Stochastic Modelling and Computational Sciences

K-SQUARE HARMONIC MEAN LABELING OF LADDER RELATED GRAPHS

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ABSTRACT
A Graph $G = (V, E)$ with p vertices and q edges is called a k- square harmonic mean labeling if there exists an injective function $h : V(G) \to \{k, k+1, k+2, \dots, k+q\}$ in such a way that an induced edge function $h^* : E(G) \to \{k, k+1, k+2, \dots, k+q-1\}$ defined by $h^*(e = uv) = \left\lceil \frac{2h(u)^2 h(v)^2}{h(u)^2 + h(v)^2} \right\rceil$ or $\left\lfloor \frac{2h(u)^2 h(v)^2}{h(u)^2 + h(v)^2} \right\rfloor$ is bijective. A graph which admits a k-square harmonic mean labeling is called a k- square harmonic mean graph. This study aims at defining the concept of k-square harmonic mean labeling of ladder related graphs.

Keywords: Square harmonic mean graph, k- square harmonic mean graph

2020 AMS subject classifications: 05C38, 05C78

1. INTRODUCTION

All graphs considered here are simple, finite, connected and undirected graph. Graph labeling plays an important role in graph theory. A graph labeling is an assignment of integers to the vertices, edges or both of a graph with certain conditions. The concept of square harmonic mean labeling has introduced by L. S. Bebisha Lenin, M. Jaslin Melbha [1]. We cite J. A. Gallian [2] for an extensive examination of graph labeling. S. Somasundaram and R. Ponraj [4] are the ones who proposed the idea of mean labeling. M. Tamilselvi and N. Revathi introduced k-harmonic mean labeling [7]. The aforementioned studies served as our inspiration as we introduced k- square harmonic mean labeling of ladder related graphs. We will provide a brief summary of definitions and other information's which are necessary for our present investigation.

Definition 1.1. Let $G = (V, E)$ be a graph having p vertices and q edges. A function h is called a *square harmonic mean labeling* of a graph G, if $h : V(G) \to \{1, 2, \dots, q+1\}$ is injection and the induced edge function $h^* : E(G) \to \{1, 2, \dots, q\}$ defined as $h^*(e = uv) = \left\lceil \frac{2h(u)^2 h(v)^2}{h(u)^2 + h(v)^2} \right\rceil$ or $\left\lfloor \frac{2h(u)^2 h(v)^2}{h(u)^2 + h(v)^2} \right\rfloor$ is bijective. A graph which admits a square harmonic mean labeling is called a *square harmonic mean graph.*

Definition 1.2. A graph obtained from a ladder with $n > 2$ by removing the edges $u_i v_i$ for $i = 1$ and n is called an *open ladder* and is denoted by OL_n.

Definition 1.3. A graph obtained from two paths $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ by joining each u_i with v_{i+1} for $1 \le i \le n-1$ is called a *slanting ladder* and is denoted by SL_n.

Definition 1.4. Let P_n be a path on n vertices denoted by $(1,1), (1,2), \dots, (1, n)$ and with $n-1$ edges denoted by $e_1, e_2, \dots, e_{n-1}$ where e_i is the edge joining the vertices $(1, i)$ and $(1, i+1)$. On each edge $e_i, 1 \le i \le n-1$, we erect a ladder with $n - (i-1)$ steps including the edge e_i. The graph obtained is called a *step ladder graph* and is denoted by $S(T_n)$.

Definition 1.5. A graph obtained from a ladder by adding the edges $u_i v_{i+1}$ for $1 \le i \le n-1$, where u_i and v_i are the vertices of ladder such that $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ are two paths of length n is L_n is called a *triangular ladder* and is denoted by $TL_n, n \ge 2$.

Definition 1.6. A graph obtained from an open ladder by joining each u_i with v_{i+1} for $1 \leq i \leq n-1$ and each u_{i+1} with v_{i+1} for $1 \leq i \leq n-2$ is called an *open triangular ladder* and is denoted by $O(TL_n)$.

Definition 1.7. A graph obtained from a ladder by joining each u_i with v_{i+1} for $1 \leq i \leq n-1$ and u_{i+1} with v_i for $1 \leq i \leq n-1$ is called *diagonal ladder* and is denoted by DL_n.

