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GENERALISED FURTENBERG TRANSFORMATION GROUP C-ALGEBRAS

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Abstract

We study K-theory of continuous fields of the generalized Anzai and Furstenberg transformation group C^*-algebras. Moreover, we consider the case of the generalized quantum Anzai and Furstenberg transformation group C^*-algebras.

Introduction

First recall that the Anzai transformation on the 2-torus $\mathbb{T}^2$ is defined by $\alpha_1^{\theta}(z, w) = (e^{2\pi i\theta}z, zw) \in \mathbb{T}^2$ for θ a real number. Also recall that the Furstenberg transformation on $\mathbb{T}^2$ is defined by $\alpha_1^{\theta, f}(z, w) = (e^{2\pi i\theta}z, e^{2\pi i f(z)}zw) \in \mathbb{T}^2$ for f a continuous real valued function on $\mathbb{T}$. By iterating these transformations as $\alpha_n^{\theta}, \alpha_n^{\theta, f}$ n-compositions of $\alpha_1^{\theta}, \alpha_1^{\theta, f}$ for an integer $n \in \mathbb{Z}$ respectively, we have the Anzai C^*-dymamical system $(C(\mathbb{T}^2), \alpha^{\theta, f}, \mathbb{Z})$ and the Furstenberg C^*-dynamical system $(C(\mathbb{T}^2), \alpha^{\theta,f}, \mathbb{Z})$, where $C(\mathbb{T}^2)$ is the C^*-algebra of continuous complex-valued functions on $\mathbb{T}^2$. Then the crossed product C^*-algebras $\mathfrak{A}^{\theta} = C(\mathbb{T}^2) \rtimes_{\alpha^{\theta}} \mathbb{Z}$ and $\mathfrak{F}^{\theta} = C(\mathbb{T}^2) \rtimes_{\alpha^{\theta,f}} \mathbb{Z}$ (or the Anzai and Furstenberg transformation group C^*-algebras) are induced from these C^*-dynamical systems in a natural way (see [Pd] for crossed products of C^*-algebras). By viewing θ as a parameter on the torus $\mathbb{T} = [0, 1]$ (mod 1), we have the continuous field C^*-algebras on $\mathbb{T}$: $\Gamma(\mathbb{T}, \{\mathfrak{A}^{\theta}\}_{\theta\in\mathbb{T}})$ and $\Gamma(\mathbb{T}, \{\mathfrak{F}^{\theta}\}_{\theta\in\mathbb{T}})$ with fibers $\mathfrak{A}^{\theta}$ and $\mathfrak{F}^{\theta}$. Our first question is the following:

Question. *What are the K-groups of the continuous fields of the Anzai and Furstenberg transformation group C*-algebras?*

In this paper we will answer this question under a more general situation as given soon below. On the other hand, recently we have considered K-theory of continuous fields of quantum (or noncommutative) tori [Sd]. We use some methods of [Sd]. However, a point different from [Sd] is that the fibers $\mathfrak{A}^\theta$ and $\mathfrak{F}^\theta$ are not quantum tori and their dynamical systems are more complicated. Thus, the situation becomes more complicated, but fortunately we can compute the K-groups as in the following.

