NAVIGATING THE LANDSCAPE OF EULERIAN AND HAMILTONIAN STRUCTURES

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ABSTRACT
This paper focuses on Eulerian and Hamiltonian graphs and extends its exploration to algebraic graphs. Eulerian graphs, allowing closed way to traverse each edge once, contribute to the understanding of graph connectivity. Hamiltonian graphs, essential in optimisation and network design, feature cycles visiting each vertex once.

Keywords: Graph Theory, Eulerian Graphs, Hamiltonian Graphs

INTRODUCTION
Graph theory is a mathematical discipline that explores the study of graphs, which are abstract representations of relationships and connections between various entities. These entities, often referred to as vertices or nodes, are interconnected by edges or links, forming a network of relationships. Graph theory provides a powerful framework for analyzing and modeling diverse real-world phenomena, ranging from social networks and communication systems to transportation networks and molecular structures.

The origins of graph theory can be traced back to the 18th century when the Swiss mathematician Leonhard Euler introduced the concept of a graph to solve the famous Seven Bridges of Königsberg problem. Since then, graph theory has evolved into a rich and versatile field of study with applications in computer science, telecommunications, biology, social sciences, and various other disciplines.

The fundamental concepts of graph theory include exploring the properties and characteristics of graphs, understanding different types of graphs such as directed and undirected graphs, and investigating various algorithms and methods for solving problems related to graphs. Graph theory has become an integral part of modern mathematics and plays a crucial role in addressing complex problems in fields as diverse as computer science, optimization, and network analysis.

As a powerful tool for modeling relationships and connectivity, graph theory continues to find new applications and inspire innovative solutions to challenges in science, engineering, and beyond. Its broad relevance and versatility make it an essential area of study for researchers, mathematicians, and professionals seeking to understand and harness the intricacies of interconnected systems.

LITERATURE REVIEW
Xia Liu et al (2022) the authors consider the problem of characterizing Hamiltonian line graphs with local degree conditions, where the degree of a vertex is only dependent on its local neighborhood. They provide a necessary condition for a graph to be a Hamiltonian line graph with such degree conditions, and also present a sufficient condition for a graph to be a Hamiltonian line graph with a stronger degree condition. The results on Hamiltonian line graphs with local degree conditions have important applications in graph theory and computer science.

Bo Zhang et al., (2021) The study of Anti-Eulerian digraphs is an important area of research in graph theory and combinatorics. The authors' findings contribute to a better understanding of the properties and behavior of these types of digraphs. "Anti-Eulerian digraphs" is a well-written and informative paper that provides a thorough study of Anti-Eulerian digraphs. The paper's contributions and insights make it a valuable addition to the existing literature on graph theory and combinatorics.

Hamiltonian and Eulerian Graph

Eulerian graphs, allowing closed walks that traverse each edge once, shed light on graph connectivity. Hamiltonian graphs, featuring cycles that visit each vertex once, are pivotal in optimization and network design. Complete algebraic graphs take connectivity further, ensuring every pair of vertices is connected by an edge. This exploration not only contributes to theoretical advancements but also finds applications in practical scenarios, from network design to logistics planning, by revealing the intricate mathematical properties within graphs.

Definition 3.1 An algebraic graph is defined on a graph $G(V,E)$ with vertex set V and edge set E. It is characterized by the existence of bijective functions $fi:Vi\to Mi$ for $i=1,2,\ldots,n$, where Vi and Mi are subsets of V. The functions must satisfy the following conditions:

(i) The degrees of the functions $d(f_1) > d(f_2) > \cdots > d(f_n)$, and the union of V_i and M_i covers the entire vertex set V, i.e., $\bigcup_{i-1}^{n} V_i \cup M_i - V$. Additionally, no function is defined on a subset M of V if the order of $M(o(M))$ is greater than the order of $V_1(o(V_1))$.
(ii) For any edge $\{a,b\} \in E_2$, there exists a unique function f_i such that $f_i(a) - b$.
(iii) If there is a path $\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{n-1},v_n\}$ in E, then there exists a function f_i such that $f_i(v_1) - v_2, f_i(v_2) - v_3, \ldots, f_i(v_{n-1}) - v_n$.

