Stochastic Modelling and Computational Sciences

ON BAYESIAN ESTIMATION OF MULTICOMPONENT SYSTEM RELIABILITY WITH NON-IDENTICAL COMPONENT STRENGTHS

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ABSTRACT

This article attempts the maximum likelihood and Bayes estimation of stress-strength reliability of a multicomponent s out of k system having non-identical component strengths which are subjected to face the random common stress. We assume that the component strengths of the system are distributed with two non-identical categories of gamma distribution and are subjected to face the common independent stress which again follow a gamma distribution. In Bayesian paradigm, tow non-informative types of priors viz. uniform and Jeffreys priors are chosen and the Bayes estimators are developed under squared error and linear-exponential loss functions. A comparative study of the proposed estimators is carried out based on simulation study employing the Markov Chain Monte Carlo approach through Metropolis-Hastings algorithm. The estimators are compared on the basis of their mean square errors and absolute biases.

Keywords: Multicomponent Stress-Strength model; Gamma distribution; S-out of-K: G system; non-identical component strengths; uniform prior, Jeffreys prior, Metropolis-Hastings algorithm.

1. INTRODUCTION

Researchers are paying close attention to the study of stress-strength reliability in multicomponent s-out of-k systems because of its general applicability in real-life scenarios. Significant amount of works has been attempted in this direction for different choices of distributions as stress-strength model. The reliability of a system depends up on the structural establishment and its inherent strength. reliability or stress-strength reliability. To study the reliability, it is always important to look at the structural establishment of the components and its operational sequence. A system is considered to be reliable, if it is strong enough to sustain imposed external loads e.g. environmental load, electrical load, pressure, temperature, etc. Then the reliability (R) of the system is defined as the probability that the strength of the system is greater than the applied stress $R = P(X > Y)$, where random variables X and Y are the strength and the random imposed stress on it, respectively.

The idea of stress-strength reliability was firstly derived by [1] and [2] for single component set-up. The formal name "stress strength" was introduced by [3]. Since then, due to applicability of the concept in different fields of day today life such as: electrical, electronic, mechanical systems to biological, medical, health service research and economic fragility models etc. (see; [4]), the literature is loaded with different assumptions on lifetime models and sampling schemes. Researchers are suggested to go through the works of [5-11] and cited references therein for deep insight.

Though the single component stress-strength reliability has wide applicability in real life scenarios, [12] noticed that a system/structure is built using combination of more than one component and each component in the system/structure has its own inherent strength to face the common stress. Therefore, the reliability and performance of a system depends upon the performance of the components. The system is known as s out of k: G system, and is assumed to consist k components having independent and identically distributed (i.i.d.) random strengths $X_1, X_2, \dots, X_k$ which are subjected to face the random stress Y. The system function as long as s out of k $(1 \leq s \leq k)$ components resist the stress. The model is defined mathematically as

$$R_{s,k} = P(at\ least\ s\ of (X_1, X_2, \dots, X_k)\ exceeds\ Y)$$

$$= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1 - F(y)]^i [F(y)]^{k-i} dG(y) \qquad (1.1)$$

where $F(.)$ is the common cdf of $X_1, X_2, \dots, X_k$ and $G(.)$ is cdf of random stress. An excellent illustration of s out of k: G system is a V-8 engine car that can only be driven while four of its eight cylinders are operating. Nonetheless, the car is said to have failed if fewer than four cylinders ignite. To gain a deeper comprehension of these models, one might look at the example of a hanging bridge, in which a sequence of k vertical cables supports the deck. Furthermore, when exposed to a common load, the bridge will only hold if at least s vertical cables remain intact (see, [13]). Other real-life examples can be viewed in [14-16].

Study of multicomponent system reliability (MSR) using the model given in (1.1) has been carried out by several authors for different choices of distributions as stress-strength model. In the above cited works, it was traditionally believed that the strengths in multi-component stress-strength models were i.i.d random variables. However, when the structures of the system's component parts diverge, this assumption becomes impractical (see [6] for additional information). Thus, the present work aims to concentrate on multicomponent stress-strength models that have different random strengths. [17] extended the concept of MSR for non-identical component strengths for exponential distributions. [18] considered the Weibull distribution for non-identical MSR and attempted Bayesian and non-Bayesian methods of estimation. Bayesian and maximum likelihood estimators were developed for non-identical MSR for exponential distribution by [19].

Let's now look at a system that has $\boldsymbol{k} = (k_1, k_2, \dots, k_m)$ components. The k_i components in this system are of type $i, i = 1,2, \dots, m$. Assume that the cdf of the strengths for the i^{th} type components is $F_i(.)$. Furthermore, all components are assumed to be influenced by a common stress Y with cdf $G(\cdot)$. The system is reliable in this scenario as long as the strength of $\boldsymbol{s} = (s_1, s_2, \dots, s_m)$ of k components is more than at applied stress. [20] improved equation (1.1) to produce the following suitable model:

$$R_{s,k} = \sum_{p_1=s_1}^{k_1} \dots \sum_{p_m=s_m}^{k_m} \left(\prod_{i=1}^{m} \binom{k}{i}\right) \int_{-\infty}^{\infty} \prod_{i=1}^{m} ([1 - F_i(y)]^{p_i} [F_i(y)]^{k_i - p_i}) dG(y) \qquad (2.1)$$

For the shake of computational complexity, in this article, we have considered a system with two types of components, i.e. $\boldsymbol{k} = (k_1, k_2)$ and $\boldsymbol{s} = (s_1, s_2)$. Assuming a system with *k*-components, of which k_1 belong to one category and that there is a common distribution function $F_1(.)$ for their strengths. The remaining components, $k_2 = k - k_1$, belong to other category and have a shared strength distribution, $F_2(.)$. The system functions properly if at least s of the k components are able to endure the common subjected stress Y with the distribution $G(.)$. The reliability in non-identical category of components strength distribution is developed in subsequent section. Due to wider applicability in real situations, the idea has attracted the authors' attention. Some of the recent contribution towards this direction for various choices of distributions under various sampling schemes can be viewed in the works of [21-25] and references therein.

On the other hand, the two-parameter gamma distribution is widely used to analyze lifetime data in the field of reliability engineering and is also used as an equivalent of Weibull, log-normal and similar distributions to analyze positively skewed data sets. One may refer to [26] and [27] for more discussion and application of two-parameter gamma distribution. Very few attempts on gamma distribution in the context of multicomponent stress-strength system reliability have been made, which may be because the distribution function, survival function and hazard rate, etc are not in a closed form for this distribution. The probability density (PD) and cumulative distribution (CD) functions are given as

$$f_X(x; \alpha, \lambda) = \frac{\alpha^{\lambda}}{\Gamma(\lambda)} x^{\lambda - 1} \, exp(-\alpha x) \quad \text{and} \quad F_X(x) = \int_0^x f(t) dt = \frac{\Gamma(\lambda, \alpha x)}{\Gamma(\lambda)} \qquad (1.3)$$

for $x > 0, \alpha, \lambda > 0$, where, λ and α are shape and scale parameters respectively. The term, $\Gamma(\lambda, \alpha x)$ is lower incomplete gamma function.

