Stochastic Modelling and Computational Sciences
Vol. 2 No. 1 (June, 2022)

Received:04th January 2022 *Revised: 06th April 2022* *Accepted: 19th April 2022*

SOME FIXED POINT THEOREM RESULTS IN TWO BANACH SPACES

Li Jun, University of Science and Technology, Zenjiang, China

Abstract

The present paper deals with the generalizations of fixed point theorems of Cho et.al. for a single mapping to the common fixed point and coincidence point results for a pair of non-linear mappings in 2-Banach space, which in turn also extend the results of Chang, Xiong *et.al.* and Zhao.

1. Introduction

The concepts of 2-metric and 2-Banach spaces are initially given by Gahler ([3] - [5]) during 1960's. Then about a decade after during 1970's some basic fixed point results in these spaces are established by Iseki ([6], [7]). Thereafter some more fixed point results are obtained in such spaces by Khan *et.al.* [8], Miczko *et.al.* [9], Rhoades [10] and many others extending the fixed point results for contractive mappings from metric space to 2-metric space and that for non expansive mappings from Banach space to 2-Banach space. In the present paper we establish some common fixed point and coincidence point results for a pair of nonlinear mappings in 2-Banach space, which mainly generalize the results of Cho *et.al.* [2]. Hereby we give some preliminary definitions and the results obtained by Cho *et.al.* [2].

Definition 1.1 Let X be a linear space and $\| \cdot, \cdot \|$ be a real valued function defined on $X \times X$ such that

(i) $\| x, y \| = 0$ if and only if x and y are linearly dependent

(ii) $\| x, y \| = \| y, x \|$ for all $x, y \in X$

(iii) $\| x, ay \| = a \| x, y \|$ for all $x, y \in X$ and real a

(iv) $\| x, y+z \| \le \| x, y \| + \| x, z \|$ for all $x, y, z \in X$

Keywords: Non-linear mapping, 2-Banach space, fixed point, coincidence point.
AMS Subject Classification No.(s) : 47H10, 54H25

Then $\| . , . \|$ is called a 2-norm and the pair $(X, \| . , . \|)$ is called a linear 2-normed space.

Some of the basic properties of 2-norms are that they are nonnegative and they satisfy

$$\| x , y+ax \| = \| x , y \| \text{ for all } x , y \in X \text{ and all real } a.$$

Definition 1.2 A sequence $\{x_n\}$ in a linear 2-normed space $(X , \| . , . \|)$ is called a Cauchy sequence if

$$\lim_{m,n \to \infty} \| x_m - x_n, y \| = 0 \text{ for all } y \in X.$$

Definition 1.3 A sequence $\{x_n\}$ in a linear 2-normed space $(X, \| . , . \|)$ is said to be convergent to a point x if $\lim_{n \to \infty} \|x_n - x, y\| = 0$ for all $y \in X$. If $\{x_n\}$ converges to x, we write $x_n \to x$.

Definition 1.4 A linear 2-normed space in which every Cauchy sequence converges is called a 2-Banach space.

Definition 1.5 Let X be a 2-Banach space and T be self mapping of X. T is said to be continuous at $x \in X$, if for every sequence $\{x_n\}$ in X, $x_n \to x$ implies that $Tx_n \to Tx$.

Theorem 1.1 Let X be 2-Banach space with $\dim X \geq 2$ and T be a continuous self mapping of X. Suppose that for any $u \in X$, there exists a function $\phi_u : [0, \infty) \to [0, \infty)$ such that

$$\| x - Tx, u \| \leq \phi_u (x) - \phi_u (Tx)$$

for all $x \in X$. Then T has a fixed point in X.

Theorem 1.2 Let T be a self mapping of a 2-Banach space X ($\dim X \geq 2$) such that $\|Tx - Ty, u \| \leq h \|x - y , u\|$ for all $x, y, u \in X$, where h is a constant in (0, 1). Then T has a unique fixed point in X.

Next we have a 2-Banach space extension of a fixed point result for non expansive mappings of Zhao [12] due to Cho *et. al.* [2].

Theorem 1.3 Let E be a nonempty closed subset of a 2-Banach space X (with $\dim X \geq 2$) and T be a self-mapping of E such that for all *x*, *y* in E and u in X

$$\|Tx - Ty, u\| \leq a \| x-y, u \| + b(\|x - Tx, u\| + \|y - Ty, u\|) + c(\|x - Ty, u\|+\|y - Tx, u\|)$$

where a, b, c all are strictly nonegative constants with $a + 2b + 2c \leq 1$. Then T has a unique fixed point z in E and for any x in E, $T^n x \to z$.

