Stochastic Modelling and Applications
Vol. 2 No. 1 (June, 2022

Received: 11th January 2022 *Revised: 25th March 2022* *Accepted: 12th April 2022*

M/M/1 QUEUEING SYSTEM AND EQUIVALENCE OF DIFFRENT EXPRESSION

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Abstract

In this paper we study the different transient solutions of the M/M/1 Queueing model by different authors. Then an attempt has been made to derive the different solutions by different authors on the transient probability distribution of the number of units in an M/M/1 Queueing System using one as base formula. While doing so we had also derived all the necessary identities, which are used in connecting these different expressions.

1. Introduction

Recently the transient distribution of $X(t)$, the number of units in the system at time 't' for the M/M/1 Queueing system remained a topic of interest, as suggested by many papers of Abate and Whitt (1987,1988,1989) [1], [2] & [3] and as well as the paper by Conolly and Langaris (1993)[9] and a number of papers based on combinatorial methods (See Baccelli-Massey (1989) [4], Kanwar Sen and Jain (1993) [19], Bohm (1997) [7], Mohanty and Panny (1990) [17], Jain, Mohanty and Jiran, 2004 [11]).

F. Pollaczek noted the importance of the transient distribution of M/M/1 queue as early as 1934. Since then a number of mathematical techniques have been used for obtaining its solution in terms of Modified Bessel functions. A common approach in most of the techniques (except by combinatorial approach) is to formulate a set of differential-difference equations governing the Queueing system and then solving them through some technique.

The distribution of $X(t)$ with constant rate parameters was first derived by Ledermann and Reuter (1954) [16] using spectral method. In the special case when the parameters are independent of the number of units in the system, distribution of $X(t)$ and Δ (the length of the busy period) have been derived by Bailey (1954, 1957) [5] and [6] by using generating function coupled with Laplace transformation.

Keywords: Transient Solution, Bessel Function, Q-function, and Generalize Q-function.

A different procedure has been used by Clarke (1956) [8] depending on the solution of a hyperbolic partial differential equation. Clarke's method is also applicable to time dependent rates; it transforms the infinite set of differential equations into a Volterra-type integral equations.

Takacs [15] in 1962 had obtained the trigonometric integral representation for the transient probability function of the M/M/1 Queueing System. He then had shown the equivalence of this trigonometric integral representation to the expression, which is expressed in terms of Modified Bessel functions [10]. There are other integral representations beside trigonometric integral representation. Some of the integral representations already exist are remarkably same and are discussed in Abate and Whitt [2]. It may be noticed that the trigonometric integral representation may look unsightly to the human eye, but the computer is pleased. Another computational procedure for the evaluation of the M/M/1 transient state probability was derived by Jones [22] using Q-function and Pierce E. Cantrell [18] using the Generalized Q-function.

Here in this paper we try to connect these different expressions by different authors taking one as a base formula and in the process proving all the necessary identities, which will be used in connecting these expressions.

2.

From the expression by Takacs [15] of the transient probability solution of the number of units in an M/M/1 Queueing system we have,

$$P_{in}(t) = f_{in}(t) + \begin{cases} (1-\rho)\rho^n & if\ \lambda < \mu \\ 0 & if\ \lambda \geq \mu \end{cases} \tag{1}$$

where;

$$f_{in}(t) = \frac{2e^{-(\lambda+\mu)t}\rho^{(n-i)/2}}{\pi} \int_0^\pi \frac{e^{2\sqrt{\lambda\,\mu}\,t\cos y}}{(1-2\sqrt{\rho}\,\cos y+\rho)} * [\sin iy - \sqrt{\rho}\,\sin(i+1)y][\sin ny - \sqrt{\rho}\,\sin(n+1)y]dy \tag{2}$$

and $\rho = \dfrac{\lambda}{\mu}$

Since the integrand of (2) is an even function of y and periodic with period 2π. Hence we can replace the integral $\int_0^\pi$ by $\frac{1}{2}\int_0^{2\pi}$ in (2). Putting $z = e^{jy}$, (where $j = \sqrt{-1}$) we have from (2).