In this work, we developed the expression for MSR considering out of k component strengths, $\boldsymbol{X_1} = (X_1, X_2, \dots, X_{k_1})$ with k_1 components follow a $gamma\ (\alpha_1, \lambda)$, while the remaining $k_2 = k - k_1$ component strengths $\boldsymbol{X_2} = (X_{k_1+1}, \dots, X_{k_2})$ follow a $gamma\ (\alpha_2, \lambda)$. Further, assuming that stress Y follow a $gamma\ (\tau, \gamma)$. The respective PDFs and CDFs are given as

$$f_1(x_1; \alpha_1, \lambda) = \frac{\alpha_1^{\lambda}}{\Gamma(\lambda)} x_1^{\lambda-1} exp(-\alpha_1 x_1) \text{ and } F_1(x_1) = \frac{\Gamma(\lambda, \alpha_1 x_1)}{\Gamma(\lambda)}; \quad x_1 > 0, \alpha_1, \lambda > 0 \tag{1.4}$$

$$f_2(x_2; \alpha_2, \lambda) = \frac{\alpha_2^{\lambda}}{\Gamma(\lambda)} x_2^{\lambda-1} exp(-\alpha_2 x_2) \text{ and } F_2(x_2) = \frac{\Gamma(\lambda, \alpha_2 x_2)}{\Gamma(\lambda)}; \quad x_2 > 0, \alpha_2, \lambda > 0 \tag{1.5}$$

$$g(y; \tau, \gamma) = \frac{\tau^{\gamma}}{\Gamma(\gamma)} y^{\gamma-1} exp(-\tau y) \quad \text{and} \quad G_Y(y) = \frac{\Gamma(\gamma, \tau y)}{\Gamma(\gamma)}; \quad y > 0, \tau, \gamma > 0 \tag{1.6}$$

where $\Gamma(\lambda, \alpha_1 x_1)$, $\Gamma(\lambda, \alpha_2 x_2)$ and $\Gamma(\gamma, \tau y)$ presented in equations (1.4), (1.5) and (1.6) respectively are defined in equation (1.3).

This is how the rest of the paper is organized. Assuming that the component strengths are of two distinct categories of strengths k_1 and $k_2 = k - k_1$ to face the same stress, the MSR is computed in Section 2. The MSR parameter's ML estimator is derived in section 3. The development of the Bayes estimators of MSR with gamma and inverted gamma priors are given in Section 4. The findings of MSR and its ML and Bayes estimators are presented in Section 5 along with a discussion. Section 6 offers an overview of the article's findings.

2. Multi-component System Reliability $(\boldsymbol{R_{s,k}})$

The MSR is derived in this section under the assumption that the system is made up of k-components with strengths $X = X_1, X_2, \dots, X_k$, of which k_1 components $X_1, X_2, \dots, X_{k_1}$ belong to a one category with a shared distribution function denoted as F_1. The remaining $k_2 = k - k_1$ components $X_{k_1+1}, \dots, X_{k_2}$ belong to another category distributed as F_2 and are all subject to an independent common random stress Y with a distribution function of G. The triplet (F_1, F_2, G) determines the system reliability, which is written as

$$R_{s,k} = \sum_{p_1=s_1}^{k_1} \binom{k_1}{p_1} \sum_{p_2=s_2}^{k_2} \binom{k_2}{p_2} \int_{-\infty}^{\infty} [1 - F_1(y)]^{p_1} [F_1(y)]^{k_1-p_1} [1 - F_2(y)]^{p_2} [F_2(y)]^{k_2-p_2} dG(y)$$

(2.1)

where the sum ranges over $0 < p_1 < k_1$, $0 < p_2 < k_2$ such that $s < p_1 + p_2 < k$. The reliability given in (2.1) was introduced by [17]. Thus, the MSR having two categories of non-identical component strengths can be found by substituting $F_1(x_1)$, $F_2(x_2)$ and $G(y)$ from equations (1.4), (1.5) and (1.6) respectively in equation (2.1), is given by

$$R_{s,k} = \sum_{p_1=s_1}^{k_1} \binom{k_1}{p_1} \sum_{p_2=s_2}^{k_2} \binom{k_2}{p_2} \int_0^{\infty} \left[1 - \frac{\Gamma(\lambda, \alpha_1 y)}{\Gamma(\lambda)}\right]^{p_1} \left[\frac{\Gamma(\lambda, \alpha_1 y)}{\Gamma(\lambda)}\right]^{k_1-p_1}$$

$$\times \left[1 - \frac{\Gamma(\lambda, \alpha_2 y)}{\Gamma(\lambda)}\right]^{p_2} \left[\frac{\Gamma(\lambda, \alpha_2 y)}{\Gamma(\lambda)}\right]^{k_2-p_2} \frac{\tau^{\gamma}}{\Gamma(\gamma)} y^{\gamma-1} exp(-\tau y)\, dy \tag{2.2}$$

Let $\tau y = z; \Rightarrow y = \frac{z}{\tau} \Rightarrow dy = \frac{dz}{\tau}$ and $\rho_1 = \frac{\alpha_1}{\tau}$ and $\rho_2 = \frac{\alpha_2}{\tau}$. Thus, the above equation can be written after simplification as

$$R_{s,k} = \frac{1}{\Gamma(\gamma)} \sum_{p_1=s_1}^{k_1} \binom{k_1}{p_1} \sum_{p_2=s_2}^{k_2} \binom{k_2}{p_2} \int_{-\infty}^{\infty} \left[1 - \frac{\Gamma(\lambda, \rho_1 z)}{\Gamma(\lambda)}\right]^{p_1} \left[\frac{\Gamma(\lambda, \rho_1 z)}{\Gamma(\lambda)}\right]^{k_1 - p_1} \left[1 - \frac{\Gamma(\lambda, \rho_2 z)}{\Gamma(\lambda)}\right]^{p_2} \times \left[\frac{\Gamma(\lambda,\rho_2 z)}{\Gamma(\lambda)}\right]^{k_2 - p_2} exp(-z) z^{\gamma - 1} dz \quad (2.3)$$

The expression of MSR given by equation (2.3) has no closed form solution. Numerical approximation approach is therefore employed to solve this equation.

