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$\mathcal{B}$-MULTIPLICATIVE FUNCTIONS

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Abstract

By a basic sequence $\mathcal{B}$ we mean a set of pairs (a, b) of positive integers with the properties.

1. (a, b) ε $\mathcal{B}$ ⇔ (b, a) ε $\mathcal{B}$
2. (a, bc) ε $\mathcal{B}$ ⇔ (a, b) ε $\mathcal{B}$ and (a, c) ε $\mathcal{B}$
3. (1, k) ε $\mathcal{B}$, for k = 1, 2, 3, ……….

In this paper we define $\mathcal{B}$-multiplicative functions.

An arithmetical function f is said to be $\mathcal{B}$-multiplicative if f is not identically zero and $f(m\ n) = f(m)\ f(n)$ for all (m, n) ε $\mathcal{B}$.

Our $\mathcal{B}$-multiplicative function is the generalization of multiplicative and completely multiplicative functions.

In this paper we have shown the following:

(i) If f is $\mathcal{B}$-multiplicative, then f (1) = 1

(ii) If f is $\mathcal{B}$-multiplicative and if $f(n) \neq 0$ then we get $f(1) \neq 0$, so the inverse of f, f^{-1} exists.

(iii) If f is $\mathcal{B}$ -multiplicative function, then f^{-1} is also a $\mathcal{B}$-multiplicative function. i.e. $f^{-1}(m\ n) = f^{-1}(m)\ f^{-1}(n)$ for all (m, n) ε $\mathcal{B}$.

(iv) If f and g are $\mathcal{B}$-multiplicative functions, then their Dirchlet product $f * g$ is also a $\mathcal{B}$-multiplicative function.

i.e. $(f * g)\ (m\ n) = (f * g)\ (m)\ (f * g)\ (n)$ for all (m, n) ε $\mathcal{B}$.

(v) If f and g are $\mathcal{B}$-multiplicative functions, then their Unitary product f x g is also a $\mathcal{B}$-multiplicative function.

i.e. $(f\ x\ g)\ (m\ n) = (f\ x\ g)\ (m)\ (f\ x\ g)\ (n)$.

We also shown that several more properties of $\mathcal{B}$-multiplicative functions.

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1. Introduction

A real or complex valued function defined on the set of all positive integers is called an Arithmetical function.

An arithmetical function f is said to be a multiplicative function if f is not identically zero and $f(m\,n) = f(m)\,f(n)$ whenever $(m, n) = 1$, f is said to be a completely multiplicative function if $f(m\,n) = f(m)\,f(n)$ for all m, n.

1.1. Definition: A set of pairs (a, b) of positive integers is said to be a Basic sequence $\mathcal{B}$, if

1. $(a, b)\ \varepsilon\ \mathcal{B} \Leftrightarrow (b, a)\ \varepsilon\ \mathcal{B}$
2. $(a, bc)\ \varepsilon\ \mathcal{B} \Leftrightarrow (a, b)\ \varepsilon\ \mathcal{B}$ and $(a, c)\ \varepsilon\ \mathcal{B}$
3. $(1, k)\ \varepsilon\ \mathcal{B}$ $(k = 1, 2, 3, \ldots\ldots\ldots)$

Examples: 1. The set $\mathcal{L}$ of all pairs of positive integers forms a basic sequence.

2. The set $\mathcal{M}$ of all pairs of relative prime positive integers forms a basic sequence.

1.2. Definition: An arithmetical function f is said to be $\mathcal{B}$-multiplicative function if f is not identically zero and $f(m\,n) = f(m)\,f(n)$ for all $(m, n)\ \varepsilon\ \mathcal{B}$.

1.3. Remark: If we take $\mathcal{L}$ as the basic sequence, then our $\mathcal{B}$-multiplicative function becomes completely multiplicative function and if we take $\mathcal{M}$ as the basic sequence, then our $\mathcal{B}$-multiplicative function becomes multiplicative function.