$$f_{in}(t) = \frac{e^{-(\lambda+\mu)t}\rho^{(n-i)/2}}{4j\pi}\int_0^{2\pi} g(z)dz \tag{3}$$

where

$$g(z) = \frac{-e^{(z+1/z)\sqrt{\lambda\mu}\;t}}{z\,[1-(z+1/z)\sqrt{\rho}+\rho]} * [(z^i - z^{-i}) - \sqrt{\rho}\,(z^{i+1} - z^{-i-1})]$$
$$* [(z^n - z^{-n}) - \sqrt{\rho}\,(z^{n+1} - z^{-n-1})] \tag{4}$$

It can be clearly seen that $z = 0$ and $z = \sqrt{\frac{\lambda}{\mu}}$ if $\lambda < \mu$ and $z = \sqrt{\frac{\mu}{\lambda}}$ if $\mu < \lambda$ are the two singularities of the integrand (3) in the unit circle $|z| \leq 1$. If $\lambda = \mu$ then $z = 0$ is the only singularity. Using the theorem of residues we have,

$$f_{in}(t) = \frac{e^{-(\lambda+\mu)t}\rho^{(n-i)/2}}{2} g_0 + \begin{cases} \frac{1}{2}(1-\rho)\rho^n & if \quad \lambda > \mu \\ \frac{1}{2}(\rho-1)\rho^n & if \quad \lambda < \mu \end{cases} \tag{5}$$

where g_0 is the residue of $g(z)$ at z = 0.

After doing some simplification, $g(z)$ can be rewritten as;

$$g(z) = \frac{-e^{(z+1/z)\sqrt{\lambda\mu}\;t}}{z(z-\sqrt{\rho})(z^{-1}-\sqrt{\rho})}\Big[-z^{n-i}(z-\sqrt{\rho})(z^{-1}-\sqrt{\rho}) - z^{i-n}(z-\sqrt{\rho})(z^{-1}-\sqrt{\rho})$$
$$+ z^{n+i+1}(z^{-1}-\sqrt{\rho})(1-z\sqrt{\rho}) + z^{-n-i-1}(z-\sqrt{\rho})(1-z^{-1}\sqrt{\rho})\Big]$$

$$= \frac{e^{(z+1/z)\sqrt{\lambda\mu}\;t}}{z}\left[z^{n-i} + z^{i-n} + \frac{\sqrt{\mu}-z\sqrt{\lambda}}{\sqrt{\lambda}-z\sqrt{\mu}} z^{n+i+1} + \frac{\sqrt{\lambda}-z\sqrt{\mu}}{\sqrt{\mu}-z\sqrt{\lambda}} z^{-n-i-1}\right]$$

$$= \frac{e^{(z+\frac{1}{z})\sqrt{\lambda\mu}\;t}}{z}\left[z^{n-i} + z^{i-n} + \sqrt{\frac{\mu}{\lambda}}\, z^{n+i+1} + \sqrt{\frac{\lambda}{\mu}}\, z^{-n-i-1} - \left(1-\frac{\lambda}{\mu}\right)\sum_{r=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^{\frac{(r-1)}{2}} z^{r-n-i-1}\right.$$

$$+\left(1-\frac{\lambda}{\mu}\right)\sum_{r=1}^{\infty}\left(\frac{\mu}{\lambda}\right)^{(r+1)/2} z^{r+n+i+1}\Bigg] \tag{6}$$

We also know that;

$$e^{(z+1/z)\sqrt{\lambda\mu}\ t} = \sum_{m=-\infty}^{\infty} I_m(2\sqrt{\lambda\mu}\ \ t)z^m \tag{7}$$

$I_n(z)$ denotes the modified Bessel function of order n, which is given by;

$$I_n(z) = \sum_{m=0}^{\infty}\frac{(z/2)^{2k+n}}{k!(n+k)!} \qquad \forall n = 0,1,2,3,\ldots\ldots\ldots.. \tag{8}$$