3. ML Estimation of $R_{s,k}$

This section performs the ML estimation of MSR under the assumptions that $Y = (y_1, y_2, \ldots, y_m)$ represents the random sample of size m from the stress distribution and $X_1 = (x_{11}, x_{12}, \ldots, x_{1n_1})$ and $X_2 = (x_{21}, x_{22}, \ldots, x_{2n_2})$ are the random samples of sizes n_1 and n_2 from their respective distributions. n_1 and n_2 are selected such a way that $n_1 + n_2 = n$. From the random observations, the likelihood function is then obtained as follows:

$$L = L(\alpha_1, \alpha_2, \lambda, \gamma, \tau \,/\, x_1, x_2, y) = \left[\prod_{i=1}^{n_1} f_{X_1}(x_{1i})\right] \times \left[\prod_{j=1}^{n_2} f_{X_2}(x_{2j})\right] \times \left[\prod_{k=1}^{m} f_Y(y_k)\right]$$

$$= \left[\frac{\alpha_1^{n_1 \lambda}}{\{\Gamma(\lambda)\}^{n_1}} \prod_{i=1}^{n_1} x_{1i}^{\lambda-1} exp\left(-\alpha_1 \sum_{i=1}^{n_1} x_{1i}\right)\right] \times \left[\frac{\alpha_2^{n_2 \lambda}}{\{\Gamma(\lambda)\}^{n_2}} \prod_{j=1}^{n_2} x_{2j}^{\lambda-1} exp\left(-\alpha_2 \sum_{j=1}^{n_2} x_{2j}\right)\right] \times \left[\frac{\tau^{m\gamma}}{\{\Gamma(\gamma)\}^m} \prod_{k=1}^{m} y_k^{\gamma-1} exp\left(-\tau \sum_{k=1}^{m} y_k\right)\right] \quad (3.1)$$

After simplification the expression becomes

$$L = \frac{\alpha_1^{n_1\lambda} \alpha_2^{n_2\lambda} \tau^{m\gamma}}{\{\Gamma(\lambda)\}^{n_1+n_2} \{\Gamma(\gamma)\}^m} \left(\prod_{i=1}^{n_1} x_{1i}^{\lambda-1}\right)\left(\prod_{j=1}^{n_2} x_{2j}^{\lambda-1}\right)\left(\prod_{k=1}^{m} y_k^{\gamma-1}\right) exp[-\alpha_1 n_1 \bar{x}_1 - \alpha_2 n_2 \bar{x}_2 - \tau m \bar{y}] \quad (3.2)$$

where $\bar{x}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} x_{1i}$, $\bar{x}_2 = \frac{1}{n_2}\sum_{j=1}^{n_2} x_{2j}$ are averages of strength samples from both the categories and $\bar{y} = \frac{1}{m}\sum_{k=1}^{m} y_k$ is the average of stress observations. The log-likelihood function is obtained by taking the logarithm of both sides in equation (3.2), given by

$$log\, L = n_1 \lambda \log \alpha_1 + n_2 \lambda \log \alpha_2 + m\gamma \log \tau - (n_1 + n_2) \log \Gamma(\lambda) - m \log \Gamma(\gamma)$$
$$+ (\lambda - 1)(s_1 + s_2) + (\gamma - 1)s_3 - (\alpha_1 n_1 \bar{x}_1 + \alpha_2 n_2 \bar{x}_2 + \tau m \bar{y}) \quad (3.3)$$

where $s_1 = \sum_{i=1}^{n_1} \log x_{1i}$, $s_2 = \sum_{j=1}^{n_2} \log x_{2j}$ and $s_3 = \sum_{k=1}^{m} \log y_k$. The ML estimators $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\lambda}, \hat{\gamma}$ and $\hat{\tau}$ of $\alpha_1, \alpha_2, \lambda, \gamma$ and τ respectively can be obtained by solving following partial derivatives, given by

$$\frac{\partial \log L}{\partial \alpha_1} = 0 \Rightarrow \hat{\alpha}_1 = \frac{\lambda}{\bar{x}_1} \quad (3.4)$$

$$\frac{\partial \log L}{\partial \alpha_2} = 0 \Rightarrow \hat{\alpha}_2 = \frac{\lambda}{\bar{x}_2} \quad (3.5)$$

$$\frac{\partial \log L}{\partial \lambda} = 0 \Rightarrow n_1 \log \alpha_1 + n_2 \log \alpha_2 - (n_1 + n_2)\psi(\lambda) + s_1 + s_2 = 0 \quad (3.6)$$

$$\frac{\partial \log L}{\partial \tau} = 0 \Rightarrow \hat{\tau} = \frac{\gamma}{\bar{y}} \quad (3.7)$$

$$\frac{\partial \log L}{\partial \gamma} = 0 \Rightarrow m \log \tau - m\psi(\gamma) + s_3 = 0 \tag{3.8}$$

where, $\psi(.)$ represent the di-gamma function and is defined as $\psi(x) = \frac{\partial \ln \Gamma(x)}{\partial x}$. Since the simultaneous solution of aforementioned likelihood equations are analytically not possible. Therefore, a numerical approximation method, such as the Newton-Raphson algorithm, is used. The ML estimate $\hat{R}_{(s,k)}$ of $R_{(s,k)}$ is obtained by substituting $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\lambda}, \hat{\gamma}$ and $\hat{\tau}$ in place of $\alpha_1, \alpha_2, \lambda, \gamma$ and τ respectively in $R_{(s,k)}$ given in (2.3), using invariance property.

4. Bayes Estimation of $R_{s,k}$

In this part of the paper, the MSR is estimated using the Bayesian method of estimation assuming the non-informative type of priors (uniform and Jeffreys priors) under the impression that the model parameters are unknown and are random variables. For a more thorough examination of the Bayesian estimators, two distinct loss functions are taken into consideration, referred to as the squared error loss (SEL) and LINEX loss (LL) functions.

4.1 Uniform Prior

We consider the model parameters $\alpha_1, \alpha_2, \lambda, \tau$ and γ are independent random variables having prior distribution as uniform prior with their respective density function as

$$h_1(\alpha_1) = \frac{1}{\alpha_1}; \alpha_1 > 0 \tag{4.1}$$

$$h_2(\alpha_2) = \frac{1}{\alpha_2}; \alpha_2 > 0 \tag{4.2}$$

$$h_3(\lambda) = \frac{1}{\lambda}; \lambda > 0 \tag{4.3}$$

$$h_4(\tau) = \frac{1}{\tau}; \tau > 0 \tag{4.4}$$

$$h_5(\gamma) = \frac{1}{\gamma}; \gamma > 0 \tag{4.5}$$

The joint prior density function of $\alpha_1, \alpha_2, \lambda, \tau$ and γ can be defined as the product of their respective marginal priors, given as

$$g(\alpha_1, \alpha_2, \lambda, \gamma, \tau) = \frac{1}{\alpha_1 \alpha_2 \lambda \tau \gamma}; \alpha_1 > 0, \alpha_2 > 0, \lambda > 0, \tau > 0, \gamma > 0. \tag{4.6}$$