Therefore our $\mathcal{B}$-multiplicative function is generalization of multiplicative and completely multiplicative functions.

1.4. Definition: If f and g are two arithmetical functions, then Donald L. Goldsmith has defined their convolution over B as

$$(f\ 0_{\mathcal{B}}\ g)\ (n) = \sum_{\substack{d\delta = n \\ (d,\,\delta)\ \varepsilon\ \mathcal{B}}} f(d)\ g(\delta) \tag{1.5}$$

1.6. Definition : If f and g are two arithmetical functions, then their Dirichlet product, denoted by $f * g$, defined as

$$(f * g)\ (n) = \sum_{d|n} f(d)\ \ g\left(\frac{n}{d}\right), \quad \text{for all } n. \tag{1.7}$$

1.8. Note: If *f* and *g* are multiplicative functions then their Dirichlet product f ∗ g is also multiplicative function.

1.9. Definition : If *f* and *g* are arithmetical functions then their Unitary Product, denoted by *f x g* defined as

$$(f\ x\ g)\ (n) = \sum_{d \| n} f(d)\ \ g\left(\frac{n}{d}\right), \text{ for all } n, \tag{1.10}$$

[Here $d \| n$ means d is a unitary divisor of n. i.e $d \mid n$ and $\left(d,\ \frac{n}{d}\right) = 1$].

1.11. Note: If *f* and *g* are multiplicative functions then their unitary product *f x g* is also a multiplicative function.

In the second section we first introduce the concept of inverse of a $\mathcal{B}$-multiplicative arithmetical function and proved that inverse of a $\mathcal{B}$-multiplicative function is also a $\mathcal{B}$-multiplicative.

In the third section we proved that "if f and g are $\mathcal{B}$-multiplicative then their Dirichlet product and unitary product are also $\mathcal{B}$-multiplicative". We conclude this section by proving that the convolution $f\ 0_{\mathcal{B}}\ g$ of f and g which is the generalization of Dirichlet and Unitary convolutions, is also a $\mathcal{B}$-multiplicative.

In the fourth section we have proved some more properties of $\mathcal{B}$-multiplicative functions.

2. In this section we first discuss the inverse of a $\mathcal{B}$-multiplicative function and then we proved that the inverse of a $\mathcal{B}$-multiplicative function is also a $\mathcal{B}$-multiplicative function.

2.1. Theorem: If *f* is a $\mathcal{B}$-multiplicative function then $f(1) = 1$.

Proof : We have $f(n) = f(1n) = f(1)\ f(n)$

[since $(1, n)\ \varepsilon\ \mathcal{B}$, for $n = 1, 2, \ldots\ldots$]

Since f is not identically zero, we have $f(n) \neq 0$ for some n.

Cancelling $f(n)$ both sides, we get $f(1) = 1$.

2.2. Definition: The inverse of $\mathcal{B}$-multiplicative function *f* is given by

$$f^{-1}_{(1)} = \frac{1}{\mathrm{f}(1)}$$

and for n > 1, $f^{-1}(n) = \dfrac{-1}{f(1)} \sum_{\substack{d|n \\ d<n}} f\left(\dfrac{n}{d}\right) f^{-1}(d)$

2.3. Theorem: If f is a $\mathcal{B}$-multiplicative function, then f^{-1} is also a $\mathcal{B}$-multiplicative function.

i.e. $f^{-1}(m\,n) = f^{-1}(m).\, f^{-1}(n)$, for all $(m, n)\ \varepsilon\ \mathcal{B}$ (2.4)

Proof: we have $(m, n)\ \varepsilon\ \mathcal{B}$

If $m = n = 1$, then $f^{-1}(m\,n) = \dfrac{1}{f(1)} = 1 = f^{-1}(m)\, f^{-1}(n)$

[since $f(1) = 1$]

Suppose that $mn \neq 1$ and assume that $f^{-1}(cd) = f^{-1}(c)\, f^{-1}(d)$ whenever $cd < mn$. If either $m = 1$ or $n = 1$, then $f^{-1}(mn) = f^{-1}(m)\, f^{-1}(n)$. Therefore, suppose that $m \neq 1$ and $n \neq 1$. Take *a* divisor d of n such that $dd' = n$.