Using (7) in (6) we have;

$$g(z)=\frac{1}{z}\Bigg[\sum_{m=-\infty}^{\infty} I_m z^{m+n-i} + \sum_{m=-\infty}^{\infty} I_m z^{m+i-n} + \sqrt{\frac{\mu}{\lambda}}\sum_{m=-\infty}^{\infty} I_m z^{m+n+i+1} + \sqrt{\frac{\lambda}{\mu}}\sum_{m=-\infty}^{\infty} I_m z^{m-n-i-1}$$

$$-\left(1-\frac{\lambda}{\mu}\right)\sum_{r=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^{\frac{r-1}{2}}\sum_{m=-\infty}^{\infty} I_m z^{m+r-n-i-1} + \left(1-\frac{\lambda}{\mu}\right)\sum_{r=1}^{\infty}\left(\frac{\mu}{\lambda}\right)^{\frac{r-1}{2}}\sum_{m=-\infty}^{\infty} I_m z^{m+r+n+i+1}\Bigg]$$

Now we obtain the value of g_0 as;

$$g_0 = \lim_{z\to 0} z\ g(z)$$

$$= 2I_{n-i} + \left(\sqrt{\frac{\mu}{\lambda}} + \sqrt{\frac{\lambda}{\mu}}\right) I_{n+i+1} - \left(1-\frac{\lambda}{\mu}\right)\sum_{r=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^{\frac{r-1}{2}} I_{r-n-i-1} + \left(1-\frac{\lambda}{\mu}\right)\sum_{r=1}^{\infty}\left(\frac{\mu}{\lambda}\right)^{\frac{r-1}{2}} I_{r+n+i+1}$$

$$= 2I_{n-i} + 2\left(\frac{\lambda}{\mu}\right)^{-\frac{1}{2}} I_{n+i+1} + 2\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{\frac{n+i}{2}}\sum_{r=n+i+2}^{\infty}\left(\frac{\lambda}{\mu}\right)^{-\frac{r}{2}} I_r - \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{\frac{n+i}{2}} e^{(\lambda+\mu)\,t}$$

$$= 2I_{n-i} + 2\,\rho^{-1/2}\ I_{n+i+1} + 2\,(1-\rho)\,\rho^{\frac{n+i}{2}}\sum_{r=n+i+2}^{\infty}\rho^{-r/2} I_r - (1-\rho)\,\rho^{\frac{n+i}{2}} e^{(\lambda+\mu)\,t} \tag{9}$$

Since $I_n = I_{-n}$ and $e^{(\lambda+\mu)\,t} = \sum_{m=-\infty}^{\infty} I_m\left(\frac{\lambda}{\mu}\right)^{m/2}$

(Suppress argument of the Bessel function is $2\sqrt{\lambda\mu}\ t$).

Using (9) in (5) we have;

$$f_{in}(t) = e^{-(\lambda+\mu)t}[\rho^{\frac{n-i}{2}} I_{n-i}(2\sqrt{\lambda\mu}\ t) + \rho^{\frac{n-i-1}{2}} I_{n+i+1}(2\sqrt{\lambda\mu}\ t) + (1-\rho)\ \rho^n$$

$$* \sum_{r=n+i+2}^{\infty} \rho^{\frac{-r}{2}} I_r(2\sqrt{\lambda\mu}\ t)] - \frac{1}{2}(1-\rho)\ \rho^n + \begin{cases} \frac{1}{2}(1-\rho)\rho^n & if \quad \lambda > \mu \\ \frac{1}{2}(\rho-1)\rho^n & if \quad \lambda < \mu \end{cases} \tag{10}$$

Using (10) in (1) we have;

$$P_{in}(t) = e^{-(\lambda+\mu)t}[\rho^{\frac{n-i}{2}} I_{n-i}(2\sqrt{\lambda\mu}\ t) + \rho^{\frac{n-i-1}{2}} I_{n+i+1}(2\sqrt{\lambda\mu}\ t)$$

$$+ (1-\rho)\rho^n \sum_{k=n+i+2}^{\infty} \rho^{-\frac{k}{2}} I_k(2\sqrt{\lambda\mu}\ t)] \tag{11}$$

Thus, we have shown, the equivalence of the expression obtained by Takacs and the well-known expression given in [10].