The joint posterior density of $\alpha_1, \alpha_2, \lambda, \tau$ and γ is obtained by combining the joint prior density $g(\alpha_1, \alpha_2, \lambda, \gamma, \tau)$ and the likelihood function $L(\alpha_1, \alpha_2, \lambda, \gamma, \tau / x_1, x_2, y)$ from equation (4.6) and (3.2) respectively. Some of the constant terms free from $\alpha_1, \alpha_2, \lambda, \gamma, \tau$ are cancelled out from both the numerator and denominator parts. The joint posterior density under uniform prior is given by

$$\Pi_1(\alpha_1, \alpha_2, \lambda, \tau, \gamma / x_1, x_2, y) = K_1\, \alpha_1^{n_1\lambda - 1} \alpha_2^{n_2\lambda - 1} \lambda^{-1} \tau^{m\gamma - 1} \gamma^{-1} \{\Gamma(\lambda)\}^{-(n_1+n_2)} \{\Gamma(\gamma)\}^{-m}$$
$$\times \exp[-\{\alpha_1 n_1 \bar{x}_1 + \alpha_2 n_2 \bar{x}_2 - \lambda(s_1 + s_2) + \tau m \bar{y} - \gamma s_3\}] \tag{4.7}$$

where K_1 is the normalizing constant are respectively defined by

$$K_1^{-1} = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \alpha_1^{n_1\lambda-1}\alpha_2^{n_2\lambda-1}\lambda^{-1}\tau^{m\gamma-1}\gamma^{-1}\{\Gamma(\lambda)\}^{-(n_1+n_2)}\{\Gamma(\gamma)\}^{-m} \times exp[-\{\alpha_1 n_1\bar{x}_1+\alpha_2 n_2\bar{x}_2-\lambda(s_1+s_2)+\tau m\bar{y}-\gamma s_3\}]d\alpha_1 d\alpha_2 d\lambda d\tau d\gamma \quad (4.8)$$

4.1.1 Bayes Estimator ${}^{sel}_{U}\hat{R}_{s,k}$ under SEL Function

Thus, under SEL function, the Bayes estimator for uniform prior ${}^{sel}_{U}\hat{R}_{s,k}$ of $R_{s,k}$ is defined as its posterior expectation, given as follow

$${}^{sel}_{U}\hat{R}_{s,k} = E[R_{s,k}/\alpha_1,\alpha_2,\lambda,\gamma,\tau; x_1,x_2,y]$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty R_{s,k}\Pi_1(\alpha_1,\alpha_2,\lambda,\gamma,\tau/x_1,x_2,y)d\alpha_1 d\alpha_2 d\lambda d\tau d\gamma \quad (4.9)$$

Substituting the value of $\Pi_1(\alpha_1,\alpha_2,\lambda,\gamma,\tau/x_1,x_2,y)$ from (4.7) in equation (4.9), we have

$${}^{sel}_{U}\hat{R}_{s,k} = K_1^{-1}\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty R_{s,k}\alpha_1^{n_1\lambda-1}\alpha_2^{n_2\lambda-1}\lambda^{-1}\tau^{m\gamma-1}\gamma^{-1}\{\Gamma(\lambda)\}^{-(n_1+n_2)}\{\Gamma(\gamma)\}^{-m} \times exp[-\{\alpha_1 n_1\bar{x}_1+\alpha_2 n_2\bar{x}_2-\lambda(s_1+s_2)+\tau m\bar{y}-\gamma s_3\}]d\alpha_1 d\alpha_2 d\lambda d\tau d\gamma \quad (4.10)$$

where $R_{s,k}$ is the multicomponent system reliability given in (2.3) and K_1 is the denominator part of joint posterior given in (4.8).

4.1.2 Bayes Estimator ${}^{ll}_{U}\hat{R}_{s,k}$ under LL Function

Under LL function, the Bayes estimator ${}^{ll}_{U}\hat{R}_{s,k}$ of $R_{s,k}$ is defined as follows

$${}^{ll}_{U}\hat{R}_{s,k} = \frac{-1}{p}ln[E\{exp(-pR_{s,k})/\alpha_1,\alpha_2,\lambda,\gamma,\tau; x_1,x_2,y\}]$$

$$= \frac{-1}{p}ln\left[\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty exp(-pR_{s,k})\,\Pi(\alpha_1,\alpha_2,\lambda,\gamma,\tau/x_1,x_2,y)d\alpha_1 d\alpha_2 d\lambda d\tau d\gamma\right] \quad (4.11)$$

where $R_{s,k}$ is the multicomponent system reliability given in (2.3). Substituting the value of $\Pi(\alpha_1,\alpha_2,\lambda,\gamma,\tau/x_1,x_2,y)$ from (4.7) in equation (4.11), we have

$${}^{ll}_{U}\hat{R}_{s,k} = \frac{-1}{p}ln\left[K_1^{-1}\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-pR_{s,k}}\alpha_1^{n_1\lambda-1}\alpha_2^{n_2\lambda-1}\lambda^{-1}\tau^{m\gamma-1}\gamma^{-1}\{\Gamma(\lambda)\}^{-(n_1+n_2)}\{\Gamma(\gamma)\}^{-m} \times exp[-\{\alpha_1 n_1\bar{x}_1+\alpha_2 n_2\bar{x}_2-\lambda(s_1+s_2)+\tau m\bar{y}-\gamma s_3\}]d\alpha_1 d\alpha_2 d\lambda d\tau d\gamma\right] \quad (4.12)$$

The expressions of Bayes estimators for uniform prior under SEL function and LL function given in equations (4.10) and (4.12) respectively cannot be solved analytically. Thus, numerical approximation technique via M-H algorithm is used to obtain the Bayes estimator of $R_{s,k}$.