1. Generalized Anzai and Furstenberg Transformation Group C*-algebras

Since $C(\mathbb{T}^2) \cong C(\mathbb{T}) \otimes C(\mathbb{T})$, we first replace $C(\mathbb{T})$ with $C(\mathbb{T}^n)$ the C^*-algebra of continuous functions on the n-torus $\mathbb{T}^n$. Now we define the generalized Anzai transformation on $\mathbb{T}^{2n}$ by $\alpha_1^\theta(z,w) = (e^{2\pi i\theta} z, zw) \in \mathbb{T}^{2n}$ for $z = (z_j)$, $w = (w_j) \in \mathbb{T}^n$, where $e^{2\pi i\theta}z = (e^{2\pi i\theta}z_j) \in \mathbb{T}^n$ and $zw = (z_jw_j) \in \mathbb{T}^n$. Also we define the generalized Furstenberg transformation on $\mathbb{T}^{2n}$ by $\alpha_1^{\theta, f}(z,w) = (e^{2\pi i f}\ z,\ e^{2\pi i f(z)}zw) \in \mathbb{T}^{2n}$ for $z = (z_j)$, $w = (w_j) \in \mathbb{T}^n$, where $e^{2\pi i f(z)}\ zw = (e^{2\pi i f_j(z_j)}\ z_jw_j) \in \mathbb{T}^n$, $f = (f_j)$ and f_j are continuous real valued functions on $\mathbb{T}$. By iterating these transformations, we have the generalized Anzai C^*-dymamical system $(C(\mathbb{T}^{2n}), \alpha^\theta, \mathbb{Z})$ and the generalized Furstenberg C^*-dynamical system $(C(\mathbb{T}^{2n}), \alpha^{\theta, f}, \mathbb{Z})$, where $C(\mathbb{T}^{2n})$ is the C^*-algebra of continuous complex valued functions on $\mathbb{T}^{2n}$. Then the crossed product C^*-algebras $\mathfrak{A}_n^\theta = C(\mathbb{T}^{2n}) \rtimes_{\alpha^{\theta,f}} \mathbb{Z}$ and $\mathfrak{F}_n^\theta = C(\mathbb{T}^{2n}) \rtimes_{\alpha^{\theta,f}} \mathbb{Z}$ (or the generalized Anzai and Furstenberg transformation group C^*-algebras) are induced from these C^*-dynamical systems in a natural way (see [Pd]). By viewing θ as a parameter on the torus $\mathbb{T} = [0,1]$ (mod 1), we have the continuous field C^*-algebras on $\mathbb{T}$: $\Gamma(\mathbb{T}, \{\mathfrak{A}_n^\theta\}_{\theta\in\mathbb{T}})$ and $\Gamma(\mathbb{T}, \{\mathfrak{F}_n^\theta\}_{\theta\in\mathbb{T}})$.

Theorem 1.1 *Let* $\mathfrak{A}_n = \Gamma(\mathbb{T}, \{\mathfrak{A}_n^\theta\}_{\theta\in\mathbb{T}})$ *and* $\mathfrak{F}_n = \Gamma(\mathbb{T}, \{\mathfrak{F}_n^\theta\}_{\theta\in\mathbb{T}}\})$ *be the continuous field C*-algebras defined above. Then for* $* = 0, 1$,

$$\begin{aligned} K_*(\mathfrak{A}_n^\theta) &\cong K_*(C(\mathbb{T}^{2n})) \oplus (K_*(C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}) / K_*(C(\mathbb{T}_n))) \\ &\cong K_*(C(\mathbb{T}^{2n})) \oplus K_*(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{2n-1}} \oplus \mathbb{Z}^{2^{n-1}} \\ K_*(\mathfrak{A}_n) &\cong K_*(C(\mathbb{T}) \otimes C(\mathbb{T}^{2n})) \oplus (K_*(C(\mathbb{T}) \otimes C^*(\mathbb{Z})) / K_*(C(\mathbb{T}))) \\ &\cong K_*(C(\mathbb{T}^{2n+1})) \oplus K_*(C(\mathbb{T})) \cong \mathbb{Z}^{2^{2n}} \oplus \mathbb{Z}, \end{aligned}$$

where $C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}$ *is the* C^**-subalgebra of* $\mathfrak{A}_n^\theta = C(\mathbb{T}^n) \times C(\mathbb{T}^n) \rtimes_{\alpha^\theta} \mathbb{Z}$ *by the canonical inclusion. Moreover, we obtain*

$$K_*(\mathfrak{A}_n^\theta) \cong K_*(\mathfrak{F}_n^\theta), \quad K_*(\mathfrak{A}_n) \cong K_*(\mathfrak{F}_n)$$

for $* = 0, 1$ *by the same reasoning and replacing* $C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}$ *with* $C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^{\theta, f}} \mathbb{Z}$.

