ISSN: 2752-3829 Vol. 3 No.2, (December, 2023)

*Stochastic Modelling and Computational Sciences*

# WEAKLY PERIODIC POINTS AND NILPOTENT SYSTEM FOR SPECIFIC LINEAR CELLULAR AUTOMATA

**Praveen Kumar Lagisetty[1] and Dr Vajha Srinivasa Kumar[2]**

[1]Research Scholar, Mathematics JNTUH College of Engineering, JNTU, Kukatpalle, Hyderabad, Telangana, India

[2]Assistant Professor, Mathematics JNTUH College of Engineering Hyderabad, JNTU, Kukatpalle, Hyderabad, Telangana, India

[1]praveenkumarl55@gmail.com and [2]vajhasrinu@gmail.com

## ABSTRACT

*Initially Cellular Automata were used in application of life as the easy structures. Later on researchers applied for fast calculations for the machines. These fast calculations and easy understanding of the complex structures were attracted by all the branches like physics. In which large problems were solved easily using Cellular Automata. It has relation with all the branches of sciences as it can be viewed as infinite array in one dimension. But here we discuss the mathematical aspects only. After introducing the space time in Cellular Automata it was classified into several concepts and scientist were interested in finding new theorem as it was not only sticked to one subject. They have been proving more results inch by inch in sub concepts like Periodicity, chaos as it is extending its relation to all topics. Cellular Automata defined in two ways as local rule and shift based and later proved that both are equivalent. Mathematically we use local rule approach as it has possibility of giving infinite number of examples on a finite set. Shift based definitions have advantages in topological and measure theoretical concept. Periodicity plays an important role it has number of applications like solar system. So far, we are unable to characterise periods of additive cellular automata on finite modulo set except prime number modulo. Weakly periodic points also defined in the same as periodic concepts but here it is the combination of CA and shift map with Integer exponents. In This paper we determine which configurations of additive group of two sided infinite sequences are weakly periodic points for the specified linear(additive)cellular automata and determine given linear cellular automata is nilpotent system or not. We also find the periodic points for the cellular automata in product form and the radius is smallest . Weakly periodic points are used in determining continuous directions and almost equicontinuous directions sets and in determination on expansive Cellular Automata.*

*Keywords: Dynamical systems, Linear cellular automata, weakly periodic points, nilpotent system.*

**AMS Subject Classification**: 37B15, 37C25

## 1. INTRODUCTION

Cellular automata is new branch of mathematics and computer science initiated by Von Neumann in fifties [1] later its applications are extended to all branches. In 1969 Hedlund [ 2] used topological properties in the theory of Cellular Automata. In the year 1984 Wolfram [3] used space time in one dimensional cellular automata and divided it into several parts like, chaos, stability and periodicity. Many authors described the behaviour of periodicity on several types of dynamical systems like interval maps [ 4], on linear operators [5], on toral automorphism [6]. In [7] The author characterised the periods of linear cellular automata on prime modulo. Weakly periodic points are different type of periodic points of dynamical Systems, which are very useful in describing equicontinuous directions and almost equicontnuous directions [8].

In this paper We are going to find

$$S_{(p,q)} = \{ x \in A^{\mathbb{Z}} : F^q \sigma^p(x) = x \} \quad (1)$$

gives weakly periodic configurations of $A^{\mathbb{Z}}$ with period $(p,q)$ where $p \in \mathbb{Z}, q \in \mathbb{N}$.

A Nilpotent system is a Cellular Automata $(A^{\mathbb{Z}}, F)$ and for $n > 0$ ,$F^n(A^{\mathbb{Z}})$ is a singleton (2)

For the linear cellular automata in the specific form for $A = \{0,1,2, \dots\dots m-1\}$

where $m \geq 2$ is and

$$F(x)_i = a x_{i+r} \pmod m \tag{3}$$

For $a \in \mathbb{Z}$ and $\sigma$ is a shift map which we are going to define.

## 2. Preliminaries

**2.1 Definition:**
*Topological Dynamical system* is a pair $(X, f)$ with $X$ being a topological space and continuous selfmap $f$ on $X$

We define basic preliminaries by assuming $(X, f)$ is a dynamical system.

**2.2 Definition:**
A point $x \in X$ is a *periodic point* if there exist a least positive integer $n$ such that $f^n(x) = x$

Then $x$ is a periodic point and its period is $n$.