Definition 1.8. A graph obtained from a diagonal ladder by removing the edges $u_i v_I$ for $i = 1$ and n joining each u_i with v_{i+1} for $1 \leq i \leq n-1$ and each u_{i+1} with v_i for $1 \leq i \leq n-2$ is called an *open diagonal ladder* and is denoted by $O(DL_n)$.

Definition 1.9. A graph obtained from the ladder by joining the opposite end points of the two copies of P_n is called *mobius ladder* and is denoted by M_n.

2. Main Results

Theorem 2.1. An open ladder $O(L_n)$ admits a k-square harmonic mean graph for all k.

Proof. Let $G = O(L_n)$ be an open ladder. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the two paths of length n in L_n. Join $u_{\alpha+1}$ with $v_{\alpha+1}, 1 \leq \alpha \leq n-2$ respectively. Let $V(G) = \{u_\alpha, v_\alpha : 1 \leq \alpha \leq n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, v_\alpha v_{\alpha+1}, : 1 \leq \alpha \leq n-1\} \cup \{u_{\alpha+1} v_{\alpha+1} : 1 \leq \alpha \leq n-2\}$. Then $|V(G)| = 2n$ and $|E(G)| = 3n-4$. A function $h : V(G) \rightarrow \{k, k+1, k+2, \ldots, k+q\}$ is defined by $h(u_1) = k$, $h(u_\alpha) = k + 3\alpha - 4, 2 \leq \alpha \leq n$, $h(v_1) = k+1$, $h(v_\alpha) = k + 3\alpha - 3, 2 \leq \alpha \leq n-1$, $h(v_n) = k + 2n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = k + 3\alpha - 3, 1 \leq \alpha \leq n-1$, $h^*(v_\alpha v_{\alpha+1}) = k + 3\alpha - 2, \quad 1 \leq \alpha \leq n-1, h^*(u_{\alpha+1} v_{\alpha+1}) = k + 3\alpha - 1, 1 \leq \alpha \leq n-2$. Thus h^* is bijective. Therefore, $O(L_n)$ admits a k-square harmonic mean graph for all k.

Illustration 2.2. The image below displays a 40-square harmonic mean labeling of $O(L_5)$.

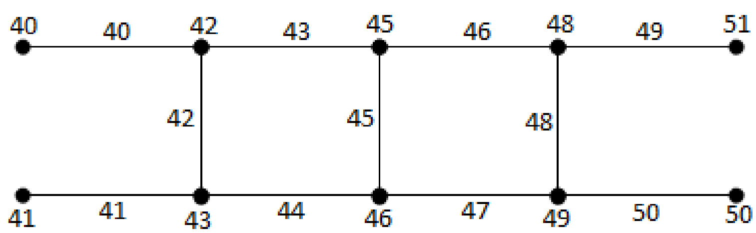

Figure I. $O(L_5)$

Theorem 2.3. A Slanting ladder SL_n admits a k-square harmonic mean graph for all k.

Proof. Let $G = SL_n$ be a slanting ladder. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the two paths of length n. Join u_α with $v_{\alpha+1}, 1 \leq \alpha \leq n-1$ respectively. Let $V(G) = \{u_\alpha, v_\alpha : 1 \leq \alpha \leq n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, v_\alpha v_{\alpha+1}, u_\alpha v_{\alpha+1} : 1 \leq \alpha \leq n-1\}$. Then $|V(G)| = 2n$ and $|E(G)| = 3n-3$. A function $h : V(G) \rightarrow \{k, k+1, k+2, \ldots, k+q\}$ is defined by $h(u_1) = k$, $h(u_\alpha) = k + 3\alpha - 3, 2 \leq \alpha \leq n$, $h(v_1) = k+1$, $h(v_\alpha) = k + 3\alpha - 4, \quad 2 \leq \alpha \leq n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = k + 3\alpha - 3, \quad 1 \leq \alpha \leq n-1$, $h^*(v_\alpha v_{\alpha+1}) = k + 3\alpha - 2, 1 \leq \alpha \leq n-1, \quad h^*(u_\alpha v_{\alpha+1}) = k + 3\alpha - 1, \quad 1 \leq \alpha \leq n-1$. Thus h^* is bijective. Therefore, SL_n admits a k-square harmonic mean graph for all k.