The last term of (11) can be rewritten as;

$$e^{-(\lambda+\mu)t}(1-\rho)\rho^n \sum_{k=n+i+2}^{\infty} \rho^{-k/2} I_k$$

$$= e^{-(\lambda+\mu)t}\rho^n \left[\sum_{k=n+i+2}^{\infty} \rho^{-k/2} I_k - \rho \sum_{k=n+i+2}^{\infty} \rho^{-k/2} I_k \right]$$

$$= e^{-(\lambda+\mu)t}\rho^n \left[\sum_{k=n+i}^{\infty} \rho^{-k/2} I_k - \rho \sum_{k=n+i+2}^{\infty} \rho^{-k/2} I_k \right] - e^{-(\lambda+\mu)t}\left[\rho^{\frac{n-i}{2}} I_{n+i} + \rho^{\frac{n-i-1}{2}} I_{n+i+1} \right]$$

$$= \rho^n \left[\sum_{k=n+i+1}^{\infty} \rho^{-(k-1)/2} I_{k-1} - \sum_{k=n+i+1}^{\infty} \rho^{-(k-1)/2} I_{k+1} \right] - e^{-(\lambda+\mu)t}\left[\rho^{\frac{n-i}{2}} I_{n+i} + \rho^{\frac{n-i-1}{2}} I_{n+i+1} \right]$$

$$= \rho^n \left[\sum_{k=n+i+1}^{\infty} \rho^{-(k-1)/2} (I_{k-1} - I_{k+1}) \right] - e^{-(\lambda+\mu)t}\left[\rho^{\frac{n-i}{2}} I_{n+i} + \rho^{\frac{n-i-1}{2}} I_{n+i+1} \right] \tag{12}$$

Now using the Bessel function property below,

$$\left(\frac{2k}{2\sqrt{\lambda\,\mu}\;t}\right)I_k(2\sqrt{\lambda\,\mu}\;t)=I_{k-1}(2\sqrt{\lambda\,\mu}\;t)-I_{k+1}(2\sqrt{\lambda\,\mu}\;t)$$

(12) can be rewritten as .

$$e^{-(\lambda+\mu)t}(1-\rho)\rho^n\sum_{k=n+i+2}^{\infty}\rho^{-k/2}I_k$$

$$=\rho^n\left[\sum_{k=n+i+1}^{\infty}\rho^{-(k-1)/2}(I_{k-1}-I_{k+1})\right]-e^{-(\lambda+\mu)t}\left[\rho^{\frac{n-i}{2}}I_{n+i}+\rho^{\frac{n-i-1}{2}}I_{n+i+1}\right] \qquad (13)$$

(Suppress argument of the Bessel function is $2\sqrt{\lambda\mu}\;t$).

Using (13) in (11), we have

$$P_{in}(t)=e^{-(\lambda+\mu)t}\rho^{\frac{n-i}{2}}[I_{n-i}(2\sqrt{\lambda\,\mu}\,t)-I_{n+i}(2\sqrt{\lambda\,\mu}\,t)]$$

$$+\rho^n e^{-(\lambda+\mu)t}\sum_{k=n+i+1}^{\infty}\frac{k\rho^{-\frac{k}{2}}}{\mu\,t}I_k(2\sqrt{\lambda\,\mu}t) \qquad (14)$$

Which is the expression obtained by Jain, Mohanty and Jiran [11].

Identity 1: $(1-\rho)=(e^{-(\lambda+\mu)t}/\mu\,t)\sum_{n=-\infty}^{\infty}n\rho^{-n/2}I_n(2\sqrt{\lambda\,\mu}\;t)$