**2.3 Definition:**
For
$A = \{0,1,2, \dots\dots m-1\}$

Is called alphabet for $m \geq 1$ and elements are called symbols

Let $\Sigma = A^{\mathbb{Z}}$ is the additive group of two sided infinite sequences modulo $m$

Given a two-sided infinite sequence $x \in A^{\mathbb{Z}}$, $x = (x_n)$, let $\sigma(x)$ be the two sided infinite

Sequence given by $\sigma(x_i) = x_{i+1}$ (4)

Here $\sigma$ is the continuous self map on $A^{\mathbb{Z}}$ and $(A^{\mathbb{Z}}, \sigma)$ is dynamical system called *Shift map*[9]. For the shift map we can get $x \in A^{\mathbb{Z}}$ *such that* $\sigma^p(x) = x$ for every $p \in \mathbb{Z}$. We write $x = u^\infty = ..uuuu..$ , where $u$ is word of length $p \in \mathbb{Z}^+$ with symbols from alphabet $A$ .[9]

If alphabet $A = \{0,1,2,3\}$ then $u = 012213$ is a word of length 6 from the alphabet $A$

**2.4 Definition:**
A dynamical system in the form of $(A^{\mathbb{Z}}, F)$ is said to be *cellular Automata* if it is commute with shift map as

$F \circ \sigma = \sigma \circ F$

It can also be defined as $F: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a map and

$r \in \mathbb{N}$ (radius) and a local rule

$f: A^{2r+1} \to A$ such that

$F(x)_i = f(x_{i-r}, \dots . x_0 \dots \dots . x_{i+r})$

For each $x \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$.

Here $x_i$ means the $i^{th}$ position of configuration $x \in A^{\mathbb{Z}}$.

Both the definitions are equivalent by Hedlund [2].

**2.5 Definition:**

We define *Linear Cellular Automata* as

$F(x) = \sum_{r=-k}^{r=k} a_i x_{i+r} \ (mod\ m')$ (5)

Where $F: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a map and $A = \{0,1,2 \ldots \ldots . m-1\} \ (mod\ m)$ for $m \geq 2$

For some natural number $m' \leq m$ and fixed $k \geq 1$ and fixed integers $a_i$.

Generally consider $m' = m$.

For Example,

$A = \{0,1,2,3\}(mod\ 4)$

Define $F: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ as

$F(x)_i = 2x_i - 3x_{i+1}(mod\ 4)$

We consider shorter form of the linear cellular automaton and determine weakly periodic points and determine possible Nilpotent systems.

**2.6 Definition:**

Euler function $\phi(m)$ denotes number of natural numbers less than $m$ and relatively prime (co prime) to $m$.

Generalisation to Fermat theorem is $a^{\phi(m)} \equiv 1 \ (mod\ m)$ where $a, m$ are relatively prime [10]. Fermat little theorem is same as Euler's theorem for prime $m$.

**3. Main problem:**

In this paper we are going to find the weakly periodic points as we defined in (1) and periods of those configurations and to determine whether the dynamical system defined by (3) is Nilpotent or not.

Weakly periodic points play a important role in signal subshift and speed of a subshift which are useful to find the Entropy of a Cellular Automata [11].

For the following theorems consider $(A^{\mathbb{Z}}, F)$ is linear Cellular Automat.

**Theorem 3.1**. Finding Weakly periodic points $\mathcal{S}_{(p,q)}$ and their periods for Linear Cellular Automata defined as $F(x)_i \equiv ax_i \ (mod\ m)$, where $a \equiv 1 \ (modm)$.

Proof. As $a \equiv 1 \ (modm)$

The Function becomes $F(x)_i \equiv x_i \ (mod\ m)$, which is identity map.

Weakly periodic points are those $x \in A^{\mathbb{Z}}$, $F^q \sigma^p(x) \equiv x$ for $p \in \mathbb{Z}, q \in \mathbb{N}$.

as $F$ is identity map $F^q(x) \equiv x$ for all $x \in A^{\mathbb{Z}}$

$\Rightarrow F^q(\sigma^p(x))_i \equiv \sigma^p(x)_i(mod\ m)$

Then weakly periodic points are satisfying $\sigma^p(x)_i = x_i(mod\ m)$

For shift map we get $x \in A^{\mathbb{Z}}$ for every $p \in \mathbb{Z}$ satisfying $\sigma^p(x)_i = x_i(mod\ m)$ [9]

In configuration of $x$ contains words $u$ of length $|p|$ repeated infinitely.

Here $x = u^{\infty}$ and $u$ is word of length $|p|$.