2. Main Results

Our first generalization goes as follows:

Theorem 2.1 Let S and T be two continuous self mappings of a 2-Banach space X. Suppose that for any u in X, there exists a function $\phi_u: [0, \infty) \rightarrow [0, \infty)$ such that

$$\| Sx - TSx, u \| \leq \phi_u (Sx) - \phi_u (TSx) \tag{2.1}$$

$$\| Tx - STx, u \| \leq \phi_u(Tx) - \phi_u(STx) \tag{2.2}$$

for all $x \in X$. Then S and T have a common fixed point.

Proof: For a given $x_o \in X$, we define a sequence recursively as

$$\begin{aligned} x_{2n+1} &= Sx_{2n}, \\ x_{2n+2} &= Tx_{2n+1} \\ n &= 0, 1, \ldots \end{aligned}$$

Using (2.1) and (2.2) we get for all $u \in X$ and $n = 0,1,\ldots\ldots$

$$\begin{aligned} 0 \leq \| x_{2n+1} - x_{2n+2}, u \| &= \| Sx_{2n} - Tx_{2n+1}, u \| \\ &= \| Sx_{2n} - TSx_{2n}, u \| \\ &\leq \phi_u (Sx_{2n}) - \phi_u (TSx_{2n}) \\ &= \phi_u (x_{2n+1}) - \phi_u (x_{2n+2}) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} 0 \leq \|x_{2n} - x_{2n+1}, u\| &= \| Tx_{2n-1} - Sx_{2n}, u \| \\ &= \| Tx_{2n-1} - STx_{2n-1}, u \| \\ &\leq \phi_u (Tx_{2n-1}) - \phi_u (STx_{2n-1}) \\ &= \phi_u (x_{2n}) - \phi_u (x_{2n+1}) \end{aligned} \tag{2.4}$$

From (2.3) and (2.4) we find that $\{\phi_u (x_n)\}$ is a monotonic decreasing sequence of real numbers and therefore there exists a number t_u such that

$$\phi_u (x_n) \rightarrow t_u \text{ as } n \rightarrow \infty \text{ for all } u \in X.$$

Further for any positive integers m and n with $m > n$ and all $u \in X$, we have from (2.3) and (2.4) that

$$\begin{aligned} \| x_n - x_m, u \| &\leq \| x_n - x_{n+1}, u \| + \| x_{n+1} - x_{n+2}, u \| + \ldots + \| x_{m-1} - x_m, u \| \\ &\leq \phi_u (x_n) - \phi_u (x_{n+1}) + \phi_u (x_{n+1}) - \phi_u (x_{n+2}) + \ldots + \phi_u (x_{m-1}) - \phi_u (x_m) \\ &= \phi_u (x_n) - f_u (x_m) \rightarrow 0 \text{ as } m,n \rightarrow \infty \end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence in X. As such there exists a point $z \in X$ such that $x_n \to z$. Now continuities of S and T gives that Tz = Sz = z. i.e. S and T have a common fixed point in X.

Remark 2.1 In case S or T is an identity mapping in the above theorem, we get theorem 1.1 of Cho *et.al.* [2] as a corollary.

Theorem 2.2 Let S and T be two commuting self mappings of a 2-Banach space X such that $S \neq I \neq T$ and

$$\| STx - STy, u \| \leq h \| Tx - Ty, u \| \tag{2.5}$$

and

$$\| TSx - TSy, u \| \leq k \| Sx - Sy, u \| \tag{2.6}$$

for all $x, y, u \in X$, where h and k are some constants in (0,1) and I is an identity mapping. Then S and T have a unique common fixed point in X.

Proof : Inequalities (2.5) and (2.6) imply that S and T are continuous mappings. For any $x, u \in X$, (2.5) gives that

$$\| STx - STSx , u \| \leq h \| Tx - TSx, u \| = h \| Tx - STx, u \|$$

and therefore

$$\|Tx - STx, u \| - h \| Tx - STx, u \| \leq \| Tx - STx , u \| - \| STx - ST Sx, u \|$$

i.e. $\| Tx - STx, u \| \leq (1-h)^{-1} (\| Tx - STx, u \| - \| STx - STSx, u\lambda)$ (2.7)

Similarly, for any $y, u \in X$, (2.6) yields

$$\| Sy - TSy, u \| \leq (1-k)^{-1} (\| Sy-TSy, u \| - \| TSy-TS Ty, u \|) \tag{2.8}$$

Now taking $\phi_u (x) = (1-h)^{-1} \| x - Sx, u \|$ and

$$\phi_u (y) = (1-k)^{-1} \| y - Ty, u \|$$

for all $x, y ,u \in X$, we have from (2.7) and (2.8) that for

$$\phi_u : [0, \infty) \to [0, \infty)$$

$$\| Tx - STx, u \| \leq \phi_u (Tx) - \phi_u (STx)$$

and $\| Sy - TSy, u \| \leq \phi_u (Sy) - \phi_u (TSy)$

for all $x, y, u \in X$. Then by theorem 2.1 S and T have a common fixed point z in X. From (2.5) or (2.6) it is easy to see that the fixed point is unique.