The number of elements in a set M is denoted by $o(M)$. An algebraic graph of $G(V,E)$ is denoted by $G(V,E,F)$, where F represents the set of bijective functions f_i satisfying the given conditions.

Now, let's represent these conditions (i)

$$\begin{aligned} d(f_1) &> d(f_2) > \cdots \\ \bigcup_{i=1}^{n} V_i \cup M_i - V &\quad \text{and } o(f_n) \\ o(M) &> o(V_1) \text{ if } M \subset V \end{aligned}$$

(ii)

$$\{a,b\} \in E \longrightarrow \exists \text{ unique } f_i \text{ such that } f_i(a) - b$$

(iii)

$$\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{n-1},v_n\} \in E \longrightarrow \exists f_i \text{ such that } f_i(v_1) - v_2, f_i(v_2)$$

Definition: 3.2 Let the algebraic graph $G(V,E,F)$, where $F=\{f_i|i=1,2,\ldots,n\}$. In this context, V represents the set of vertices, E is the set of edges, and F is a collection of functions f_i indexed from 1 to n.

The degree of a function f in F is defined as the number of elements in the domain of f. Mathematically, if f_i is a function in F, then the degree of f_i, denoted as $\deg(f_i)$, is given by:

$\deg(f_i)=|\text{dom}(f_i)|$

Here, $\text{dom}(f_i)$ represents the domain of the function f_i, and $|\cdot|$ denotes the cardinality or the number of elements in a set.

In the context of algebraic graphs, these functions f_i may represent various algebraic operations or mappings associated with the vertices and edges of the graph. The degree of a function provides insight into the complexity or richness of its domain, indicating how many distinct elements it operates on.

Algebraic graphs are valuable in representing mathematical structures, and the incorporation of functions in F adds a layer of abstraction, allowing for a more nuanced understanding of the relationships within the graph.

Definition: 3.3 An algebraic graph, denoted as $G(V,E,F)$, consists of a set of vertices V, a set of edges E, and a set of functions $F=\{f_i|i=1,2,\ldots,n\}$ associated with the edges. Here, n represents the number of edges in the graph.

The size of an algebraic graph is defined as the cardinality of the edge set E, denoted as $|E|$. In mathematical terms:

Size of $G=|E|=n$

This means that the size of the algebraic graph is equal to the number of edges it contains.

The order of an algebraic graph is defined as the cardinality of the vertex set V, denoted as $|V|$. In mathematical terms:

Order of $G=|V|$

This implies that the order of the algebraic graph is equal to the number of vertices it comprises.

Definition 3.4: algebraic graph $G(V,E,F)$ where $F=\{f_i|i=1,2,\ldots,n\}$ and $d(f_1)>\mathrm{d}(f_2)>\ldots>d(f_n)$, where $d(f_i)$ is the degree of f_i.

In an algebraic graph, vertices are associated with polynomials, and the edges are defined by the common roots of these polynomials. The degree of a polynomial is related to the number of its distinct roots.

The diameter of a graph is defined as the maximum distance between any two vertices in the graph. In the context of an algebraic graph, the distance can be measured by the number of edges in the shortest path between two vertices.

Now, let's consider $F=\{f_i|i=1,2,\ldots,n\}$, where f_i is associated with vertex V_i. The degrees of these polynomials are given by $d(f_1)>d(f_2)>\ldots>d(f_n)$.

In an algebraic graph, the degree of a polynomial is related to the number of roots it has. Therefore, $d(f_i)$ represents the number of roots of f_i and indirectly the number of edges connected to the corresponding vertex V_i.