Proof: Considering the generating function of $I_n(y)$, we have

$$\sum_{n=-\infty}^{\infty}x^nI_n(y)=e^{\frac{y}{2}(x+\frac{1}{x})}$$

Differentiating with respect to x on both sides we have,

$$\sum_{n=-\infty}^{\infty}nx^{n-1}I_n(y)=e^{\frac{y}{2}(x+\frac{1}{x})}\frac{y}{2}\left(1-\frac{1}{x^2}\right)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} n\left(\frac{\mu}{\lambda}\right)^{(n-1)/2} I_n(2\sqrt{\lambda\,\mu}\,t) = e^{(\lambda+\mu)\,t}\mu\,t(1-\rho) \quad ;where\ x=\sqrt{\frac{\mu}{\lambda}}\ \&\ y = 2\sqrt{\lambda\,\mu}\,t$$

$$\Rightarrow (1-\rho) = \left(\frac{e^{-(\lambda+\mu)t}}{\mu\,t}\right)\sum_{n=-\infty}^{\infty} n\rho^{-n/2} I_n(2\sqrt{\lambda\,\mu}\ t)$$

Identity 2: $\sum_{k=-\infty}^{n+i} k\rho^{-\frac{k}{2}} I_k(2\sqrt{\lambda\,\mu}\,t) = \sum_{m=0}^{\infty}\frac{(\lambda\,t)^m}{m!}\sum_{k=0}^{m+n+i}(k-m)\frac{(\mu\,t)^k}{k!}$

Proof: Considering L.H.S., we have

$$\begin{aligned}
&\sum_{k=-\infty}^{n+i} k\rho^{-k/2} I_k(2\sqrt{\lambda\,\mu}\,t)\\
&= \sum_{k=-\infty}^{n+i} k\left(\frac{\lambda}{\mu}\right)^{-k/2}\sum_{m=0}^{\infty}\frac{(\lambda\,\mu)^{(k+2m)/2}t^{k+2m}}{m!(k+m)!}\\
&= \sum_{m=0}^{\infty}\frac{(\lambda\,t)^m}{m!}\sum_{k=-m}^{n+i} k\,\frac{(\mu\,t)^{k+m}}{(k+m)!}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty}\frac{(\lambda\,t)^m}{m!}\sum_{l=0}^{m+n+i}(l-m)\frac{(\mu\,t)^l}{l!} \qquad where\ \ l=k+m\\
&= R.H.S.
\end{aligned}$$

Using Identity1, equation (14) can be rewritten as;

$$\begin{aligned}
P_{in}(t) \quad &= \quad (1-\rho)\rho^n + e^{-(\lambda+\mu)t}\rho^{\frac{n-i}{2}}[I_{n-i}(2\sqrt{\lambda\,\mu}\,t) - I_{n+i}(2\sqrt{\lambda\,\mu}\,t)]\\
&- \quad \frac{\rho^n e^{-(\lambda+\mu)t}}{\mu\,t}\sum_{k=-\infty}^{n+i} k\rho^{-\frac{k}{2}}\ I_k(2\sqrt{\lambda\mu}\ t) \qquad (15)
\end{aligned}$$

Again using Identity 2 the above expression can be rewritten as,

$$\begin{aligned}
P_{in}(t) \quad &= \quad (1-\rho)\rho^n + e^{-(\lambda+\mu)t}\rho^{(n-i)/2}[I_{n-i}(2\sqrt{\lambda\,\mu}\,t) - I_{n+i}(2\sqrt{\lambda\,\mu}\,t)]\\
&- \quad \rho^n\left(\frac{e^{-(\lambda+\mu)t}}{\mu\,t}\right)\sum_{k=0}^{\infty}\frac{(\lambda\,t)^k}{k!}\sum_{m=0}^{k+n+i}(m-k)\frac{(\mu\,t)^m}{m!} \qquad (16)
\end{aligned}$$

When the above expression is further simplified, we have the expression given by Sharma [20] in his recent book, where he discussed the problem using a two-dimensional state model and his solution is;

$$P_{in}(t) = (1-\rho)\rho^n + e^{-(\lambda+\mu)\,t}\rho^n \sum_{k=0}^{\infty}\frac{(\lambda\,t)^k}{k!}\sum_{m=0}^{k+n+i}(k-m)\frac{(\mu\,t)^{m-1}}{m!}$$
$$+\ e^{-(\lambda+\mu)t}\sum_{k=0}^{\infty}(\lambda\,t)^{n+k-i}(\mu\,t)^k\left(\frac{1}{k!(n+k-i)!}-\frac{1}{(n+k)!(k-i)!}\right) \tag{17}$$