$\mathcal{S}_{(p,q)} = \{u^{\infty}\}$

Periods = $\{(p, 1): p \in \mathbb{Z}\}$.

That is any word $u$ of length $p$ has period $(p, 1)$.

In the shift map, configuration of $x \in A^{\mathbb{Z}}$ is same for $\sigma^{p}(x) = x$ and $\sigma^{-p}(x) = x$

**Theorem 3.2**. Finding Weakly periodic points $\mathcal{S}_{(p,q)}$ and their periods for Linear Cellular Automata defined as $F(x)_i \equiv ax_i \ (mod\ m)$, where $a \equiv -1 \ (modm)$.

Proof: As $a \equiv -1 \ (modm)$ then $F(x)_i \equiv -x_i \ (mod\ m)$

$F^2(x)_i \equiv x_i \ (mod\ m)$ and $F^3(x)_i \equiv -x_i \ (mod\ m)$.

We can write

$F^2(\sigma^p(x))_i \equiv \sigma^p(x)_i (mod\ m)$

By the theorem 3.1 we can go for the same procedure as shift map has periodic points of every period $p$.

Here $x = u^{\infty}$ and $u$ is word of length $|p|$.

$\mathcal{S}_{(p,q)} = \{u^{\infty}\}$

Periods = $\{(p, 2): \ p \ \in \mathbb{Z}\}$.

**Theorem 3.3**. Finding Weakly periodic points $\mathcal{S}_{(p,q)}$ and their periods for Linear Cellular Automata defined as $F(x)_i \equiv ax_i \ (mod\ m)$, where $a$ is idempotent element.

Proof: $a \in \mathbb{Z} \ (mod\ m)$ is idempotent element means $a^2 \equiv a \ \ (mod\ m)$.

Then $F^2(x)_i \equiv a^2 x_i \ \equiv ax_i(mod\ m)$ and $F^3(x)_i \equiv a^3 x_i \ \equiv a^2 ax_i \equiv ax_i(mod\ m)$.

So, for every natural number $q$, $F^q(x)_i \ \equiv ax_i(mod\ m)$

To get the weakly periodic points we find solution for $F^q(\sigma^p(x))_i \equiv x_i(mod\ m)$

$\Rightarrow a\sigma^p(x)_i \equiv x_i(mod\ m)$

$\Rightarrow ax_{i+p} \equiv x_i(mod\ m)$

Case1: If $a \equiv 0 \ (mod\ m)$ then the given function is zero map then the only weakly periodic point is zero sequence $0^{\infty} \ \in A^{\mathbb{Z}}$ of period $(1,0)$

Case 2 : If $a \equiv 1 \ (mod\ m)$ then the function becomes identity map and same as theorem 3.1.

Case 3: If $a \not\equiv 0 \ (mod\ m)$ and $a \not\equiv 1 \ (mod\ m)$ then it does not have any other solution except zero sequence $0^{\infty} \ \in A^{\mathbb{Z}}$ of period $(1,0)$

**Theorem 3.4**. Finding Weakly periodic points $\mathcal{S}_{(p,q)}$ and their periods for Linear Cellular Automata defined as $F(x)_i \equiv ax_i(mod\ m)$, where $a$ is nilpotent element.

Proof: By the definition of nilpotent element there exist some natural number $n$ such that $a^n \equiv 0 \ (mod\ m)$.

To get the weakly periodic if we choose the above $n$ then

$F^n(\sigma^p(x))_i \equiv x_i(mod\ m)$

$\Rightarrow a^n \sigma^p(x)_i \equiv x_i(mod\ m)$

$\Rightarrow 0 \equiv x_i(mod\ m)$

$\Rightarrow x = 0^{\infty} \ \in A^{\mathbb{Z}}$

In this case we get zero sequence is the only weakly periodic point of period $(1,0)$.

**Theorem 3.5**. Finding Weakly periodic points $\mathcal{S}_{(p,q)}$ and their periods for Linear Cellular Automata defined as $F(x)_i \equiv ax_i(mod\ m)$, $a$ and $m$ are relatively prime $(a, m) = 1$.

Proof: If $a$ and $m$ are relatively prime then by the Generalised Fermat theorem $a^{\phi(m)} \equiv 1\ (mod\ m)$

where $\phi(m)$ is the number of natural numbers less than $m$ and relatively prime to it.

We have $F(x)_i \equiv ax_i(mod\ m)$ and $(a, m) = 1$.