So $(m, dd')\ \varepsilon\ \mathcal{B} \Leftrightarrow (m, d)\ \varepsilon\ \mathcal{B}$ and $(m, d') = \left(m, \dfrac{n}{d}\right) \varepsilon\ \mathcal{B}$

Now, $(m, d)\ \varepsilon\ \mathcal{B} \Leftrightarrow (d, m)\ \varepsilon\ \mathcal{B}$

Take *a* divisor c of m such that $cc' = m$.

So $(d, cc')\ \varepsilon\ \mathcal{B} \Leftrightarrow (d, c)\ \varepsilon\ \mathcal{B}$ and $(d, c') = \left(d, \dfrac{m}{c}\right) \varepsilon\ \mathcal{B}$

Now, $(d, c)\ \varepsilon\ \mathcal{B} \Leftrightarrow (c, d)\ \varepsilon\ \mathcal{B}$ ...(1)

Again $\left(m, \dfrac{n}{d}\right) \varepsilon\ \mathcal{B} \Leftrightarrow \left(cc', \dfrac{n}{d}\right) \varepsilon\ \mathcal{B}$

$\Leftrightarrow \left(\dfrac{n}{d}, cc'\right) \varepsilon\ \mathcal{B}$

$\Leftrightarrow \left(\dfrac{n}{d}, c\right) \varepsilon\ \mathcal{B}$ and $\left(\dfrac{n}{d}, c'\right) = \left(\dfrac{n}{d}, \dfrac{m}{c}\right) \varepsilon\ \mathcal{B}$

$\left(\dfrac{n}{d}, \dfrac{m}{c}\right) \varepsilon\ B \Leftrightarrow \left(\dfrac{m}{c}, \dfrac{n}{d}\right) \varepsilon\ \mathcal{B}$ ...(2)

By the formula of inverse of an arithmetical function.

we have $f^{-1}(m\,n) = -\dfrac{1}{f(1)} \sum_{\substack{c|m \\ d|n \\ cd>1}} f(cd)\, f^{-1}\left(\dfrac{mn}{cd}\right)$.

Since $f(1) = 1$, $cd < mn$, $\left(\left(\dfrac{m}{c}\right)\left(\dfrac{n}{d}\right)\right) < mn$ and from (1) & (2), also by our assumption,

we have $f^{-1}(m\,n) = -\sum_{\substack{c|m \\ d|n \\ cd>1}} f(c)\, f(d)\, f^{-1}\left(\dfrac{m}{c}\right) f^{-1}\left(\dfrac{n}{d}\right)$

$$= -f^{-1}(m) \sum_{\substack{d|n \\ d>1}} f(d)\, f^{-1}\left(\frac{n}{d}\right) - f^{-1}(n) \sum_{\substack{c|m \\ c>1}} f(c)\, f^{-1}\left(\frac{m}{c}\right)$$

$$\left[-\sum_{\substack{c|m \\ c>1}} f(c)\, f^{-1}\left(\frac{m}{c}\right)\right] \left[-\sum_{\substack{d|n \\ d>1}} f(d)\, f^{-1}\left(\frac{n}{d}\right)\right]$$

$$= (-f^{-1}(m))\,(-f^{-1}(n)) - (f^{-1}(n))\,(-f^{-1}(m)) - (-f^{-1}(m))\,(-f^{-1}(n))$$

$$= f^{-1}(m)\, f^{-1}(n) + f^{-1}(m)\, f^{-1}(n) - f^{-1}(m)\, f^{-1}(n)$$

$$= f^{-1}(m)\, f^{-1}(n).$$

Thus, $f^{-1}(mn) = f^{-1}(m)\, f^{-1}(n)$, for all $(m, n) \in \mathcal{B}$

i.e. f^{-1} is a $\mathcal{B}$-multiplicative function.