Which can also be rewritten as;

$$P_{in}(t) = (1-\rho)\rho^n + e^{-(\lambda+\mu)t}\rho^n \sum_{k=0}^{\infty}\frac{(\lambda\,t)^k}{k!}\sum_{m=0}^{k+n+i}(k-m)\frac{(\mu\,t)^{m-1}}{m!}$$
$$+\ e^{-(\lambda+\mu)t}\sum_{k=0}^{\infty}\frac{(\mu\,t)^k(\lambda\,t)^{n+k-i}}{(2k+n-i)!}\left[\binom{2k+n-i}{k}-\binom{2k+n-i}{k-i}\right] \tag{18}$$

Conolly and Langaris[9] give an alternative expression to Sharma formula as;

$$P_{in}(t) = (1-\rho)\rho^n + e^{-(\lambda+\mu)t}\rho^n \sum_{k=0}^{\infty}\left(\frac{(\lambda\,t)^k}{k!}\sum_{m=0}^{k+n+i}(k-m)\frac{(\mu\,t)^{m-1}}{m!}\right)$$
$$+\ e^{-(\lambda+\mu)t}\rho^n\left(\frac{(\lambda\,t)^{k+1}(\mu\,t)^{k+V}}{k!}\right)\left(\frac{(\lambda\,t)^{-(U+1)}}{(k+|i-n|)!}-\frac{(\mu\,t)^{U+1}}{(k+i+n+2)!}\right) \tag{19}$$

where $U = \min(i,n),\ and\ V = \max(i,n)$

When we carefully study (11), (14), (18) and (19), it can be seen that these expressions involves an infinite sum because of which we need to have some approximation while doing the mathematical computation which in fact is a major disadvantage of these expressions, but this can be easily overcome by introducing another function call Q-function, which is generally denoted by $Q(a,b)$ where a and b are the parameters and is define as;

$$Q(a,b) = \int_b^{\infty} e^{-\frac{a^2+x^2}{2}} I_0(ax)x\,dx \tag{20}$$

It can also be shown that

$$Q(a,b) = e^{-\frac{a^2+b^2}{2}}\sum_{k=0}^{\infty}\left(\frac{a}{b}\right)^k I_k(ab) \tag{21}$$

(In the literature $1-Q(a,b)$ is known as Circular Coverage function).

Consider the R.H.S. of (20), we have

$$= \int_b^\infty e^{-\frac{a^2+x^2}{2}} I_0(ax)xdx$$

$$= \int_b^\infty e^{-\frac{a^2+x^2}{2}} \left(\sum_{m=0}^\infty \frac{(ax/2)^{2m}}{(m!)^2}\right) x\,dx$$

$$= \sum_{m=0}^\infty \frac{(a/2)^{2m}}{(m!)^2} \left(\int_b^\infty e^{-\frac{a^2+x^2}{2}} x^{2m} x\,dx\right)$$

$$= \sum_{m=0}^\infty \frac{(a/2)^{2m}}{(m!)^2} \int_{\frac{a^2+b^2}{2}}^\infty e^{-y}(2y-a^2)^m\,dy \quad where \quad y=\frac{a^2+x^2}{2}$$

After successive integration, we have

$$= \sum_{m=0}^\infty \frac{(a/2)^{2m}}{(m!)^2} \left[e^{-\frac{a^2+b^2}{2}} \{b^{2m} + 2mb^{2(m-1)} + 2^2 m(m-1)b^{2(m-2)} + \ldots\ldots\} \right]$$

$$= e^{-\frac{a^2+b^2}{2}} \left[\sum_{m=0}^\infty \frac{(ab/2)^{2m}}{(m!)^2} + \left(\frac{a}{b}\right) \sum_{m=0}^\infty \frac{(ab/2)^{2m-1}}{(m-1)!m!} + \left(\frac{a}{b}\right)^2 \sum_{m=0}^\infty \frac{(ab/2)^{2m-2}}{(m-2)!m!} + \ldots\ldots \right]$$