Proof. First note the following inclusion:

$$C(\mathbb{T}) \otimes C(\mathbb{T}^{2n}) \subset \mathfrak{A}_n = \Gamma(\mathbb{T}, \{\mathfrak{A}_n^\theta\}_{\theta \in \mathbb{T}}),$$

where $C(\mathbb{T})$ corresponds to the base space $\mathbb{T}$, and $C(\mathbb{T}^{2n})$ is the canonical C^*-subalgebra of $\mathfrak{F}_n^\theta = C(\mathbb{T}^{2n}) \rtimes_{\alpha^\theta} \mathbb{Z}$. Then it is easy to see the following inclusions in the K-level:

$$K_*(C(\mathbb{T}^{2n})) \subset K_*(\mathfrak{A}_n^\theta), \quad K_*(C(\mathbb{T}) \otimes C(\mathbb{T}^{2n})) \subset K_*(\mathfrak{A}_n)$$

for $* = 0, 1$. It is well known that $K_*(C(\mathbb{T}^k)) \cong \mathbb{Z}^{2^{k-1}}$ for $* = 0,1$ and $k \geq 1$ (cf. [Wo]). Furthermore, note that the following inclusion:

$$C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes \alpha^\theta \mathbb{Z} \subset C(\mathbb{T}^n) \otimes C(\mathbb{T}^n) \rtimes_{\alpha^\theta} \mathbb{Z} = \mathfrak{A}_n^\theta$$

implies the following inclusion in the K-level:

$$K_*(C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}) \subset K_*(\mathfrak{A}_n^\theta)$$

for $* = 0, 1$. Note also that $C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}$ is a noncommutative $(n+1)$-torus. It is well known that $K_*(C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}) \cong \mathbb{Z}^{2^n}$ for $* = 0, 1$ (cf. [Rf]). Since $C(\mathbb{T}^n) \otimes \mathbb{C}$ of $C(\mathbb{T}^{2n}) \cong C(\mathbb{T}^n) \otimes C(\mathbb{T}^n)$ is identified with that of $C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}$ in $\mathfrak{A}_n^\theta$, we in fact have

$$K_*(\mathfrak{A}_n^\theta) \supset K_*(C(\mathbb{T}^{2n})) \oplus (K_*(C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}) / K_*(C(\mathbb{T}_n)))$$
$$\cong K_*(C(\mathbb{T}^{2n})) \oplus K_*(C(\mathbb{T}^n)).$$

Keeping in mind this inclusion, by the aid of the Pimsener-Voiculesce six-term exact sequence ([Bl]):

$$\begin{array}{ccccc} K_0(C(\mathbb{T}^{2n})) & \xrightarrow{id - \alpha_*^\theta} & K_0(C(\mathbb{T}^{2n})) & \xrightarrow{i_*} & K_0(\mathfrak{A}_n^\theta) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A}_n^\theta) & \xleftarrow{i_*} & K_1(C(\mathbb{T}^{2n})) & \xleftarrow{id - \alpha_*^\theta} & K_1(C(\mathbb{T}^{2n})) \end{array}$$

where $\mathrm{id} - \alpha_*^\theta$ is induced from the identity map id on $C(\mathbb{T}^{2n})$ and α_1^θ, and i_* is induced from the canonical inclusion i from $C(\mathbb{T}^{2n})$ to $\mathfrak{A}_n^\theta$, we have

$$K_*(\mathfrak{A}_n^\theta) \cong K_*(C(\mathbb{T}^{2n})) \oplus (K_*(C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}) / K_*(C(\mathbb{T}))).$$