3. In this section we have proved that if f and g are $\mathcal{B}$-multiplicative then their Dirichlet product $f * g$ and Unitary product $f\, x\, g$ are also $\mathcal{B}$-multiplicative.

3.1. Theorem: If f and g are *B*-multiplicative, then their dirichlet product $f * g$ is also a $\mathcal{B}$-multiplicative,

i.e. $(f * g)(mn) = (f * g)(m)\,(f * g)(n)$, for all $(m, n) \in \mathcal{B}$

Proof: Write $h = f * g$ and

we prove

$h(m\,n) = h(m)\,h(n)$, for all $(m, n)\ \varepsilon\ \mathcal{B}$.

If one of m and n is 1, then the result is obvious.

Now, suppose that $m > 1$ and $n > 1$, and $(m, n)\ \varepsilon\ \mathcal{B}$.

Take a divisor b of n such that $bb' = n$,

So $(m, bb')\ \varepsilon\ \mathcal{B} \Leftrightarrow (m, b)\ \varepsilon\ \mathcal{B}$ and $(m, b') = \left(m,\ \frac{n}{b}\right)\ \varepsilon\ \mathcal{B}$.

Now $(m, b) \Leftrightarrow (b, m)\ \varepsilon\ \mathcal{B}$

Take a divisor a of m such that $aa' = m$,

So $(b, aa')\ \varepsilon\ \mathcal{B} \Leftrightarrow (b, a)\ \varepsilon\ \mathcal{B}$ and $(b, a')\ \varepsilon\ \mathcal{B}$

Now, $(b, a)\ \varepsilon\ \mathcal{B} \Leftrightarrow (a, b)\ \varepsilon\ \mathcal{B}$ ...(1)

Also $$\left(m,\ \frac{n}{b}\right)\ \varepsilon\ \mathcal{B} \quad \Leftrightarrow \left(\frac{n}{b}, m\right) = \left(\frac{n}{b}, aa'\right)\varepsilon\ \mathcal{B}$$

$$\Leftrightarrow \left(\frac{n}{b}, a\right)\varepsilon\ \mathcal{B} \text{ and } \left(\frac{n}{b}, a'\right)\varepsilon\ \mathcal{B}$$

so, $$\left(\frac{n}{b}, a'\right) = \left(\frac{n}{b}, \frac{m}{a}\right)\varepsilon\ \mathcal{B} \Leftrightarrow \left(\frac{m}{a}, \frac{n}{b}\right)\varepsilon\ \mathcal{B} \qquad \text{...(2)}$$

Therefore, by definition,

$$h(m\,n) = \sum_{ab|mn} f(a\,b)\ g\left(\frac{mn}{ab}\right)$$

$$= \sum_{ab|mn} f(a)\,f(b)\,g\left(\frac{m}{a}\right)\ g\left(\frac{n}{b}\right) \text{ (from (1) \& (2))}$$

$$= \left(\sum_{a|m} f(a)\ g\left(\frac{m}{a}\right)\right)\left(\sum_{b|n} f(b)\ g\left(\frac{n}{b}\right)\right)$$

$$= h(m)\,h(n)$$

Hence $h(m\,n) = h(m)\,h(n)$, for all $(m, n)\ \varepsilon\ \mathcal{B}$

i.e. $(f * g)(mn) = (f * g)(m)\,(f * g)(n)$, for all $(m, n)\ \varepsilon\ \mathcal{B}$, and the theorem is proved.