Illustration 2.4. The image below displays a 10-square harmonic mean labeling of SL_5.

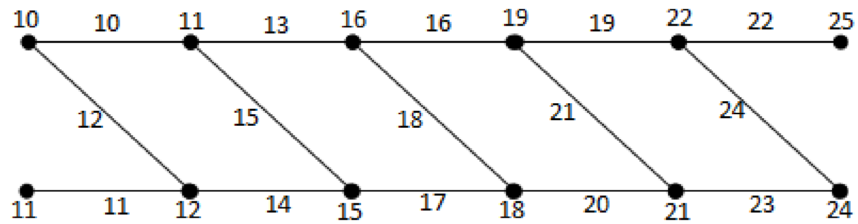

Figure II. SL_5

Theorem 2.5. The Step ladder $S(T_n)$ admits a k-square harmonic mean graph for all k.

Proof. Let $G = S(T_n)$ be the given step ladder. Let P_n be a path on n vertices denoted by $(1,1),(1,2),\dots,(1,n)$ with $n-1$ edges denoted by $e_1,e_2,\dots,e_{n-1}$ where e_α is the edge joining the vertices $(1,\alpha)$ and $(1,\alpha+1)$. The step ladder has vertices denoted by $V(G) = \{(1,1),\dots,(1,n),(2,1),\dots,(2,n),(3,1),\dots,(3,-1),\dots,(n,1),(n,2)\}$. A function $h: V(G) \to \{k, k+1, k+2, \dots, k+q\}$ is defined by $h(\alpha,1) = k + n^2 + \alpha - 2$; $1 \le \alpha \le n$, $h(1,\beta) = k + (n-\beta+1)^2 - 1$; $2 \le \beta \le n$, $h(\alpha,\beta) = k + (n-\beta+1)^2 + \alpha - 2$; $2 \le \alpha \le n, 2 \le \beta \le n-\beta+2$. The corresponding induced edge labels are
$h^*((\alpha,1),(\alpha+1,1)) = k + n^2 + \alpha - 2\,; 1 \le \alpha \le n-1$,
$h^*((1,\beta),(1,\beta+1)) = k + (n-\beta)(n-\beta+1) - \alpha - 1\,; 1 \le \beta \le n-1$,

$h^*((\alpha,\beta),(\alpha,\beta+1)) = k + (n-\beta)(n-\beta+1) + \alpha - 2\,; 2 \le \alpha \le n, 1 \le \beta \le n-\beta+1$,
$h^*((\alpha,\beta),(\alpha+1,\beta)) = k + (n-\beta+1)^2 + \alpha - 2\,; 2 \le \beta \le n, 1 \le \alpha \le n-\beta+1$. Thus h^* is bijective. Therefore, $S(T_n)$ admits a k-square harmonic mean graph.

Illustration 2.6. The image below displays a 100-square harmonic mean labeling of $S(T_7)$.