Indeed, let u_j , v_j $(1 \leq j \leq n)$ be the coordinate functions defined by $u_j(z,w) = z_j$ and $v_j(z,w) = w_j$ for $z = (z_k)_{k=1}^n$, $w = (w_k)_{k=1}^n \in \mathbb{T}^n$ respectively. Since they are unitaries that generate $C(\mathbb{T}^{2n})$, let $[u_j]$, $[v_j]$ $(1 \leq j \leq n)$ be the classes of $K_1(C(\mathbb{T}^{2n}))$ corresponding to u_j, v_j. Then

$$(id - \alpha_*^\theta)[u_j] = [u_j][(\alpha_1^\theta(u_j))^{-1}] = [u_j][e^{-2\pi i\theta}u_j^{-1}] = [u_j][u_j^{-1}] = [u_j u_j^{-1}] = [1],$$

$$(id - \alpha_*^\theta)[v_j] = [v_j][(u_j v_j)^{-1}] = [v_j][v_j^{-1}u_j^{-1}] = [v_j v_j^{-1} u_j^{-1}] = [u_j^{-1}].$$

Therefore, $[u_j]$ are in the kernel of id $-\alpha_*^\theta$. Furthermore, note that the other generators of $K_1(C(\mathbb{T}^{2n}))$ are given by the classes of the following unitaries:

$$S_j = I_2 + (u_j - 1) \otimes P_k, \text{ or } T_j = I_2 + (v_j - 1) \otimes P_k \in M_2(C(\mathbb{T}^{2k}))$$

where I_2 is the identity matrix of $M_2(\mathbb{C})$, 1 is the unit of $C(\mathbb{T})$ and P_k are the (generalized) Bott projections of $M_2(C(\mathbb{T}^{2k}))$ corresponding to $2k$ components of $\mathbb{T}^{2n}$ (see [AP] or [Sd]), and $M_2(C(\mathbb{T}^{2k}))$ is naturally identified with elements of $M_2(C(\mathbb{T}^{2n}))$ as coordinate functions. In fact, P_k are defined by taking the inner automorphisms on the rank 1 projection of $M_2(\mathbb{C})$ by certain k unitaries of two variables on $\mathbb{T}^2$ (different components of $\mathbb{T}^{2n}$) as the usual Bott projection for $k = 1$. Since id $-\ \alpha_*^\theta$ is the zero map on $K_0(C(\mathbb{T}^{2n}))$, so is on P_k. Thus S_j are zero under the map id $-\ \alpha_*^\theta$ when P_k comes from $2k$-components of $\mathbb{T}^n$ of $C(\mathbb{T}^n) \otimes \mathbb{C}$, and T_j are nonzero (otherwise). Hence it follows from the six-term exact sequence above that

$$0 \to K_0(C(\mathbb{T}^{2n})) \to K_0(\mathfrak{A}_n^\theta) \to K_1(C(\mathbb{T}^n)) \to 0,$$

and moreover, it follows that

$$0 \to K_1(C(\mathbb{T}^n)) \to K_1(\mathfrak{A}_n^\theta) \to K_0(C(\mathbb{T}^{2n})) \to 0.$$

To determine $K_*(\mathfrak{A}_n)$, we note that any element of the quotient $K_*(C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}) / K_*(C(\mathbb{T}^n))$ for $C(\mathbb{T}^n)$ the canonical subalgebra of $C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}$ is associated with the generalized Rieffel projections defined in [Sd] involving the implementing unitary w of the action α^θ by $\mathbb{Z}$. In fact, the projections are defined by two unitaries coming from some products of generating unitaries on different components of $\mathbb{T}^n$ and the unitary w involving the same relation as two unitaries for the usual Rieffel projections (see [Wo]). Since those projections are not definable at $\theta = 0$ as the usual Rieffel projections, elements of $K_*(C(\mathbb{T}^n) \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}) / K_*(C(\mathbb{T}^n))$ do not produce elements of $K_*(\Gamma(\mathbb{T}, \{\mathfrak{A}_n^\theta\}_{\theta\in\mathbb{T}}))$. Also, since the unitary w commutes with $C(\mathbb{T})$ for $\mathbb{T}$ the base space of $\mathfrak{A}_n$, we have $K_*(C(\mathbb{T}) \otimes C^*(\mathbb{Z}))/K_*(C(\mathbb{T}))$ is contained in $K_*(\mathfrak{A}_n)$.