4.2 Jeffreys Prior

In this subsection, a non-informative type of prior, proposed by [28], is assumed for independent random variables $\alpha_1,\alpha_2,\lambda,\tau$ and γ. The Jeffreys prior is defined as

$$J(\alpha_1,\alpha_2,\lambda,\tau,\gamma) = \sqrt{det\, I(\alpha_1,\alpha_2,\lambda,\tau,\gamma)} \quad (4.13)$$

where, $I(\alpha_1,\alpha_2,\lambda,\tau,\gamma)$ is Fisher information matrix, defined as

$$I(\alpha_1,\alpha_2,\lambda,\tau,\gamma) = -E\begin{bmatrix} \frac{\partial^2 \log L}{\partial \alpha_1^2} & \frac{\partial^2 \log L}{\partial \alpha_1 \partial \alpha_2} & \frac{\partial^2 \log L}{\partial \alpha_1 \partial \lambda} & \frac{\partial^2 \log L}{\partial \alpha_1 \partial \tau} & \frac{\partial^2 \log L}{\partial \alpha_1 \partial \gamma} \\ \frac{\partial^2 \log L}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 \log L}{\partial \alpha_2^2} & \frac{\partial^2 \log L}{\partial \alpha_2 \partial \lambda} & \frac{\partial^2 \log L}{\partial \alpha_2 \partial \tau} & \frac{\partial^2 \log L}{\partial \alpha_2 \partial \gamma} \\ \frac{\partial^2 \log L}{\partial \lambda \partial \alpha_1} & \frac{\partial^2 \log L}{\partial \lambda \partial \alpha_2} & \frac{\partial^2 \log L}{\partial \lambda^2} & \frac{\partial^2 \log L}{\partial \lambda \partial \tau} & \frac{\partial^2 \log L}{\partial \lambda \partial \gamma} \\ \frac{\partial^2 \log L}{\partial \tau \partial \alpha_1} & \frac{\partial^2 \log L}{\partial \tau \partial \alpha_2} & \frac{\partial^2 \log L}{\partial \tau \partial \lambda} & \frac{\partial^2 \log L}{\partial \tau^2} & \frac{\partial^2 \log L}{\partial \tau \partial \gamma} \\ \frac{\partial^2 \log L}{\partial \gamma \partial \alpha_1} & \frac{\partial^2 \log L}{\partial \gamma \partial \alpha_2} & \frac{\partial^2 \log L}{\partial \gamma \partial \lambda} & \frac{\partial^2 \log L}{\partial \gamma \partial \tau} & \frac{\partial^2 \log L}{\partial \gamma^2} \end{bmatrix}$$

$$= -E\begin{bmatrix} \frac{-n_1\lambda}{\alpha_1^2} & 0 & \frac{n_1}{\alpha_1} & 0 & 0 \\ 0 & \frac{n_2\lambda}{\alpha_2^2} & \frac{n_2}{\alpha_2} & 0 & 0 \\ \frac{n_1}{\alpha_1} & \frac{n_2}{\alpha_2} & (n_1+n_2)\psi'(\lambda) & 0 & 0 \\ 0 & 0 & 0 & \frac{m\gamma}{\tau^2} & \frac{m}{\tau} \\ 0 & 0 & 0 & \frac{m}{\tau} & m\psi'(\gamma) \end{bmatrix} \quad (4.14)$$

Thus, determinant of Fisher information matrix i.e., $det\, I\,(\alpha_1,\alpha_2,\lambda,\tau,\gamma)$ is obtained as

$$det[I(\alpha_1,\alpha_2,\lambda,\tau,\gamma)] = n_1 n_2 m^2 \lambda \alpha_1^{-2}\alpha_2^{-2}\tau^{-2}\{\gamma\psi'(\gamma)-1\}\{(n_1+n_2)(\lambda\psi'(\lambda)-1)\} \quad (4.15)$$

Substituting the value of $det[I(\alpha_1,\alpha_2,\lambda,\tau,\gamma)]$ from (4.15) in equation (4.13), the Jeffreys prior of $\alpha_1,\alpha_2,\lambda,\tau$ and γ is given as

$$J(\alpha_1,\alpha_2,\lambda,\tau,\gamma) = (n_1+n_2)^{\frac{1}{2}}(m^2 n_1 n_2)^{\frac{1}{2}}(\alpha_1\alpha_2\tau)^{-1}\lambda^{\frac{1}{2}}[\{\gamma\psi'(\gamma)-1\}\{\lambda\psi'(\lambda)-1\}]^{\frac{1}{2}} \quad (4.16)$$

The joint posterior distribution of random variables $\alpha_1,\alpha_2,\lambda,\tau$ and γ under Jeffreys prior is obtained by combining the joint Jeffreys prior $J(\alpha_1,\alpha_2,\lambda,\tau,\gamma)$ and the likelihood function $L(\alpha_1,\alpha_2,\lambda,\gamma,\tau/x_1,x_2,y)$ from equation (4.16) and (3.2) respectively via Bayes rule. Some of the constant terms free from $\alpha_1,\alpha_2,\lambda,\gamma,\tau$ are cancelled out from both the numerator and denominator parts. The joint posterior density is given by

$$\Pi_2(\alpha_1,\alpha_2,\lambda,\tau,\gamma/x_1,x_2,y) = K_2\alpha_1^{n_1\lambda-1}\alpha_2^{n_2\lambda-1}\lambda^{\frac{1}{2}}\tau^{m\gamma-1}\gamma^{-1}\{\Gamma(\lambda)\}^{-(n_1+n_2)}\{\Gamma(\gamma)\}^{-m}$$
$$\times\{\gamma\psi'(\gamma)-1\}^{\frac{1}{2}}\{\lambda\psi'(\lambda)-1\}^{\frac{1}{2}} exp[-\{\alpha_1 n_1\bar{x}_1+\alpha_2 n_2\bar{x}_2 - \lambda(s_1+s_2)+\tau m\bar{y}-\gamma s_3\}] \quad (4.17)$$

where K_2 is the normalizing constant defined by

$$K_2^{-1} = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \alpha_1^{n_1\lambda-1}\alpha_2^{n_2\lambda-1}\lambda^{\frac{1}{2}}\tau^{m\gamma-1}\gamma^{-1}\{\Gamma(\lambda)\}^{-(n_1+n_2)}\{\Gamma(\gamma)\}^{-m}\{\gamma\psi'(\gamma)-1\}^{\frac{1}{2}}$$
$$\times\{\lambda\psi'(\lambda)-1\}^{\frac{1}{2}} exp[-\{\alpha_1 n_1\bar{x}_1+\alpha_2 n_2\bar{x}_2 - \lambda(s_1+s_2)+\tau m\bar{y}-\gamma s_3\}]d\alpha_1 d\alpha_2 d\lambda d\tau d\gamma \quad (4.18)$$

4.2.1 Bayes Estimator ${}^{sel}_{j}\hat{R}_{s,k}$ under SEL function

The Bayes estimator under SELF ${}^{sel}_{j}\hat{R}_{s,k}$ of $R_{s,k}$ is defined by its posterior expectation, given by

$$ {}^{sel}_{j}\hat{R}_{s,k} = E[R_{s,k}/\alpha_1,\alpha_2,\lambda,\gamma,\tau; x_1,x_2,y] $$

$$ = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} R_{s,k}\Pi_4(\alpha_1,\alpha_2,\lambda,\gamma,\tau/x_1,x_2,y)d\alpha_1 d\alpha_2 d\lambda d\tau d\gamma \quad (4.19) $$

Substituting the value of $\Pi_2(\alpha_1,\alpha_2,\lambda,\gamma,\tau/x_1,x_2,y)$ from (4.17) in equation (4.19), we get

$$ {}^{sel}_{j}\hat{R}_{s,k} = K_2\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} R_{s,k}\alpha_1^{n_1\lambda-1}\alpha_2^{n_2\lambda-1}\lambda^{\frac{1}{2}}\tau^{m\gamma-1}\gamma^{-1}\{\Gamma(\lambda)\}^{-(n_1+n_2)}\{\Gamma(\gamma)\}^{-m} $$
$$ \times\{\lambda\psi'(\lambda)-1\}^{\frac{1}{2}} exp[-\{\alpha_1 n_1\bar{x}_1+\alpha_2 n_2\bar{x}_2-\lambda(s_1+s_2)+\tau m\bar{y}-\gamma s_3\}]d\alpha_1 d\alpha_2 d\lambda d\tau d\gamma \quad (4.20) $$

where $R_{s,k}$ is the multicomponent system reliability given in (2.3) and K_2 is the denominator part of joint posterior given in (4.18).