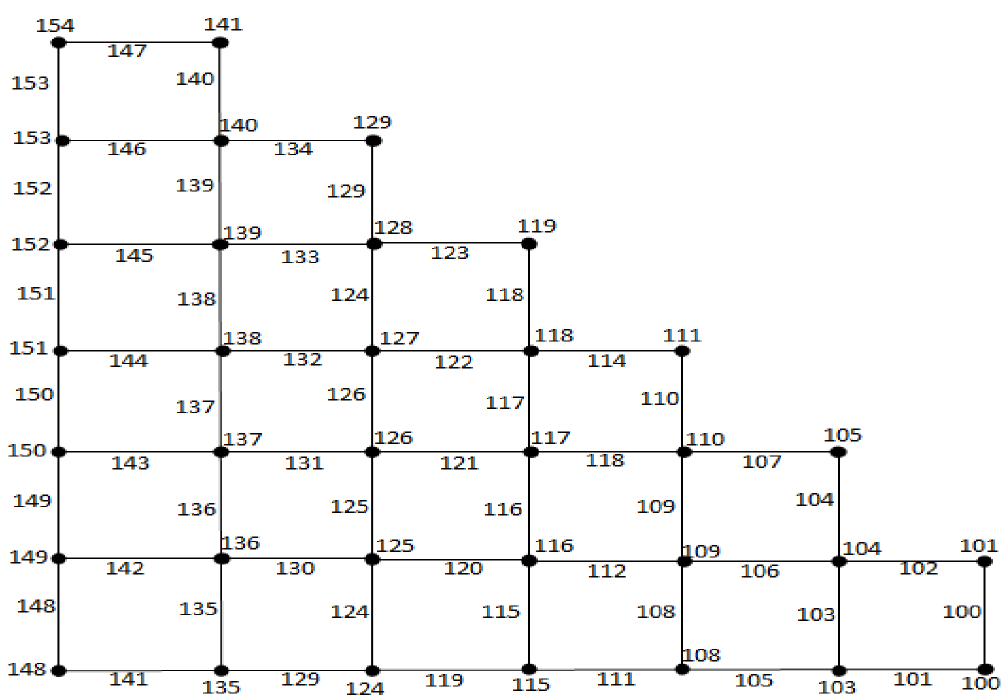

Figure III. $S(T_7)$

Theorem 2.7. The Triangular ladder TL_n admits a k-square harmonic mean graph for all k and $n \geq 2$.

Proof. Let $G = TL_n$ be a triangular ladder. Let $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ be the two paths of length n in L_n. Join u_α with $v_{\alpha+1}, 1 \leq \alpha \leq n-1$ respectively. Let $V(G) = \{u_\alpha, v_\alpha : 1 \leq \alpha \leq n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, v_\alpha v_{\alpha+1}, u_\alpha v_{\alpha+1} : 1 \leq \alpha \leq n-1\} \cup \{u_\alpha v_\alpha : 1 \leq \alpha \leq n\}$. Then $|V(G)| = 2n$ and $|E(G)| = 4n-3$. A function $h : V(G) \rightarrow \{k, k+1, k+2, \dots, k+q\}$ is defined by $h(u_1) = k$, $h(u_\alpha) = k + 4\alpha - 5$, $2 \leq \alpha \leq n$, $h(v_\alpha) = k + 4\alpha - 3, 1 \leq \alpha \leq n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = k + 4\alpha - 3, 1 \leq \alpha \leq n-1$, $h^*(v_\alpha v_{\alpha+1}) = k + 4\alpha - 1, 1 \leq \alpha \leq n-1$, $h^*(u_\alpha v_{\alpha+1}) = k + 4\alpha - 2, 1 \leq \alpha \leq n-1$, $h^*(u_\alpha v_\alpha) = k + 4\alpha - 4, 1 \leq \alpha \leq n$. Thus h^* is bijective. Therefore, TL_n admits a k-square harmonic mean graph for all k and $n \geq 2$.

Illustration 2.8. The image below displays a 400-square harmonic mean labeling of TL_6.

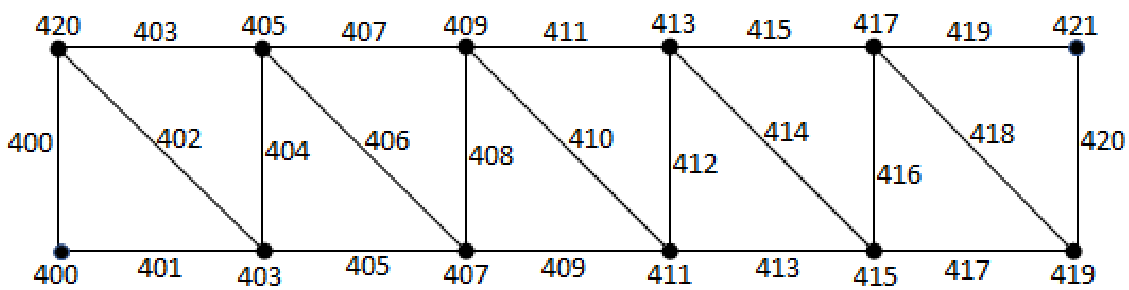

Figure IV. TL_6

Theorem 2.9. An open triangular ladder $O(TL_n)$ admits a k-square harmonic mean graph for all k