$$= e^{-\frac{a^2+b^2}{2}} \left[\left(\frac{a}{b}\right)^0 I_0(ab) + \left(\frac{a}{b}\right)^1 I_{-1}(ab) + \left(\frac{a}{b}\right)^2 I_{-2}(ab) + \ldots\ldots. \right]$$

$$= e^{-\frac{a^2+b^2}{2}} \left[\left(\frac{a}{b}\right)^0 I_0(ab) + \left(\frac{a}{b}\right)^1 I_1(ab) + \left(\frac{a}{b}\right)^2 I_2(ab) + \ldots\ldots\ldots. \right] \quad (\because \; I_k = I_{-k})$$

$$= e^{-\frac{a^2+b^2}{2}} \sum_{m=0}^\infty \left(\frac{a}{b}\right)^m I_m(ab) \quad = \quad equation\ (21)$$

By putting $a=\sqrt{2\lambda t}$ and $b=\sqrt{2\mu t}$, we get

$$Q(\sqrt{2\lambda t},\sqrt{2\mu t}) = e^{-(\lambda+\mu)t}\sum_{k=0}^{\infty}\rho^{-\frac{k}{2}} I_k(2\sqrt{\lambda\mu}\,t) \tag{22}$$

Using the above result (22) in (11), we have;

$$\begin{aligned} P_{in}(t) &= (1-\rho)\rho^n Q(\sqrt{2\lambda t},\sqrt{2\mu t}) + e^{-(\lambda+\mu)t}\{\rho^{(n-i)/2} I_{n-i}(2\sqrt{\lambda\mu}\,t) \\ &+ \rho^{(n-i-1)/2} I_{n+i+1}(2\sqrt{\lambda\mu}\,t) - (1-\rho)\rho^n \sum_{k=0}^{n+i+1}\rho^{-k/2} I_k(2\sqrt{\lambda\mu}\,t)\} \end{aligned} \tag{23}$$

which is the solution obtained by S.K. Jones, R.K. Cavin, III and D.A. Johnston [22].

We can further write the above equation (23) in more compact form by introducing another function call Generalized Q-function, which we define as:

$$Q_m(a,b) = \frac{1}{a^{m-1}}\int_b^{\infty} x^m e^{-\frac{a^2+x^2}{2}} I_{m-1}(ax)dx \tag{24}$$

If m = 1 this will give (20). This can be shown to be equal to,

$$Q_m(a,b) = e^{-\frac{a^2+b^2}{2}}\sum_{k=1-m}^{\infty}\left(\frac{a}{b}\right)^k I_k(ab) \tag{25}$$

and

$$1 - Q_m(a,b) = e^{-\frac{a^2+b^2}{2}}\sum_{k=m}^{\infty}\left(\frac{b}{a}\right)^k I_k(ab) \tag{26}$$

Taking the R.H.S. of (24) we have,

$$\begin{aligned} &\frac{1}{a^{m-1}}\int_b^{\infty} x^m e^{-\frac{a^2+x^2}{2}} I_{m-1}(ax)dx \\ &= \frac{1}{a^{m-1}}\int_b^{\infty} x^m e^{-\frac{a^2+x^2}{2}}\left(\sum_{k=0}^{\infty}\frac{(ax/2)^{2k+m-1}}{k!(k+m-1)!}\right) dx \\ &= \frac{1}{a^{m-1}}\sum_{k=0}^{\infty}\left(\frac{(a/2)^{2k+m-1}}{k!(k+m-1)!}\int_b^{\infty} e^{-\frac{a^2+x^2}{2}} x^{2k+2(m-1)} x \quad dx\right) \end{aligned}$$

which after repeated integration, reduces to

$$= \frac{e^{-\frac{a^2+b^2}{2}}}{a^{m-1}} \sum_{k=0}^{\infty} \frac{(a/2)^{2k+m-1}}{k!(k+m-1)!} [b^{2k+2(m-1)} + 2(k+m-1)b^{2k+2(m-2)}$$