4.2.2 Bayes Estimator ${}^{ll}_{j}\hat{R}_{s,k}$ under LL function

Under LL function, the Bayes estimator ${}^{ll}_{j}\hat{R}_{s,k}$ of MSR $R_{s,k}$ is defined by

$$ {}^{ll}_{j}\hat{R}_{s,k} = \frac{-1}{p} ln[E\{exp(-pR_{s,k})/\alpha_1,\alpha_2,\lambda,\gamma,\tau; x_1,x_2,y\}] $$

$$ = \frac{-1}{p} ln\left[\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} exp(-pR_{s,k})\,\Pi_2(\alpha_1,\alpha_2,\lambda,\gamma,\tau/x_1,x_2,y)d\alpha_1 d\alpha_2 d\lambda d\tau d\gamma\right] \quad (4.21) $$

where $R_{s,k}$ is the multicomponent system reliability given in (2.3). Substituting the value of $\Pi_2(\alpha_1,\alpha_2,\lambda,\gamma,\tau/x_1,x_2,y)$ from (4.17) in equation (4.21), we get the required Bayes estimator of $R_{s,k}$ under LL function as

$$ {}^{ll}_{j}\hat{R}_{s,k} = \frac{-1}{p} ln\left[K_2^{-1}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-pR_{s,k}}\alpha_1^{n_1\lambda-1}\alpha_2^{n_2\lambda-1}\lambda^{\frac{1}{2}}\tau^{m\gamma-1}\gamma^{-1}\{\Gamma(\lambda)\}^{-(n_1+n_2)}\right. $$
$$ \times\{\Gamma(\gamma)\}^{-m}\{\gamma\psi'(\gamma)-1\}^{\frac{1}{2}}\{\lambda\psi'(\lambda)-1\}^{\frac{1}{2}} $$
$$ \left.\times exp[-\{\alpha_1 n_1\bar{x}_1+\alpha_2 n_2\bar{x}_2-\lambda(s_1+s_2)+\tau m\bar{y}-\gamma s_3\}]d\alpha_1 d\alpha_2 d\lambda d\tau d\gamma\right] \quad (4.22) $$

It is to be noticed that the expressions of Bayes estimators for uniform prior under SEL and LL function given in (4.10) and (4.12) respectively as well as for the Jefrreys priors under both the loss functions given in (4.20) and (4.22) consist multiple integrals. Thus, the analytic solutions of Bayes estimators are not possible and therefore, the MCMC technique via M-H algorithm is employed to solve the integrals in order to find out the Bayes estimates of MSR.

5. SIMULATION STUDY

A simulation-based analysis of suggested ML and Bayesian estimators of proposed MSR is carried out in this section. The effectiveness of estimators of MSR under Bayesian paradigm for both the priors are compared with that of ML estimator for different numerical choices of model parameters. To compare the various estimators, their MSEs and ABs for various combinations of model parameters and sample sizes are considered. A thousand replications are used to analyze the ABs and MSEs.

Since the Bayes estimators for the uniform and Jeffreys priors under both the SEL and LL functions are not in closed form and consist many integrals, therefore, the Metropolis-Hastings algorithm of MCMC approach, is used to generate samples from the joint posterior densities. Below is a discussion of the basic M-H method's [see; 29] step-by-step algorithm for drawing N random samples from any given posterior distribution:

While using Metropolis-Hastings algorithm, we consider proposal distribution is asymptotically normal distributed with initial values as the ML estimates of parameters i.e. ${\alpha_1}^{(0)} = \hat{\alpha}_1, {\alpha_2}^{(0)} = \hat{\alpha}_2,\ \lambda^{(0)} = \hat{\lambda}, \tau^{(0)} = \hat{\tau}$ and $\gamma^{(0)} = \hat{\gamma}$ with their asymptotic covariance matrix. The basic Metropolis-Hastings algorithm involves following steps to generate random sample from a posterior distribution $\pi(\boldsymbol{\theta}/data);\ \boldsymbol{\theta} = (\alpha_1, \alpha_2, \lambda, \tau, \gamma)'$.

1. Set the start value of parameter $\boldsymbol{\theta}^{(0)} = \left({\alpha_1}^{(0)}, {\alpha_2}^{(0)}, \lambda^{(0)}, \tau^{(0)}, \gamma^{(0)}\right)$.
2. For $t = 1,2,\ldots,N$ repeating the following steps

i. Set $\boldsymbol{\theta} = \boldsymbol{\theta}^{(j-1)}$

ii. Generate a 'candidate' value $\boldsymbol{\theta}^c$ from a proposal density, say $q(\boldsymbol{\theta}^c/\boldsymbol{\theta})$.

iii. Draw 'U' from $U(0,1)$.

iv. Calculate $R = \frac{\pi(\boldsymbol{\theta}^c/\boldsymbol{data})q(\boldsymbol{\theta}/\boldsymbol{\theta}^c)}{\pi(\boldsymbol{\theta}/\boldsymbol{data})q(\boldsymbol{\theta}^c/\boldsymbol{\theta})}$.

v. If $u \le min(1, R)$, accept the candidate point $(\boldsymbol{\theta}^c)$, otherwise set $\boldsymbol{\theta}^{(j)} = \boldsymbol{\theta}$.

3. Repeat step 2 for $t = 1,2,\ldots,N$.

Here $q(\boldsymbol{\theta}^c/\boldsymbol{\theta})$ is the probability of transitions from state $\boldsymbol{\theta}$ to $\boldsymbol{\theta}^c$ of the Markov chain. The M-H algorithm was performed with eleven thousand of intermediate iterations for evaluating the Bayes estimate of MSR. First one thousand iterations are dropped as burn-in period of the Markov chain. Later, we further discarded every second simulation to reduce the autocorrelation within the chain.