3.2. Theorem: If f and g are $\mathcal{B}$-multiplicative then, their Unitary product is also $\mathcal{B}$-multiplicative.

i.e. $(f \times g)(mn) = (f \times g)(m)\,(f \times g)(n)$, for all $(m, n) \in \mathcal{B}$

Proof: Write $u = f \times g$

We show that $u(m\,n) = u(m)\,u(n)$ for all $(m, n) \in \mathcal{B}$.

Take a divisor b of n such that $bb' = n$ and $(b, b') = 1$

We have $(m, n) \in \mathcal{B}$

So $(m, bb') \in \mathcal{B} \Leftrightarrow (m, b) \in \mathcal{B}$ and $(m, b') \in \mathcal{B}$

So, $(m, b) \Leftrightarrow (b, m) \in \mathcal{B}$

Take a divisor a of m such that $aa' = m$ and $(a, a') = 1$

As $(b, m) = (b, aa') \in \mathcal{B} \Leftrightarrow (b, a) \in \mathcal{B}$ and $(b, a') \in \mathcal{B}$

We have, $(b, a) \in \mathcal{B} \Leftrightarrow (a, b) \in \mathcal{B}$ ...(1)

Also $(m, b') \in \mathcal{B} \Leftrightarrow (b', m) = (b', aa') \in \mathcal{B}$

$\Leftrightarrow (b', a) \in \mathcal{B}$ and $(b', a') \in \mathcal{B}$,

So $(b', a') \in \mathcal{B} \Leftrightarrow (a', b') \in \mathcal{B}$ ...(2)

Therefore, by definition,

$$u(m\,n) = \sum_{\substack{aba'b' = mn \\ (ab, a'b') = 1}} f(a\,b)\,g(a'\,b')$$

$$= \sum_{\substack{aba'b' = mn \\ (ab, a'b') = 1}} f(a)\,f(b)\,g(a')\,g(b')$$

$$= \sum_{\substack{aa' = m \\ (a, a') = 1 \\ bb' = n \\ (b, b') = 1}} f(a)\,f(b)\,g(a')\,g(b')$$

$$= \left(\sum_{\substack{aa'=m \\ (a,a')=1}} f(a)\, g(a') \right) \left(\sum_{\substack{bb'=n \\ (b,b')=1}} f(b)\, g(b') \right)$$

$= u(m)\, u(n)$

Hence u(m n) = u(m) u(n) for all (m, n) ε $\mathcal{B}$ and the theorem is proved.

3.3. Theorem: If *f* and *g* are $\mathcal{B}$-multiplicative then $f_{0\mathcal{B}}\, g$ is also a $\mathcal{B}$-multiplicative

i.e. $(f\ 0_{\mathcal{B}}\ g)(mn) = (f\ 0_{\mathcal{B}}\ g)(m)\,(f\ 0_{\mathcal{B}}\ g)(n)$, for all (m, n) ε $\mathcal{B}$

Proof: The Proof of this result follows at once from the theorems 3.1 and 3.2.

4. In this section we have proved some properties of $\mathcal{B}$-multiplicative functions in which the divisors of m and n are involved.

4.1. Theorem: If both *g* and $f{*}g$ are $\mathcal{B}$-multiplicative then *f* is also $\mathcal{B}$-multiplicative.

Proof: Assuming that f is not $\mathcal{B}$-multiplicative, we show that $f * g$ is also not $\mathcal{B}$-multiplicative.

Write $h = f * g$.

Since f is not $\mathcal{B}$-multiplicative, we can find positive integers *m* and *n* with (*m*, *n*) ε $\mathcal{B}$

Such that $f(m\,n) \neq f(m)\, f(n)$.

We choose such a pair m and n for which the product mn is as small as possible.

Suppose $mn = 1$

So $f(1) = f(1.1) \neq f(1)\, f(1)$ [since (1, 1) ε $\mathcal{B}$]

So $f(1) \neq 1$

Now $h(1) = f(1)\, g(1)$
$= f(1)$

$\neq 1$ [since *g* is $\mathcal{B}$-multiplicative $g(1) = 1$]

which shows $h(1) \neq 1$ and *h* is not $\mathcal{B}$-multiplicative

Now, suppose $mn > 1$.