$$+ \quad 2^2(k+m-1)(k+m-2)b^{2k+2(m-3)} + ...]$$

$$= e^{-\frac{a^2+b^2}{2}} \left[\left(\frac{b}{a}\right)^{m-1} \sum_{k=0}^{\infty} \frac{(ab/2)^{2k+m-1}}{k!(k+m-1)!} + \left(\frac{b}{a}\right)^{m-2} \sum_{k=0}^{\infty} \frac{(ab/2)^{2k+m-2}}{k!(k+m-2)!} + \left(\frac{b}{a}\right)^{m-3} \sum_{k=0}^{\infty} \frac{(ab/2)^{2k+m-3}}{k!(k+m-3)!} + ... \right]$$

$$= e^{-\frac{a^2+b^2}{2}} \left[\left(\frac{b}{a}\right)^{m-1} I_{m-1}(ab) + \left(\frac{b}{a}\right)^{m-2} I_{m-2}(ab) + \left(\frac{b}{a}\right)^{m-3} I_{m-3}(ab) + ... \right]$$

$$= e^{-\frac{a^2+b^2}{2}} \left[\left(\frac{a}{b}\right)^{1-m} I_{1-m}(ab) + \left(\frac{a}{b}\right)^{2-m} I_{2-m}(ab) + \left(\frac{a}{b}\right)^{3-m} I_{3-m}(ab) + ... \right] \qquad [\because \ I_k = I_{-k}]$$

$$= e^{-\frac{a^2+b^2}{2}} \sum_{k=1-m}^{\infty} \left(\frac{a}{b}\right)^k I_k(ab) = equation\ (25)$$

Using equation (25), we can easily verify (26), so from (25) we have;

$$Q_m(a,b) = e^{-\frac{a^2+b^2}{2}} \sum_{k=1-m}^{\infty} \left(\frac{a}{b}\right)^k I_k(ab)$$

$$= e^{-\frac{a^2+b^2}{2}} \sum_{k=-\infty}^{m-1} \left(\frac{b}{a}\right)^k I_k(ab)$$

$$= e^{-\frac{a^2+b^2}{2}} \left[e^{\frac{a^2+b^2}{2}} - \sum_{k=m}^{\infty} \left(\frac{b}{a}\right)^k I_k(ab) \right] \qquad [\because e^{\frac{z}{2}(t+t^{-1})} = \sum_{k=-\infty}^{\infty} t^k I_k(z)]$$

$$\Rightarrow 1 - Q_m(a,b) = e^{-\frac{a^2+b^2}{2}} \sum_{k=m}^{\infty} \left(\frac{b}{a}\right)^k I_k(ab)$$

Putting $a = \sqrt{2\lambda\ t} \quad and \quad b = \sqrt{2\mu t}$ in (26), we have

$$1 - Q_m(\sqrt{2\lambda\ t}, \sqrt{2\mu\ t}) = e^{-(\lambda+\mu)t} \sum_{k=m}^{\infty} \rho^{-k} I_k(\sqrt{2\lambda\ \mu}\ t) \qquad (27)$$

Using (27) in (11), we have;

$$P_{in}(t) \;=\; e^{-(\lambda+\mu)\,t}[\rho^{\frac{n-i}{2}} I_{n-i}(2\sqrt{\lambda\,\mu}\,t) \;+\rho^{\frac{n-i-1}{2}} I_{n+i+1}(2\sqrt{\lambda\,\mu}\,t)]$$

$$+\quad (1-\rho)\rho^{n}(1-Q_{n+i+2}(\sqrt{2\lambda\,t},\sqrt{2\mu\,t})\,) \qquad (28)$$

Now the finite sum of Bessel function and Q- function in (23) are being replace by the Generalize Q-function. This expression (28) is the solution given by Pierce E. Cantrell [18].

3. Conclusion

In this paper an attempt has been made to derive the different solutions by different authors on the transient probability distribution of the number of units in an M/M/1 Queueing System using one as base formula. While doing so we had also derived all the necessary identities, which are used in connecting these different expressions. The computational aspects of differential expressions are discussed in [12] and [13] by the authors, which are under communication.

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