To find the diameter of the algebraic graph, we need to consider the vertices with the highest degrees. Since $d(f_1)>d(f_2)>\ldots>d(f_n)$, the vertex V_1 has the highest degree.

The diameter of the algebraic graph G is determined by the distance between V_1 and the vertex V_j (where $j\neq 1$) such that $d(f_j)$ is the second-highest degree.

In mathematical terms, the diameter δ can be expressed as:

$\delta(G)=\max j\neq 1 \mathrm{dist}(V_1,V_\mathrm{j})$

where $\mathrm{dist}(V_1,V_\mathrm{j})$ is the shortest path distance between V_1 and V_j.

RESULT

THEOREM 3.1: A graph G is considered a line graph L (H) if and only if its lines can be partitioned into complete subgraphs in such a way that no point lies in more than two of the subgraphs.

From the definition of a line graph and Theorem 3.1, it is evident that if a graph $G=L(H)$, then:

(i) Every point P in H gives rise to a complete subgraph C_P in $L(H)$, determined by the set of all lines of H incident with point P. Furthermore, C_P has even order if P has even degree. Mathematically, this can be expressed as:

$\mathrm{C_P}=\{L\in L(H)|\mathrm{P}$ is incident with $L\}$

and

$|C_P|\equiv 0(\mathrm{mod}\ 2)$ if $\deg(P)\equiv 0(\mathrm{mod}\ 2)$

(ii) Every line L in G corresponds to the adjacency of two lines L_1 and L_2 in H. Alternatively, every line L_1, L_2 in G corresponds uniquely to the point in common between the two lines L_1 and L_2 in H. Mathematically, this can be represented as:

$L \leftrightarrow \{L_1, L_2\}$ in H

LEMMA

If a graph G is Eulerian, meaning it has a closed walk traversing every edge exactly once, then its line graph $L(G)$ is Hamiltonian, possessing a cycle that visits each edge once. Recognizing G as the line graph of an Eulerian graph implies that G is Hamiltonian, highlighting a direct relationship between Eulerian and Hamiltonian properties in graph theory. This insight enables efficient identification of Hamiltonian cycles by examining the Eulerian nature of the original graph and its line graph.

THEOREM 3.2 A connected graph G is the line graph of an Eulerian graph if and only if its lines can be partitioned into even complete subgraphs such that each point lies in exactly two of these subgraphs.

To elaborate, an Eulerian graph is a graph where every vertex has an even degree, meaning that the number of edges incident to each vertex is an even number. The given statement asserts that a connected graph G is the line graph of such an Eulerian graph if certain conditions on its line partitions are met.

The proof of this statement can be derived from observations (i) and (ii) made earlier:

COROLLARY 3.2a if a graph G has a subgraph partition, as specified in Theorem 3.2, then G is Hamiltonian. In Theorem 3.2, a subgraph partition refers to a partitioning of the graph into connected subgraphs $G_1, G_2, \ldots, G_k$ such that each subgraph G_i has a dominating vertex, and certain conditions are met regarding the connections between the dominating vertices. The conclusion drawn from this theorem is that the original graph G is Hamiltonian, implying the existence of a Hamiltonian cycle.

COROLLARY 3.2b a graph G is Hamiltonian if and only if it possesses a spanning subgraph H that is the line graph of an Eulerian graph. This means that the Hamiltonian property, where the graph contains a Hamiltonian cycle visiting each vertex exactly once, is intimately connected to the existence of a particular type of subgraph in G. Specifically, this subgraph H is constructed as the line graph of an Eulerian graph, where every vertex has an even degree. The "if and only if" nature of the statement signifies a bidirectional relationship: if G is Hamiltonian, then there exists such a spanning subgraph H; conversely, the presence of this specific subgraph H implies the Hamiltonian nature of G. This provides a powerful criterion for recognizing Hamiltonian graphs based on the structural properties of their spanning subgraphs.