Take a divisor *a* of *m* and *b* of *n* so that $f(ab) = f(a)\, f(b)$ such that (*a*, *b*) ε $\mathcal{B}$ and $ab < mn$. (such *a*, *b* exist, for example we may take $a = b = 1$.)

Now, we apply the argument used in the above theorem 4.1.

i.e. if $mn > 1$, then we have $f(ab) = f(a) f(b)$ for all positive integers a and b whenever $(a, b) \varepsilon \mathcal{B}$ and $ab < mn$.

Here in the sum defining $h(mn)$, we seperate the term corresponding to $a = m$ and $b = n$.

Therefore, we have

$$h(mn) = \sum_{\substack{a|m \\ b|n \\ ab<mn}} f(ab)\, g\left(\frac{mn}{ab}\right) + f(mn)\, g(1)$$

$$= \sum_{\substack{a|m \\ b|n \\ ab<mn}} f(a)\, f(b)\, g\left(\frac{m}{a}\right) g\left(\frac{n}{b}\right) + f(mn)$$

[since g is $\mathcal{B}$-multiplicative and $\left(\frac{m}{a}, \frac{n}{b}\right) \varepsilon \mathcal{B}$]

$$= \left(\sum_{a|m} f(a)\, g\left(\frac{m}{a}\right)\right) \left(\sum_{b|n} f(b)\, g\left(\frac{n}{b}\right)\right) - f(m)\, f(n) + f(mn)$$

So $h(mn) = h(m)\, h(n) - f(m)\, f(n) + f(mn)$

Since $f(mn) \neq f(m)\, f(n)$, this shows that $h(mn) \neq h(m)\, h(n)$ and h is not $\mathcal{B}$-multiplicative.

This contradiction completes the proof.

4.2. Note: We can prove the inverse of $\mathcal{B}$-multiplicative function is also a $\mathcal{B}$-multiplicative function directly by using the above theorem 4.1 as follows.

4.3. Theorem : If g is $\mathcal{B}$-multiplicative function, then its dirichlet inverse g^{-1} is also $\mathcal{B}$-multiplicative.

Proof: In the above theorem 4.1 we proved that if both g and $f * g$ are $\mathcal{B}$-multiplicative, then f is also $\mathcal{B}$-multiplicative.

Writing g^{-1} in place of f, we get if both g and $g^{-1} * g = I$ are $\mathcal{B}$-multiplicative functions. So we get g^{-1} is $\mathcal{B}$-multiplicative function.

4.4. Theorem : Let k be a positive integer.

Write $$h(n) = \sum_{d^k | n} f(d)\, g\left(\frac{n}{d^k}\right) \tag{4.5}$$

where f and g are $\mathcal{B}$-multiplicative functions then h is also a $\mathcal{B}$-multiplicative function.

Proof: Let $(m, n) \varepsilon \mathcal{B}$

If $m = n = 1$, then $h(mn) = h(1) = 1 = 1.\ 1 = h(m)\, h(n)$

If one of m and n is 1, then $h(1n) = h(1)\, h(n)$ and $h(m1) = h(m)\, h(1)$

[since $(1, n)\ \varepsilon\ \mathcal{B}$ for $n = 1, 2, \ldots$..

and $(1, m)\ \varepsilon\ \mathcal{B} \Leftrightarrow (m, 1)\ \varepsilon\ \mathcal{B}$ for $m = 1, 2, \ldots$.]

Now, suppose that m > 1 and $n > 1$

Take a divisor b^k of n such that $b^k\, b' = n$

So, $(m, b^k\, b')\ \varepsilon\ \mathcal{B} \Leftrightarrow (m, b^k)\ \varepsilon\ \mathcal{B}$ and $(m, b') = \left(m, \frac{n}{b^k}\right) \varepsilon\ \mathcal{B}$

So, $(m, b^k)\ \varepsilon\ \mathcal{B} \Leftrightarrow (b^k, m)\ \varepsilon\ \mathcal{B}$.