To get weakly periodic points

$F^{\phi(m)}(\sigma^p(x))_i \equiv x_i(mod\ m)$

$\Longrightarrow a^{\phi(m)}\sigma^p(x)_i \equiv x_i(mod\ m)$

$\Longrightarrow 1.\sigma^p(x)_i \equiv x_i(mod\ m)$

So, It is true for $x = u^\infty \in A^Z$, here $u$ is word of length $|p| \in \mathbb{Z}$.

$\mathcal{S}_{(p,q)} = \{u^\infty\}$

Periods = $\{(p, \phi(m)) \colon p \in \mathbb{Z}\}$.

If $m$ is a prime number then $\phi(m) = m - 1$ then weakly periodic points are

$\mathcal{S}_{(p,q)} = \{u^\infty\}$

Periods = $\{(p, m - 1) \colon p \in \mathbb{Z}\}$.

**Theorem 3.6** Determine the Nilpotent system $(A^{\mathbb{Z}}, F)$ if it is in the form of $F(x)_i \equiv ax_i(mod\ m)$

Proof: By the definition of Nilpotent system $F^n(A^{\mathbb{Z}})$ is singleton for some $n > 0$.

For every $x \in A^Z$, $F^n(x)_i \equiv k_i(mod\ m)$ for every $i \in \mathbb{Z}$ and $k = (k_i) \in A^Z$ be a fixed singleton.

$\Longrightarrow a^n x_i \equiv k_i(mod\ m)$

It has to be true for every $x \in A^Z$ . The possibilities are $a \equiv 0\ (mod\ m)$ or $a^n \equiv 0\ (mod\ m)$ for $k = 0^\infty$

$(A^{\mathbb{Z}}, F)$ is nilpotent system if $a \equiv 0\ (mod\ m)$ or $a^n \equiv 0\ (mod\ m)$ for the Linear Cellular Automata is the form $(A^{\mathbb{Z}}, F)$ .

## 4. Periodic points

In This section we are going to find the periods for the cellular automata of the form

$F: A^{\mathbb{Z}} \times A^{\mathbb{Z}} \to A^{\mathbb{Z}} \times A^{\mathbb{Z}}$

With radius zero and $A = \{0,1,2 \ldots . m - 1\}(mod\ m)$

**Theorem 4.1:** If $F(x_i, y_i) = (ax_i, bx_i)$ and $A = \{0,1,2 \ldots . p - 1\}(mod\ p)$ where $p$ is a prime number.

Proof: For any two configurations $x = (x_i), y = (y_i) \in A^{\mathbb{Z}}$ are two infinite sequence with symbols form $A$, then

$F^2(x_i, y_i) = (a^2 x_i, b^2 y_i)$

Keep on doing we get $F^{p-1}(x_i, y_i) = (a^{p-1}x_i, b^{p-1}y_i) = (x_i, y_i)$.

Here $a^{p-1} \equiv 1(mod\ p)$ and $b^{p-1} \equiv 1\ (mod\ p)$ by the Fermat's little theorem as $a, b \in A$ so $a$ and $b$ doesn't divide $p$.

We can conclude the period is $p - 1$. We can apply it to the finite product also

**Proposition 4.1:** $F: A^{\mathbb{Z}} \times A^{\mathbb{Z}} \times \dots A^{\mathbb{Z}}. \to A^{\mathbb{Z}} \times A^{\mathbb{Z}} \times \dots A^{\mathbb{Z}}$ as $F((x_1)_i, (x_2)_i \dots (x_n)_i) = ((a_i x_1)_i, (a_2 x_2)_i \dots (a_n x_n)_i, (mod\ p)$ where $p$ is a prime.

Proof: From the theorem 4.1, We can conclude it has the period $p-1$.

**Theorem 4.2: :** If $F(x_i, y_i) = (ax_i, bx_i)$ and $A = \{0,1,2 \dots . p-1\}(mod\ m)$. Where $(a, m) = 1 = (b, m)$.

Proof: By the Euler's Theorem $a^{\emptyset(m)} \equiv 1\ (mod\ m)$ and $b^{\emptyset(m)} \equiv 1\ (mod\ m)$.

So $F^{\emptyset(m)}(x_i, y_i) = (a^{\emptyset(m)} x_i, b^{\emptyset(m)} y_i) = (x_i, y_i)$.

Then the period is $\emptyset(m)$.