The Bayes estimates of $R_{s,k}$ under SEL and LL functions are approximated respectively by M-H algorithm as follows

$$ {}^{sel}_{U}\hat{R}_{s,k} = \frac{1}{N-M}\sum_{t=1}^{N-M}\left(R_{s,k}\right)_{\alpha_1=\alpha_{1:t};\ \alpha_2=\alpha_{2:t};\ \lambda=\lambda_t;\ \tau=\tau_t; \gamma=\gamma_t} \tag{5.1} $$

$$ {}^{ll}_{U}\hat{R}_{s,k} = \frac{-1}{p}ln\left[exp\left\{-p\frac{1}{N-M}\sum_{t=1}^{N-M}\left(R_{s,k}\right)_{\alpha_1=\alpha_{1:t};\ \alpha_2=\alpha_{2:t};\ \lambda=\lambda_t;\ \tau=\tau_t; \gamma=\gamma_t}\right\}\right] \tag{5.2} $$

and similarly, for Jeffreys prior also. $\alpha_{1:t}, \alpha_{2:t}, \lambda_t, \tau_t, \gamma_t;\ t = 1,2,\ldots,N$ are independently generated random samples from their respective marginal posteriors through M-H algorithm and $M < N$ being the burn-in period of Markov chain.

Based on their MSEs and ABs, the proposed ML and Bayes estimators of multicomponent stress-strength reliability are compared for various combinations of stress-strength parameters $\alpha_1, \alpha_2, \lambda, \gamma$ and τ as well as for various sample size selections (n_1, n_2, m) with various combinations of (s_1, k_1) and (s_2, k_2); such that $s_1 + s_2 = s$ and $k_1 + k_2 = k$.

The average estimates, MSEs, and ABs for ML and Bayes estimators for both the priors under SEL and LL function are shown in Tables 1 and 2. We consider two sets of variations for model parameters $\alpha_1, \alpha_2, \lambda, \gamma$ and τ with selected values of (s_1, k_1) and (s_2, k_2) by keeping hyper-parameters constant as $a_1 = 0.30, b_1 = 0.45, a_2 = 0.75, b_2 = 0.85, a_3 = 0.25, b_3 = 0.30, c_1 = 0.65, d_1 = 0.55,$

$c_2 = 0.35,\ d_2 = 0.40$ and $p = 2.5$.

.

(i) In Table 1, three non-identical combinations $(k = 8, s = 2)$ with $(k_1 = 4, s_1 = 1; k_2 = 4, s_2 = 1)$, $(k = 8, s = 4)$ with $(k_1 = 4, s_1 = 2;\ k_2 = 4, s_2 = 2)$ and $(k = 8, s = 6)$ with $(k_1 = 4, s_1 = 3; k_2 = 4, s_2 = 3)$ are considered for fixed set of values of model parameters as $\alpha_1 = 0.70, \alpha_2 = 0.75, \lambda = 1.50, \tau = 0.75, \gamma = 1.50.$ for uniform and Jeffreys priors. The following are observed from the table.

Findings: The Table shows that for increasing sets of (k_1, s_1) and (k_2, s_2), the MSR values steadily decrease. Furthermore, for all three sets of (k_1, s_1) and (k_2, s_2), it is shown that the Bayes estimators outperform the ML estimator with reduced MSEs and ABs for both priors (uniform and Jeffreys priors). In general, the uniform prior under both the loss functions (SEL and LL) is observed to provide more precise estimates of MSR with lower ABs and MSEs than the ML estimators and the Jeffreys prior.

(ii) Similarly, in Table 2, the effects of three different choices of non-identical sequences are considered as $(k_1, s_1; k_2, s_2)$ as $(k_1 = 4, s_1 = 1; k_2 = 4, s_2 = 1)$, $(k_1 = 4, s_1 = 2;\ k_2 = 4, s_2 = 2)$ and $(k_1 = 4, s_1 = 3; k_2 = 4, s_2 = 3)$ for another fixed sets of values of $\alpha_1 = 2.50, \alpha_2 = 1.75, \lambda = 2.75, \tau = 1.50, \gamma = 2.50$. The followings were observed from the table.

Findings: here also, the table shows that the Bayes estimates for both uniform and Jeffreys priors have less MSEs and ABs as compared with ML estimates of MSR. In comparison to its other equivalents, the uniform prior predominates with least MSEs and ABs. Additionally, it is seen that when the sets of (k_1, s_1) and (k_2, s_2) increase, the multicomponent values drop.

6. CONCLUSION

The MSR in this article is derived under the assumption that, out of the k-component strengths, k_1 strengths belong to one distribution type and follow a two-parameter gamma distribution with parameters (α_1, λ), and $k_2 = k - k_1$ strengths belong to a different distribution type and distributed as $amma(\alpha_2, \lambda)$. Additionally, all the components experience independent stress with a gamma distribution with (τ, γ) as parameters. If s_1 out of k_1 and s_2 out of k_2 components can bear the common stress such that $s_1 + s_2 = s$ and $k_1 + k_2 = k$, then the system is said to be operating. This assumption is made in light of the possibility that the components of a system have distinct structures in many real-world scenarios, making the assumption of equivalent strength distributions implausible.

Simulation-based samples are used to perform and compare the ML and Bayes estimators of MSR based on their MSEs and ABs. uniform and Jeffreys priors under non-informative forms of priors are taken into account in the Bayesian paradigm, and the Bayes estimators are computed for each prior under both SEL and LL functions.

The findings displayed in the following tables demonstrate that, when compared to the ML estimator of MSR, the Bayes estimators for both types of priors perform better with fewer MSEs and ABs. Specifically, for non-informative prior set-up, the uniform prior choice is more appropriate than the Jeffreys prior. It is also observed that the performance of the two loss functions is nearly equal. The MSR values are also shown to be steadily declining for increasing sets of (k_1, s_1) and (k_2, s_2).

Table 1: Average estimates, MSE and ABs of the estimators of MSR ($R_{s,k}$) under SEL and LL functions for uniform and Jeffreys priors with varying combinations of (s_1, k_1), (s_2, k_2) and (n_1, n_2, m) when $\alpha_1 = 0.70, \alpha_2 = 0.75, \lambda = 1.50, \tau = 0.75, \gamma = 1.50$ and $p = 2.5$.