Take a divisor a^k of m such that $a^k\, a' =$ m

So $(b^k, m) = (b^k, a^k\, a')\ \varepsilon\ \mathcal{B} \Leftrightarrow (b^k, a^k)\ \varepsilon\ \mathcal{B}$ and $(b^k, a')\ \varepsilon\ \mathcal{B}$

Now $(b^k, a^k)\ \varepsilon\ \mathcal{B} \Leftrightarrow (a^k, b^k)\ \varepsilon\ \mathcal{B}$

also $$\left(m, \frac{n}{b^k}\right) \varepsilon\ \mathcal{B} \Leftrightarrow \left(\frac{n}{b^k}, m\right) = \left(\frac{n}{b^k}, a^k a'\right) \varepsilon\ \mathcal{B}$$

$$\Leftrightarrow \left(\frac{n}{b^k}, a^k\right) \varepsilon\ \mathcal{B} \text{ and} \left(\frac{n}{b^k}, a'\right) = \left(\frac{n}{b^k}, \frac{m}{a^k}\right) \varepsilon\ \mathcal{B}$$

Now, $$\left(\frac{n}{b^k}, \frac{m}{a^k}\right) \varepsilon\ \mathcal{B} \Leftrightarrow \left(\frac{m}{a^k}, \frac{n}{b^k}\right) \varepsilon\ \mathcal{B}.$$

We have $(a^k, b^k)\ \varepsilon\ \mathcal{B} \Leftrightarrow (a^k, bb^{k-1})\ \varepsilon\ \mathcal{B} \Leftrightarrow (a^k, b)\ \varepsilon\ \mathcal{B}$ and $(a^k, b^{k-1})\ \varepsilon\ \mathcal{B}$

Now, $(a^k, b)\ \varepsilon\ \mathcal{B} \Leftrightarrow (b, a^k)\ \varepsilon\ \mathcal{B}$

$$\Leftrightarrow (b, aa^{k-1})\ \varepsilon\ \mathcal{B} \Leftrightarrow (b, a)\ \varepsilon\ \mathcal{B} \text{ and } (b, a^{k-1})\ \varepsilon\ \mathcal{B}$$

Now $\qquad (b, a)\ \varepsilon\ \mathcal{B} \Leftrightarrow (a, b)\ \varepsilon\ \mathcal{B}$

Thus, we have $(a, b)\ \varepsilon\ \mathcal{B}$ and $\left(\frac{m}{a^k}, \frac{n}{b^k}\right)\ \varepsilon\ \mathcal{B}$.

Therefore, by definition,

$$h(mn) = \sum_{(ab)^k | mn} f(ab)\ g\left(\frac{mn}{(ab)^k}\right)$$

$$= \sum_{a^k b^k | mn} f(ab)\ g\left(\frac{mn}{a^k b^k}\right)$$

$$= \sum_{a^k b^k | mn} f(a)\ f(b)\ g\left(\frac{m}{a^k}\right)\ g\left(\frac{n}{b^k}\right)$$

$$= \sum_{\substack{a^k | m \\ b^k | n}} f(a)\ g\left(\frac{m}{a^k}\right) f(b)\ g\left(\frac{n}{b^k}\right)$$

$$= \sum_{a^k | m} f(a)\ g\left(\frac{m}{a^k}\right) \sum_{b^k | n} f(b)\ g\left(\frac{m}{b^k}\right)$$

$$= h(m)\ h(n)$$

Thus, $\quad h(mn) = h(m)\ f(n)$, for all $(m, n)\ \varepsilon\ \mathcal{B}$

i.e. h is $\mathcal{B}$-multiplicative function.

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