Stochastic Modelling and Computational Sciences

THE LATTICE OF CONVEX SUBLATTICES OF $S^3(B_n)$

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Abstract
In this paper, we prove that $CS[S^3(B_n)]$ is an Eulerian lattice under the set inclusion relation and it is neither simplicial nor dual simplicial, if $n > 1$.

Keywords: Convex sublattice; Simplicial Eulerian lattice; Dual simplicial.

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1 Introduction

The lattice of sublattices of a lattice with convex sublattices has been studied in some detail by K. M. Koh [3] in the year 1972. He had investigated the internal structure of a lattice L, in relation to $CS(L)$, like so many other authors for various algebraic structures such as groups, Boolean algebras, directed graphs and so on. In 1992, V. K. Santhi [12] constructed a new Eulerian lattice $S(B_n)$ from a Boolean algebra B_n of rank n. In 2012, R. Subbarayan and A. Vethamanickam [15] have proved in their paper that the lattice of convex sublattices of a Boolean algebra B_n, of rank n, $CS(B_n)$ with respect to the set inclusion relation is a dual simplicial Eulerian lattice. Neither simplicity nor dual simplicity are characteristics associated with the set inclusion relation.

In this paper, we are going to look at the structure of $CS[S^3(B_n)]$ and prove it to be Eulerian under $'\subseteq'$ relation. $S(B_2)$ is shown in figure 1. We note that $S(B_2)$ contains three copies of B_2, we call them left copy, right copy and middle copy of $S(B_2)$.

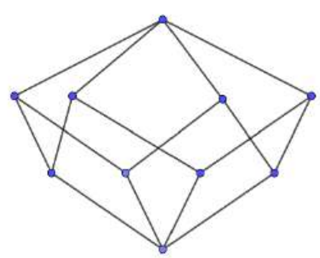

Figure 1

Lemma 1.1. [8] A finite graded poset P is Eulerian if and only if all intervals$[x, y]$ of length $l \geq 1$ in P contain an equal number of elements of odd and even rank.

Lemma 1.2. [13] If L_1 and L_2 are two Eulerian lattices then $L_1 \times L_2$ is also Eulerian.

There is no way to contain a three element chain as an interval. In the case that an undefined term needs to be referred to, we use [2], [11] and [12].