We can use the same argument above for computing $K_*(\mathfrak{F}_n^\theta)$ and $K_*(\mathfrak{F}_n)$. In particular, note that for u_j, v_j the coordinate functions of $C(\mathbb{T}^{2n})$ defined above,

$$(\mathrm{id} - \alpha_*^{\theta,f})[u_j] = [1] \text{ by the same calculation, and}$$

$$(\mathrm{id} - \alpha_*^{\theta,f})[v_j] = [v_j][(e^{2\pi i f_j(u_j)}u_j v_j)^{-1}]$$

$$= [v_j][v_j^{-1}u_j^{-1}e^{-2\pi i f_j(u_j)}] = [u_j^{-1}e^{-2\pi i f_j(u_j)} = [u_j^{-1}]$$

since the unitary $e^{-2\pi i f_j(u_j)}$ is homotopic to the identity 1.

Remark. For instance, see [Ko] for the case $n = 1$.

2. Generalized Quantum Anzai and Furstenberg Transformation Group C*–algebras

We next replace $C(\mathbb{T}^n)$ with the quantum (or noncommutative) n-torus $\mathbb{T}_\Theta^n$ generated by n unitaries U_j with the commutation relation $U_jU_i = e^{2\pi i\theta_{ij}}U_iU_j$ with $\Theta = (\theta_{ij})$ a skew-adjoint matrix over $\mathbb{R}$. We say that $\mathbb{T}_\Theta^n$ is irrational if all θ_{ij} are rationally independent each other. We define the generalized quantum Anzai transformation on the tensor product $\mathbb{T}_\Theta^n \otimes \mathbb{T}_\Theta^n$ by $\alpha_1^\theta(U_j \otimes U_j) = e^{2\pi i\theta}U_j \otimes U_j^2 \in \mathbb{T}_\Theta^n \otimes \mathbb{T}_\Theta^n$ for $1 \le j \le n$ and θ a real number. Also deifne the generalized quantum Furstenberg transformation on $\mathbb{T}_\Theta^n \otimes \mathbb{T}_\Theta^n$ by $\alpha_1^{\theta,f}(U_j \otimes U_j) = e^{2\pi i\theta}U_j \otimes e^{2\pi i f_j(z_j)}U_j^2 \in \mathbb{T}_\Theta^n \otimes \mathbb{T}_\Theta^n$ $1 \le j \le n$, where $f = (f_j)$ and f_j are continuous real valued functions on $\mathbb{T}$. By iterating these transformations, we have the generalized quantum Anzai C^*-dymamical system $(\mathbb{T}_\Theta^n \otimes \mathbb{T}_\Theta^n, \alpha^\theta, \mathbb{Z})$ and the generalized quantum Furstenberg C^*-dynamical system $(\mathbb{T}_\Theta^n \otimes \mathbb{T}_\Theta^n, \alpha^{\theta,f}, \mathbb{Z})$. Then the crossed product C^*-algebras $\mathfrak{B}_n^\theta = \mathbb{T}_\Theta^n \otimes \mathbb{T}_\Theta^n \rtimes_{\alpha^\theta} \mathbb{Z}$ and $\mathfrak{G}_n^\theta = \mathbb{T}_\Theta^n \otimes \mathbb{T}_\Theta^n \rtimes_{\alpha^{\theta,f}} \mathbb{Z}$ (or the generalized quantum Anzai and Furstenberg transformation group C^*-algebras) are induced from these C^*-dynamical systems in a natural way (see [Pd]). By viewing θ as a parameter on the torus $\mathbb{T} = [0,1]$ (mod 1), we have the continuous field C^*-algebras on $\mathbb{T}$: $\Gamma(\mathbb{T}, \{\mathfrak{B}_n^\theta\}_{\theta\in\mathbb{T}})$ and $\Gamma(\mathbb{T}, \{\mathfrak{G}_n^\theta\}_{\theta\in\mathbb{T}})$.