Proof. Let $G = O(TL_n)$ be an open triangular ladder. Let $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ be the two paths of length n. Join u_α with $v_{\alpha+1}, 1 \leq \alpha \leq n-1$ and $u_{\alpha+1}$ with $v_{\alpha+1}$, $1 \leq \alpha \leq n-2$ respectively. Let $V(G) = \{u_\alpha, v_\alpha : 1 \leq \alpha \leq n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, v_\alpha v_{\alpha+1}, u_\alpha v_{\alpha+1} : 1 \leq \alpha \leq n-1\} \cup \{u_{\alpha+1} v_{\alpha+1} : 1 \leq \alpha \leq n-2\}$. Then $|V(G)| = 2n$ and $|E(G)| = 4n-5$. A function $h : V(G) \rightarrow \{k, k+1, k+2, \dots, k+q\}$ is defined by $h(u_1) = k$, $h(u_\alpha) = k + 4\alpha - 5, 2 \leq \alpha \leq n$, $h(v_1) = k+1$, $h(v_\alpha) = k + 4\alpha - 6, 2 \leq \alpha \leq n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = k + 4\alpha - 4, 1 \leq \alpha \leq n-1$, $h^*(v_\alpha v_{\alpha+1}) = k + 4\alpha - 2, 1 \leq \alpha \leq n-1$, $h^*(u_\alpha v_{\alpha+1}) = k + 4\alpha - 3, 1 \leq \alpha \leq n-1$, $h^*(u_{\alpha+1} v_{\alpha+1}) = k + 4\alpha - 1, 1 \leq \alpha \leq n-2$. Thus h^* is bijective. Therefore, $O(TL_n)$ admits a k-square harmonic mean graph for all k.

Illustration 2.10. The image below displays a 140-square harmonic mean labeling of $O(TL_7)$.

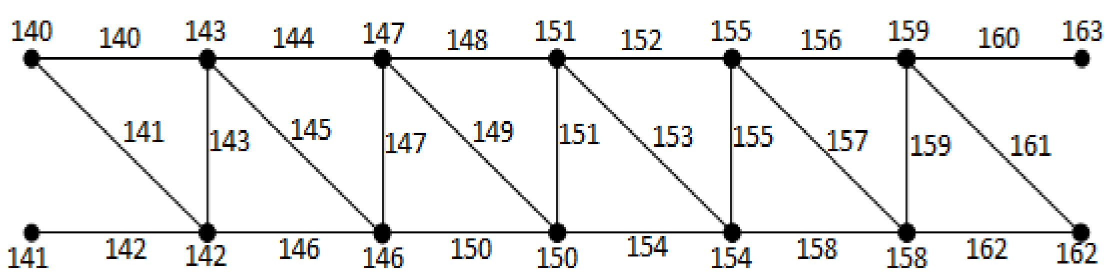

Figure V. $O(TL_7)$

Theorem 2.11. The diagonal ladder DL_n admits a k-square harmonic mean graph for all k.

Proof. Let $G = DL_n$ be a diagonal ladder. Let $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ be the two paths of length n. Join u_α with $v_{\alpha+1}, 1 \le \alpha \le n-1$ and $u_{\alpha+1}$ with $v_\alpha, 1 \le \alpha \le n-1$ respectively. Let $V(G) = \{u_\alpha, v_\alpha : 1 \le \alpha \le n\}$ and $E(G) = \{u_\alpha v_\alpha : 1 \le \alpha \le n\} \cup \{u_\alpha u_{\alpha+1}, v_\alpha v_{\alpha+1}, u_\alpha v_{\alpha+1}, u_{\alpha+1} v_\alpha : 1 \le \alpha \le n-1\}$. Then $|V(G)| = 2n$ and $|E(G)| = 5n - 4$. A function $h : V(G) \to \{k, k+1, k+2, \dots, k+q\}$ is defined by $h(u_1) = k$, $h(u_\alpha) = k + 5\alpha - 4, 2 \le \alpha \le n$, $h(v_1) = k + 1$, $h(v_\alpha) = k + 5\alpha - 5,\ 2 \le \alpha \le n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = k + 5\alpha - 4,\ 1 \le \alpha \le n-1$, $h^*(v_\alpha v_{\alpha+1}) = k + 5\alpha - 3, 1 \le \alpha \le n-1$, $h^*(u_\alpha v_{\alpha+1}) = k + 5\alpha - 1,\ 1 \le \alpha \le n-1$, $h^*(u_\alpha v_\alpha) = k + 5\alpha - 5, 1 \le \alpha \le n$, $h^*(u_{\alpha+1} v_\alpha) = k + 5\alpha - 2,\ 1 \le \alpha \le n-1$. Thus h^* is bijective. Therefore, DL_n admits a k-square harmonic mean graph for all k.