LEMMA

In a line graph G, any group containing four or more vertices must correspond to one of the complete subgraphs identified in Theorem 3.1. Theorem 3.1 characterizes line graphs by associating each point in the original graph with a complete subgraph in the line graph. Therefore, the presence of larger groups in the line graph is inherently linked to the structural properties outlined in Theorem 3.1, offering a clear relationship between the order of groups and the underlying characteristics of line graphs.

Proof: The theorem considers a set of complete subgraphs in a graph G, denoted as C_1, C_2, ..., C_k. It introduces conditions for a group C in G with at least four points, distinct from other groups. The theorem highlights a specific complete subgraph C_n, sharing an edge with C, and introduces a point of C not present in C_1. The theorem explores relationships and distinctions between cliques and complete subgraphs, contributing to the understanding of structural properties in graph theory.

THEOREM 3.3 A graph G, representing the lines of a Hamiltonian graph. It asserts that G can be partitioned into complete subgraphs in such a way that every point in G belongs to exactly two subgraphs. Furthermore, G contains a cycle incorporating one line from each subgraph. In simpler terms, if you imagine cities connected by roads (Hamiltonian graph), G's lines are these roads. The statement suggests organizing these roads into subnetworks so that each city connects to exactly two roads, and collectively, there's a cycle passing through every city, using one road from each subnetwork. This arrangement has implications for structured network organization and optimization.

Proof: Assume G is the line graph of a Hamiltonian graph H. Let's denote a Hamiltonian cycle in H as u_1u_2 ... u_mu_1 and correspondingly, the complete subgraphs in G as C_1, C_2, ...,C_m. These complete subgraphs partition the lines in G, as each line in G is uniquely associated with a point v_j in H, and thus, with the complete subgraph C_{Uj}.

Now, for each point v_i in G, there is a unique correspondence to a line α in H, incident to two points, say u_1 and u_2, in H. These two points, in turn, correspond uniquely to complete subgraphs G_1 and G_2 of G. Since α is incident to both u_1 and up in H, it implies that the corresponding point v_i in G belongs to both G_1, and G_2. Therefore, every point in G is precisely in two of the complete subgraphs (G_i), demonstrating a clear partitioning of the graph G based on its Hamiltonian cycle and associated complete subgraphs.

Assume that:

(1) The lines of G can be partitioned into complete subgraphs $\{G_i\}$, i = l,..., m, in such a way that

(2) Every point of G lies in exactly two of these complete subgraphs, and

(3) G contains a cycle having exactly one line in each of these complete subgraphs. Then (1) and (2) and Theorem 3.1 imply that G is the line graph of some graph H.

We have seen that the points of H correspond l-l with the complete subgraphs $\{G_i\}$, i = l,...,m of G. Furthermore, we know that two points in H are adjacent if and only if the corresponding subgraphs in G have a point in common. A single edge in one of these G_i then corresponds to two incident edges through the corresponding point in H. Therefore, the cycle in G having exactly one line in each ($G_i\}$ corresponds to a cycle in H which passes into and out of each point in H exactly once. Hence, H is Hamiltonian.

The cycle described in Theorem 3.3 and its corollary is a Hamiltonian cycle itself only if G is a cycle. It is usually much shorter than a Hamiltonian cycle.

CONCLUSION

This research has unveiled the intricate relationships within graphs, particularly emphasising the significance of Eulerian and Hamiltonian properties. The introduction of algebraic graphs with bijective functions adds a layer of abstraction, allowing for a more nuanced understanding of graph structures. The study not only contributes to theoretical advancements in graph theory but also showcases practical applications in network design and logistics planning. The resulting theorems and corollaries give us a lot of useful information by connecting Eulerian and Hamiltonian properties and giving us a way to tell Hamiltonian graphs apart based on their structural properties. This research, therefore, establishes a foundation for further exploration in the diverse and dynamic field of graph theory.

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