In what follows we give a coincidence point result for two mappings with a different mapping condition.

Theorem 2.3 Let S and T be two continuous and commuting self-mappings of a 2-Banach space X with $S(X) \subset T(X)$. Suppose for any $u \in X$, there exists a function

$\phi_u : [0, \infty) \to [0, \infty)$ such that

$$\|Tx - Sx, u\| \le \phi_u(Tx) - \phi_u(Sx) \tag{2.9}$$

for all $x \in X$. Then S and T have a coincidence point in X.

Proof: Let $x_O \in X$. Since $S(X) \subset T(X)$, therefore there exist $x_1, x_2, \ldots$ in X such that

$Sx_o = Tx_1 = y_1$ (say), $Sx_1 = Tx_2 = y_2$ (say)…and $Sx_{n-1} = Tx_n = y_n$ (say),… . Then from (2.9) we have

$$\begin{aligned} 0 \le \| y_n - y_{n+1}, u \| &= \| Tx_n - Sx_n, u \| \\ &\le \phi_u(Tx_n) - \phi_u(Sx_n) \\ &= \phi_u(y_n) - \phi_u(y_{n+1}) \end{aligned}$$

for all $u \in X$ and $n = 1,2,\ldots\ldots$ Clearly $\{\phi_u(y_n)\}$ is a monotone decreasing sequence of real numbers. Therefore there exists a number t_u such that

$$\phi_u(y_n) \to t_u \text{ as } n \to \infty \text{ for all } u \in X.$$

Further for any positive integers m and n with $m > n$, we have for all $u \in X$

$$\begin{aligned} \| y_n - y_m, u \| &\le \| y_n - y_{n+1}, u \| + \| y_{n+1} - y_{n+2}, u \| + \ldots + \| y_{m-1} - y_m, u \| \\ &\le \phi_u(y_n) - \phi_u(y_{n+1}) + \phi_u(y_{n+1}) - \phi_u(y_{n+2}) + \ldots + \phi_u(y_{m-1}) - \phi_u(y_m) \\ &= \phi_u(y_n) - \phi_u(y_m) \to 0 \text{ as } m, n \to \infty. \end{aligned}$$

Therefore $\{y_n\}$ is a Cauchy sequence in X and so there exists a point $z \in X$ such that $y_n \to z$ i.e. $Tx_n \to z$ and $Sx_n \to z$. Now continuities of S and T give that

$$ST\,x_n = Sy_n \to Sz \text{ and } TSx_n = Ty_{n+1} \to Tz.$$

Then $STx_n = TSx_n \Rightarrow Sz = Tz$ in limits and hence S and T have a coincidence point z.

Remark 2.2 By taking T to be an identity mapping in the above theorem, we get theorem 1.2 of Cho *et.al.* [2] as a corollary.

REFERENCES

[1] S.S. Chang and N.J. Huang, On the generalized 2-metric spaces and probabilistic 2-metric spaces with applications to fixed point theory, Math. Japan. **34**(1989), 885-900.

[2] Y.J. Cho, N. Huang and X. Long, Some fixed point theorems for non-linear mappings in 2-Banach spaces, Far East J. Math. Sci. **3(2)** (1995), 125-133.

[3] S. Gahler, 2-metric Raume and ihre topologische Strucktur, Math. Nachr. **26**(1963), 115-143.

[4] ——, Lineare 2-normietre Raume, Math. Nachr. **28**(1965), 1-43.

[5] ——, Uber die uniformisieberkeit 2-metrischer Raume, Math. Nachr. **28**(1965), 235-244.

[6] K. Iseki, Fixed point theorems in 2-metric spaces, Math. Seminar Notes, Kobe Univ. **3**(1975), 133-136.

[7] ——, Fixed point theorems in Banach spaces, Math. Seminar Notes, Kobe Univ. **2**(1976), 11-13.

[8] M.S. Khan and M.D. Khan, Involutions with fixed points in 2-Banach spaces, Internat. J. Math. Math. Sci. **16**(1993), 429-434.

[9] A. Miczko and B. Palczewski Common fixed points of contractive type mappings in 2-metric spaces, Math. Nachr. **124**(1985), 341-355.

[10] B.E. Rhoades, Contractive type mappings on a 2-metric space, Math. Nachr. **91**(1979), 151-155.

[11] D.T. Xiong, Y.T. Li and N.J. Huang, 2-Metric Spaces Theory, Shanxi Normal University Publishing House, Xian, P.R. China, 1992.

[12] H.B. Zhao, Existence theorems of fixed point of average nonexpansive mappings in Banach spaces, Acta Math. Sinica **22**(1979), 459-470.