Illustration 2.12. The image below displays a 90-square harmonic mean labeling of DL_4.

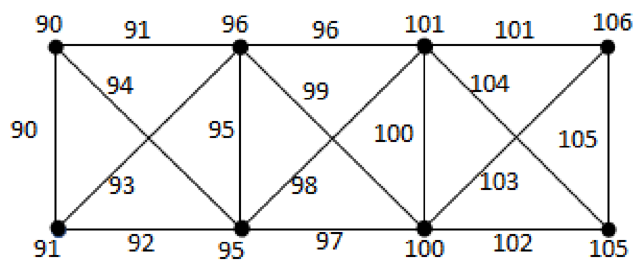

Figure VI. DL_4

Theorem 2.13. An open diagonal ladder $O(\mathrm{DL}_n)$ admits a k-square harmonic mean graph for all k.

Proof. Let $G = O(DL_n)$ be an open diagonal ladder. Let $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ be the two paths of length n. Join u_α with $v_{\alpha+1}, 1 \le \alpha \le n-1$, $u_{\alpha+1}$ with $v_\alpha, 1 \le \alpha \le n-1$ and $u_{\alpha+1}$ with $v_{\alpha+1}, 1 \le \alpha \le n-2$ respectively. Let $V(G) = \{u_\alpha, v_\alpha : 1 \le \alpha \le n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, v_\alpha v_{\alpha+1}, u_\alpha v_{\alpha+1}, u_{\alpha+1} v_\alpha : 1 \le \alpha \le n-1\} \cup \{u_{\alpha+1} v_{\alpha+1} : 1 \le \alpha \le n-2\}$. Then $|V(G)| = 2n$ and $|E(G)| = 5n - 6$. A function $h : V(G) \to \{k, k+1, k+2, \dots, k+q\}$ is defined by $h(u_1) = k$, $h(u_\alpha) = k + 5\alpha - 6, 2 \le \alpha \le n$, $h(v_1) = k + 1$, $h(v_\alpha) = k + 5\alpha - 7, 2 \le \alpha \le n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = k + 5\alpha - 5, 1 \le \alpha \le n-1$, $h^*(v_\alpha v_{\alpha+1}) = k + 5\alpha - 3, 1 \le \alpha \le n-1$, $h^*(u_\alpha v_{\alpha+1}) = k + 5\alpha - 4, 1 \le \alpha \le n-1$, $h^*(u_{\alpha+1} v_{\alpha+1}) = k + 5\alpha - 1, 1 \le \alpha \le n-2$, $h^*(u_{\alpha+1} v_\alpha) = k + 5\alpha - 2, 1 \le \alpha \le n-1$. Thus h^* is bijective. Therefore, $O(DL_n)$ admits a k-square harmonic mean graph for all k.

Illustration 2.14. The image below displays a 150-square harmonic mean labeling of $O(DL_6)$.

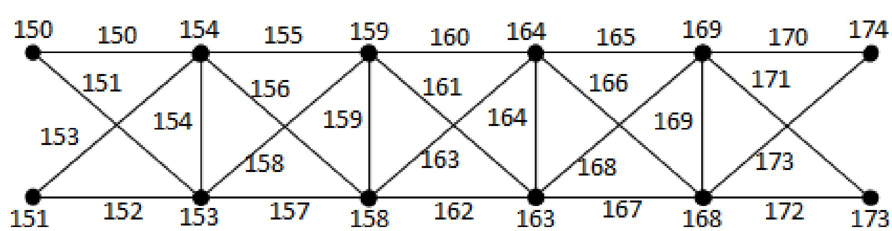

Figure VII. $O(\mathrm{DL}_6)$

Theorem 2.15. A mobius ladder M_n admits a k-square harmonic mean graph for all k.