Theorem 2.1 *Let* $\mathfrak{B}_n = \Gamma(\mathbb{T}, \{\mathfrak{G}_n^\theta\}_{\theta\in\mathbb{T}})$ *and* $\mathfrak{G}_n = \Gamma(\mathbb{T}, \{\mathfrak{G}_n^\theta\}_{\theta\in\mathbb{T}})$ *be the continuous field C^*-algebras defined above. Suppose that* $\mathbb{T}_\Theta^n$ *of* $\mathfrak{G}_n^\theta$ are *irrational in the sense above. Then for* $* = 0, 1$,

$$K_*(\mathfrak{B}_n^\theta) \cong K_*(\mathbb{T}_\Theta^n \otimes \mathbb{T}_\Theta^n) \oplus (K_*(\mathbb{T}_\Theta^n \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}) / K_*(\mathbb{T}_\Theta^n))$$

$$\cong \mathbb{Z}^{2^{2n-1}} \oplus K_*(\mathbb{T}_\Theta^n) \cong \mathbb{Z}^{2^{2n-1}} \oplus \mathbb{Z}^{2^{n-1}}$$

$$K_*(\mathfrak{B}_n) \cong K_*(C(\mathbb{T})) \oplus (\oplus^{n^2} K_*(C(\mathbb{T}^2))/K_*(C(\mathbb{T})))$$

$$\oplus (K_*(C(\mathbb{T}) \otimes C^*(\mathbb{Z})) / K_*(C(\mathbb{T})))$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}^{n^2} \oplus \mathbb{Z},$$

where $\mathbb{T}_\Theta^n \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}$ *is the C^*-subalgebra of* $\mathfrak{B}_n^\theta = \otimes^2 \mathbb{T}_\Theta^n \rtimes_{\alpha^\theta} \mathrm{Z}$ *by the canonical inclusion. Moreover, we obtain*

$$K_*(\mathfrak{B}_n^\theta) \cong K_*(\mathfrak{G}_n^\theta), \quad and \quad K_*(\mathfrak{B}_n) \cong K_*(\mathfrak{G}_n)$$

for $* = 0, 1$ *by the same reasoning and replacing* $\mathbb{T}_\Theta^n \otimes \mathbb{C} \rtimes_{\alpha^\theta} \mathbb{Z}$ *with* $\mathbb{T}_\Theta^n \otimes \mathbb{C} \rtimes_{\alpha^{\theta,f}} \mathbb{Z}$.

Proof. We use the same argument of the proof of Theorem 1.1 for computing $K_*(\mathfrak{B}_n^\theta)$ for $* = 0, 1$ by replacing $C(\mathbb{T}^n)$ with $\mathbb{T}_\Theta^n$. Note that K-groups of $\mathbb{T}_\Theta^n$ are the same as $C(\mathbb{T}^n)$, and their generators of K_0 and K_1 are given by the generalized Rieffel projections and the unitaries involving those projections and generating unitaries of $\mathbb{T}_\Theta^n$ (see [Sd] and [Rf, p. 330]).

For computing $K_*(\mathfrak{B}_n)$ for $* = 0, 1$, we use the similar argument of the proof of Theorem 1.1. The first $K_*(C(\mathbb{T}))$ in the statement corresponds to the base space $\mathbb{T}$. The second $\oplus^{n^2} K_*(C(\mathbb{T}^2))$ correspond to choosing one generating unitary from $\mathbb{T}_\Theta^n \otimes \mathbb{C}$ in $\mathbb{T}_\Theta^n \otimes \mathbb{T}_\Theta^n$ and one generating unitary from $\mathbb{C} \otimes \mathbb{T}_\Theta^n$ in it. The third $K_*(C(\mathbb{T}) \otimes C^*(\mathbb{Z}))$ corresponds to the base space $\mathbb{T}$ and the implementing unitary of the action α^θ.

Furthermore, we can use the same argument above for computing $K_*(\mathfrak{G}_n^\theta)$ and $K_*(\mathfrak{G}_n)$.

Remark. More other variations could be obtained by using our methods given in this paper.

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