}(u,x)du\right] \\ &\le V_x^{z-}(g_x)\frac{\alpha\lambda x(1+x)}{n(z-x)^2} + \frac{\alpha\lambda x(1+x)}{n}\int_z^{2x-}\frac{1}{(x-t)^2}d_t(-V_x^t(g_x)) \\ &\quad + \frac{1}{2}\left[V_x^{2x-}(g_x)\frac{\alpha\lambda x(1+x)}{nx^2}\right]. \end{aligned}$$

Thus arguing similarly as in estimate of E_1, we get

$$\left|E_{31}\right| \leq \frac{2\alpha\lambda(1+x)}{nx}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x).$$

Again by Lemma 2, we get

$$\left|E_{32}\right| \leq \frac{\alpha\lambda(1+x)}{nx}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x).$$

Finally for $n > r$, we can write

$$\left|E_{33}\right| \leq M\int_{2x}^{\infty} W_{n,\alpha}(t,x)[(1+t)^r + (1+x)^r]dt.$$

Using the inequalities $(1+t)^r \leq 2^r \dfrac{(1+x)^r}{x^{2r}}(t-x)^{2r}$ and $(1+x)^r \leq 2^r \dfrac{(1+x)^r}{x^{2r}}(t-x)^{2r}$ for $t \geq 2x$, Lemma 1 and Lemma 2, we obtain $\left|E_{33}\right| \leq M2^{r+1}\dfrac{\alpha(1+x)^r}{x^{2r}}T_{n,2r}(x) \leq M2^{r+1}\dfrac{\alpha(1+x)^r}{x^{2r}}M_1 n^{-r} \leq K(f,\alpha,r,x)n^{-r}$.

Combining the estimates of (3) we reach the require result. This completes proof of the theorem.

Remarks

1. If we take $\alpha = 1$, then our result reduces to the main result of Gupta [2].
2. The analogous estimates can be obtained the general sequence of Srivastava and Gupta [6], the methods are different we shall discuss them elsewhere.

REFERENCES

[1] Agrawal, P. N. and Thamer, J., Approximation of unbounded functions by a new sequence of linear positive operators, *J. Math. Anal. Appl.*, **225** (1998), 660-672.

[2] Gupta, V., Rate of approximation by a new sequence of linear positive operators, *Comput. Math. Appl.* **45 (12)** (2003), 1895-1904.

[3] Gupta, V., Gupta, M. K. And Vasishtha, V., Simultaneous approximations by summation-integral type operators, Nonlinear Funct. Anal. Appl., **8 (3),** (2003), 399-412.

[4] Shivyayev, A. N., Probability, Springer-Verlag, New-York, 1984.

[5] Srivastava, H. M. And Gupta, V., A certain family of summation integral type operators, *Math. Comput. Modelling,* **37** (2003), 1307-1315.

[6] Phillips, R. S., An inversion formula for semi-groups of linear operators, *Ann. of Math.*, **59 (2)**, (1954), 352-356.

[7] Zeng, X. M. and Chen, W. On the rate of convergence of generalized Durrmeyer type operators for function of bounded variation*, J. Approx. Theory*, **102** (2000), 1-12.

[8] Zeng, X. M. and Gupta, V. Rate of convergence of Baskakov-Bezier type operators for locally bounded functions, *Comput. Math. Appl.*, **44**(10-11) (2002), 1445-1453.