Proof. Let $G = M_n$ be a mobius ladder obtained from the ladder $P_n \times P_2$ by joining the opposite end points of the two copies of P_n. Let $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ be the two copies of path P_n. Join u_α with v_α, $1 \le \alpha \le n$. Let $V(G) = \{u_\alpha, v_\alpha : 1 \le \alpha \le n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, v_\alpha v_{\alpha+1}, u_\alpha v_\alpha, u_n v_n : 1 \le \alpha \le n-1\} \cup \{u_n v_1, v_n u_1\}$. Then $|V(G)| = 2n$ and $|E(G)| = 3n$. A function $h : V(G) \to \{k, k+1, k+2, \dots, k+q\}$ is defined by $h(u_1) = k$, $h(u_\alpha) = k + 3\alpha, 2 \le \alpha \le n$, $h(v_1) = k+1$, $h(v_\alpha) = k + 3\alpha - 2, 2 \le \alpha \le n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = k + 3\alpha - 1, 1 \le \alpha \le n-1$, $h^*(v_\alpha v_{\alpha+1}) = k + 3\alpha - 2, 1 \le \alpha \le n-1$, $h^*(u_\alpha v_\alpha) = k + 3\alpha - 3, 1 \le \alpha \le n$, $h^*(v_n u_1) = k + 3n - 2$, $h^*(u_n v_1) = k + 3n - 1$. Thus h^* is bijective. Therefore, M_n admits a k-square harmonic mean graph for all k.

Illustration 2.16. The image below displays a 500-square harmonic mean labeling of M_8.

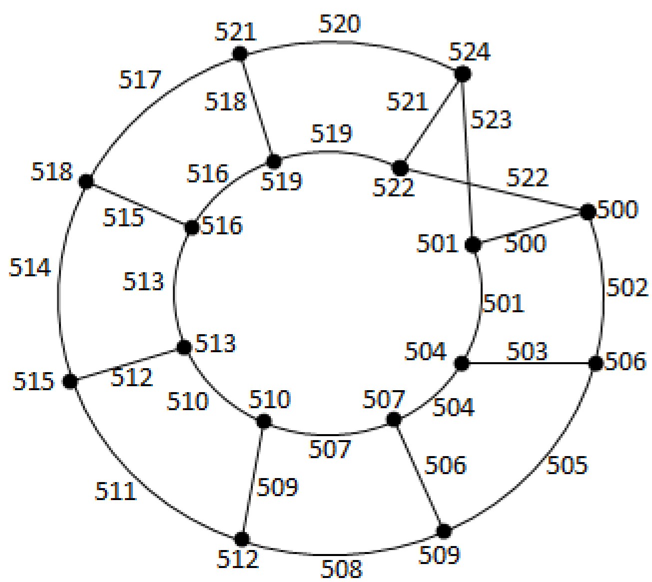

Figure VIII. M_8

REFERENCES

[1] L. S. Bebisha Lenin and M. Jaslin Melbha, "Square Harmonic Mean Labeling of Simple Graphs", European Chemical Bulletin, Volume 12 (S3), (2023), 5472-5479.

[2] J.A. Gallian, "A Dynamic Survey of Graph Labeling", The Electronic Journal of Combinatorics,17# DS6, 2014.

[3] F. Harary, "Graph Theory", Narosa Publishing House, New Delhi, 1988.

[4] S. Somasundaram, R. Ponraj, "Mean Labeling of Graphs", National Academy of Science Letter, Volume 26, (2003), 210-213.

[5] S. S. Sandhya, S. Somasundaram and R. Ponraj, "Some Results on Harmonic Mean Graphs", International Journal of Contemporary Mathematical Sciences, Volume 7 (4), (2012), 197-208.

[6] S. S. Sandhya, S. Somasundaram and R. Ponraj, "Some More Results on Harmonic Mean Graphs", Journal of Mathematics Research, Volume 4 (1), (2012), 21-29.

[7] M. Tamilselvi and N. Revathi, "k- harmonic mean labeling of some graphs", International Journal of Mathematics Trends and Technology, Volume 5 (14), (2017), 216-222.

[8] M. Tamilselvi and N. Revathi, "k- harmonic mean labeling of some special graphs", International Journal of Scientific Research, Volume 7 (14), (2018), 458-465.