**Proposition 4.2:** $F: A^{\mathbb{Z}} \times A^{\mathbb{Z}} \times \dots A^{\mathbb{Z}}. \to A^{\mathbb{Z}} \times A^{\mathbb{Z}} \times \dots A^{\mathbb{Z}}$ as $F((x_1)_i, (x_2)_i \dots (x_n)_i) = ((a_1 x_1)_i, (a_2 x_2)_i \dots (a_n x_n)_i, (mod\ m)$ where $(a_i, m) = 1$ for each $i = 1,2 \dots n$.

Proof: From the Theorem 4.2 , we can conclude it has period $\emptyset(m)$.

**Theorem 4.3: :** If $F(x_i, y_i) = (ax_i, bx_i)$ and $A = \{0,1,2 \dots . p-1\}(mod\ m)$. Where $a^r = 1\ and\ b^s = 1(mod\ m)$.

Proof: Here $a^r = 1\ and\ b^s = 1(mod\ m)$. But both of these values will be at $k = lcm[r, s]$

So here the period is $k$.

**Summary:**
In this paper we have shown that weakly periodic points for Linear Cellular Automata $(A^{\mathbb{Z}}, F)$ in the form $F(x)_i \equiv ax_i(mod\ m)$ are in the form of $\{0^{\infty}, u^{\infty}\}$ where $u$ is any word of length $|p|$ from the alphabet $A = \{0,1,2 \dots m-1\}(mod\ m)$ and the possible periods are

1. $\{(p, 1):\ p\ \in \mathbb{Z}\}$ for $a \equiv 1\ (mod\ m)$
2. $\{(p, 2):\ p\ \in \mathbb{Z}\}$. for $a \equiv -1\ (mod\ m)$
3. $(1,0)$ for $a$ is idempotent element.
4. $(1,0)$ for $a$ is nilpotent.
5. $\{(p, \phi(m)):\ p\ \in \mathbb{Z}\}$ for $(a, m) = 1$

$(A^{\mathbb{Z}}, F)$ is nilpotent system if $a \equiv 0\ (mod\ m)$ or $a^n \equiv 0\ (mod\ m)$ for the Linear Cellular Automata is the form $(A^{\mathbb{Z}}, F)$ . We also found the periods in product form of cellular automata.

**REFERENCES**

[1] J. von Neumann, "The institute for advance study " in The general and logical theory of automata. In L. A. Jefferss, editor, Cerebral Mechanics of Behaviour. Wiley, New York, 1951, pp 1-41.

[2] Hedlund G.A., "Endomorphisms and automorphisms of the shift dynamical system". *Math. Systems Theory* **3**, pp. 320–375, 1969. https://doi.org/10.1007/BF01691062.

[3] S. Wolfram. Computation theory of cellular automata. Comm. Math. Phys., Vol 96, pp. 15–57, https://doi.org/10.1007/BF01217347.

[4] L.S.Block, W.A.Coppel, Dynamics in one dimension, Lecture Notes in Math- ematics, 1513, Springer, 1992, pp 230-235.

[5] K. Ali Akbar, V. Kannan, Sharan Gopal, P. Chiranjeevi, "The set of periods of periodic points of a linear operator", Linear algebra and its applications., 431, pp. 241-246. doi:10.1016/j.laa.2009.02.027

[6] V. Kannan, I. Subramania Pillai, K. Ali Akbar ,B. Sankararao, “ The set of periods of periodic points of a toral automorphism”, Topology proceedings., Vol. 37, pp 219-232. https://topology.nipissingu.ca/tp/reprints/v37/tp37014.pdf

[7] T.K. Subrahmonian Moothathu, “Set of periods of additive cellular automata,” Theoretical Computer Science, Volume 352, Issues 1–3, pp. 226-231,2006. DOI.org/10.1016/j.tcs.2005.10.050

[8] P. K°urka. “Topological dynamics” in Topological and symbolic dynamics, volume 11 of Cours sp´ecialis´es Soci´et´e Math´ematique de France, Paris, 2003, pp 60-66.

[9] R. Holmgren,” Symbolic dynamics and chaos,” A first course in discrete dynamical systems, 1 st ed, Springervelag, 1994, pp 70-90.

[10] Apostol, Tom M,”Arithmetical Functions and Dirichlet Multiplication ” in Introduction to Analytic number theory, Narosa ,1998, pp 25-27

[11] E. Formenti, P. K°urka. “A search algorithm for the maximal attractor of a cellular automaton” In Wolfgang Thomas and PascalWeil, STACS 2007, volume 4393 of Lecture Notes in Computer Science, pp 356-366 https://doi.org/10.1007/978-3-540-70918-3_31