(n_1, n_2, m)	Measure	MLE	Bayes			
			uniform prior		Jeffreys prior	
			SEL	LL	SEL	LL
$k_1 = 4, s_1 = 1; k_2 = 4, s_2 = 1;$ $R_{2,8}$= 0.483676						
(20,20,10)	AVG.	0.538297	0.452610	0.451846	0.458618	0.456767
	MSE	0.039848	0.018356	0.018484	0.016354	0.017869
	AB	0.020105	0.013354	0.014542	0.013032	0.013456
(20,40,10)	AVG.	0.514634	0.464523	0.461375	0.476852	0.473295
	MSE	0.025869	0.017754	0.018357	0.015244	0.015609
	AB	0.018728	0.017356	0.017584	0.013551	0.014197
(40,20,10)	AVG.	0.518357	0.473427	0.475027	0.476475	0.478745
	MSE	0.020653	0.010354	0.009775	0.009024	0.009125
	AB	0.017831	0.009567	0.010453	0.003154	0.003457
(50,50,10)	AVG.	0.493571	0.486574	0.488451	0.483157	0.484125
	MSE	0.009352	0.008751	0.008618	0.006452	0.006464
	AB	0.010542	0.005147	0.005134	0.004570	0.004575
$k_1 = 4, s_1 = 2; k_2 = 4, s_2 = 2;$ $R_{4,8}$ = 0.434682						
(20,20,10)	AVG.	0.453574	0.455345	0.455347	0.440245	0.442216
	MSE	0.033584	0.023546	0.022543	0.018654	0.018765
	AB	0.016864	0.013461	0.012654	0.010321	0.011256
(20,40,10)	AVG.	0.443858	0.443567	0.447548	0.441347	0.441657
	MSE	0.023461	0.016523	0.016457	0.010354	0.010452
	AB	0.013212	0.013546	0.013254	0.009654	0.009754
(40,20,10)	AVG.	0.439658	0.438951	0.439564	0.433575	0.438675
	MSE	0.006585	0.000986	0.000994	0.000318	0.000394
	AB	0.010685	0.007548	0.008145	0.004136	0.001253
(50,50,10)	AVG.	0.436584	0.434154	0.435014	0.434591	0.434645
	MSE	0.000903	0.000065	0.000071	0.000037	0.000041
	AB	0.008341	0.000142	0.000153	0.000121	0.000135
$k_1 = 4, s_1 = 3; k_2 = 4, s_2 = 3;$ $R_{6,8}$ = 0.396572						
(20,20,10)	AVG.	0.425374	0.385416	0.386458	0.393253	0.396351
	MSE	0.026257	0.009756	0.009835	0.004751	0.004865
	AB	0.012355	0.009025	0.009611	0.002145	0.002357
(20,40,10)	AVG.	0.402852	0.392864	0.393754	0.397385	0.398545
	MSE	0.010231	0.000231	0.000230	0.000135	0.000151
	AB	0.015751	0.004128	0.004538	0.003155	0.003275
(40,20,10)	AVG.	0.395176	0.396027	0.395963	0.396523	0.395972

	MSE	0.000221	0.000143	0.000140	0.000105	0.000107
	AB	0.005138	0.002175	0.002463	0.002045	0.002075
(50,50,10)	AVG.	0.396587	0.396503	0.396396	0.396569	0.396565
	MSE	0.000131	0.000051	0.000065	0.000043	0.000075
	AB	0.001291	0.001035	0.001075	0.000985	0.000994

Table 2: Average estimates, MSE and ABs of the estimators of MSR $(R_{s,k})$ under SEL and LL functions for uniform and Jeffreys priors with varying combinations of (s_1, k_1), (s_2, k_2) and (n_1, n_2, m) when $\alpha_1 = 2.50, \alpha_2 = 1.75, \lambda = 2.75, \tau = 1.50, \gamma = 2.50$ and $p = 2.5$.

(n_1, n_2, m)	Measure	MLE	Bayes			
			Uniform prior		Jeffreys prior	
			SEL	LL	SEL	LL
$k_1 = 4, s_1 = 1; k_2 = 4, s_2 = 1; R_{2,8} = 0.731837$						
(20, 20,20)	AVG.	0.765232	0.743567	0.746452	0.741253	0.741375
	MSE	0.004435	0.002142	0.002455	0.002041	0.002086
	AB	0.052115	0.021568	0.022548	0.020574	0.021164
(20,40,20)	AVG.	0.746609	0.747245	0.748258	0.739675	0.739864
	MSE	0.003565	0.003120	0.003220	0.000964	0.000973
	AB	0.021850	0.014563	0.016201	0.008643	0.008964
(40,20,20)	AVG.	0.737816	0.736458	0.736754	0.731763	0.731985
	MSE	0.000352	0.000317	0.000319	0.000115	0.000117
	AB	0.004185	0.002136	0.002235	0.001935	0.002038
(50,50,20)	AVG.	0.733652	0.734461	0.736558	0.731371	0.731754
	MSE	0.000189	0.000476	0.000480	0.000150	0.000180
	AB	0.002385	0.001525	0.001604	0.001385	0.001435
$k_1 = 4, s_1 = 2; k_2 = 4, s_2 = 2; R_{4,8} = 0.667515$						
(20,20,20)	AVG.	0.681584	0.647675	0.648564	0.651563	0.651415
	MSE	0.031420	0.025643	0.028354	0.009382	0.009631
	AB	0.018654	0.013861	0.014652	0.014264	0.015672
(20,40,20)	AVG.	0.676528	0.650185	0.650058	0.659141	0.658318
	MSE	0.016535	0.005645	0.006754	0.004028	0.004256
	AB	0.010325	0.013452	0.014030	0.009735	0.009865
(40,20,20)	AVG.	0.662417	0.667496	0.663150	0.667521	0.667539
	MSE	0.000466	0.000037	0.000087	0.000019	0.000021
	AB	0.003634	0.001624	0.001768	0.001235	0.001504
(50,50,20)	AVG.	0.667429	0.667598	0.667610	0.667505	0.667501
	MSE	0.000063	0.000033	0.000035	0.000021	0.000022
	AB	0.000933	0.000348	0.000385	0.000124	0.000127
$k_1 = 4, s_1 = 3; k_2 = 4, s_2 = 3; R_{6,8} = 0.593568$						

(20,20,20)	AVG.	0.616581	0.580167	0.583587	0.588648	0.586385
	MSE	0.021475	0.009856	0.009868	0.007685	0.008693
	AB	0.018354	0.010672	0.010864	0.009856	0.009968
(20,40,20)	AVG.	0.600345	0.589646	0.586374	0.593129	0.590054
	MSE	0.012654	0.000633	0.000657	0.000124	0.000135
	AB	0.010658	0.004107	0.004463	0.003025	0.003586
(40,20,20)	AVG.	0.592130	0.592568	0.595785	0.593754	0.593568
	MSE	0.000064	0.000056	0.000058	0.000021	0.000050
	AB	0.000210	0.000195	0.000198	0.000161	0.000172
(50,50,20)	AVG.	0.593157	0.593689	0.593896	0.593482	0.593491
	MSE	0.000005	0.000006	0.000007	0.000003	0.000003
	AB	0.000103	0.000075	0.000078	0.000056	0.000057

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