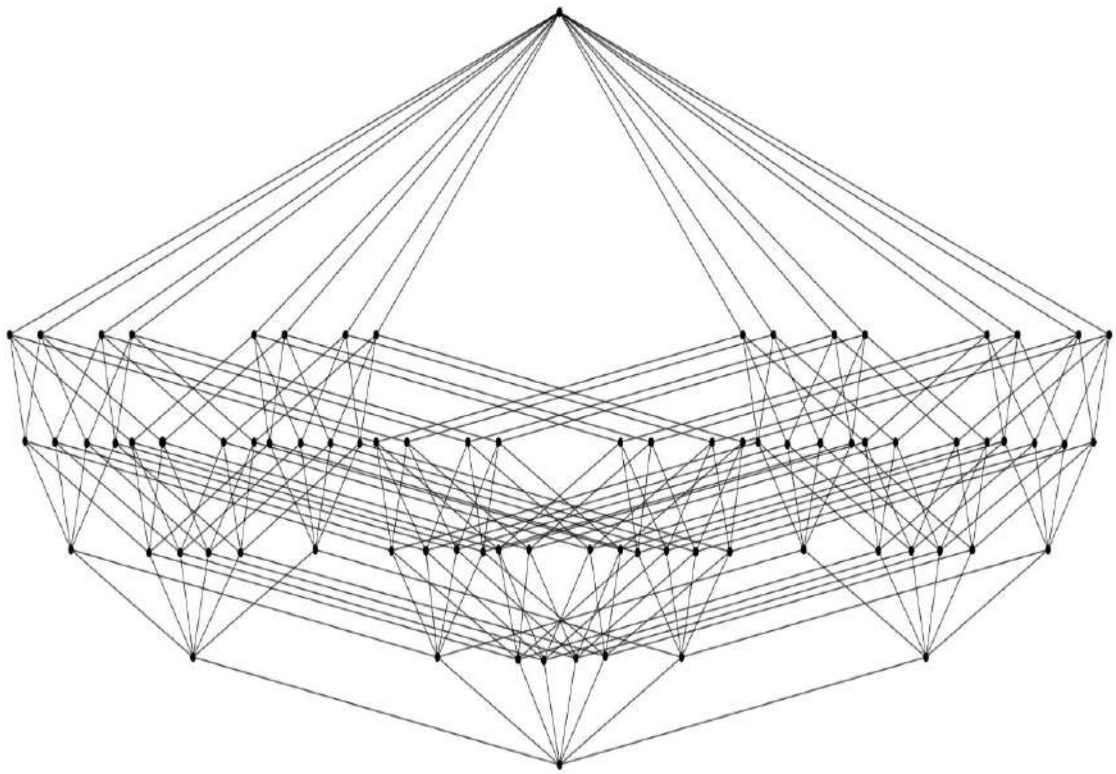

Figure 2-$S^3(B_2)$

2 The Eulerian property of the lattice $CS[S^3(B_n)]$

Lemma 2.1. For $n \geq 1$, we have

$$1+2+\binom{n}{1}+2+2+2[2+\binom{n}{1}+2]+2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}+22[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}+2[2\binom{n}{1}+\binom{n}{2}]+2\binom{n}{2}+\binom{n}{3}+22[2\binom{n}{1}+\binom{n}{2}]+2\binom{n}{2}+\binom{n}{3}+2[2\binom{n}{2}+\binom{n}{3}]+2\binom{n}{3}+\binom{n}{4}+\cdots+22[2\binom{n}{n-3}+\binom{n}{n-2}]+2\binom{n}{n-2}+\binom{n}{n-1}+2[2\binom{n}{n-2}+\binom{n}{n-1}]+2\binom{n}{n-1}+22[2\binom{n}{n-2}+\binom{n}{n-1}]+2\binom{n}{n-1}+2[2\binom{n}{n-1}]+22[2\binom{n}{n-1}]+1=3^3.2^n-26.$$

Theorem 2.2 $CS[S^3(B_n)]$, the lattice of convex sublattices of $S^3(B_n)$with respect to the set inclusion relation is an Eulerian lattice.

Proof

We first note that, the number of elements of ranks $0,1,2,\ldots,n+1$ in$S(B_n)$are, $1,2+\binom{n}{1},2\binom{n}{1}+\binom{n}{2},2\binom{n}{2}+\binom{n}{3},\ldots,2\binom{n}{n-2}+\binom{n}{n-1},2\binom{n}{n-1},1$ respectively.

The number of elements of ranks $0,1,2,\ldots,n+2$ in $S[S(B_n)]$ are,
$1,2+\binom{n}{1}+2,2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2},2[2\binom{n}{1}+\binom{n}{2}]+2\binom{n}{2}+\binom{n}{3},2[2\binom{n}{2}+\binom{n}{3}]+2\binom{n}{3}+\binom{n}{4},\ldots,2[2\binom{n}{n-2}+\binom{n}{n-1}]+2\binom{n}{n-1},2[2\binom{n}{n-1}],1$
respectively.

The number of elements of ranks $0,1,2,\dots,n+3$ in $S^3(B_n)$ are,
$1, 2+\binom{n}{1}+2+2, 2[2+\binom{n}{1}+2]+2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}, 22[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}+2[2\binom{n}{1}+\binom{n}{2}]+2\binom{n}{2}+\binom{n}{3}, 22[2\binom{n}{1}+\binom{n}{2}]+2\binom{n}{2}+\binom{n}{3}+2[2\binom{n}{2}+\binom{n}{3}]+2\binom{n}{3}+\binom{n}{4}, \dots, 22[2\binom{n}{n-3}+\binom{n}{n-2}]+2\binom{n}{n-2}+\binom{n}{n-1}+2[2\binom{n}{n-2}+\binom{n}{n-1}]+2\binom{n}{n-1}, 22[2\binom{n}{n-2}+\binom{n}{n-1}]+2\binom{n}{n-1}+2[2\binom{n}{n-1}], 22[2\binom{n}{n-1}], 1$
respectively.

It is clear that the rank of $CS[S^3(B_n)]$, is $n+4$.

We are going to prove that $CS[S^3(B_n)]$, is Eulerian.

That is, to prove that this interval $[\varphi, S^3(B_n)]$ has the same number of elements of odd and even rank.

Let A_i be the number of elements of rank i in $CS[S^3(B_n)]$, $i=1,2,\dots,n+3$.

A_1= The number of singleton subsets of $S^3(B_n)$

=
$1+2+\binom{n}{1}+2+2+2[2+\binom{n}{1}+2]+2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}+22[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}+2[2\binom{n}{1}+\binom{n}{2}]+2\binom{n}{2}+\binom{n}{3}+22[2\binom{n}{1}+\binom{n}{2}]+2\binom{n}{2}+\binom{n}{3}+2[2\binom{n}{2}+\binom{n}{3}]+2\binom{n}{3}+\binom{n}{4}+\cdots+22[2\binom{n}{n-3}+\binom{n}{n-2}]+2\binom{n}{n-2}+\binom{n}{n-1}+2[2\binom{n}{n-2}+\binom{n}{n-1}]+2\binom{n}{n-1}+22[2\binom{n}{n-2}+\binom{n}{n-1}]+2\binom{n}{n-1}+2[2\binom{n}{n-1}]+22[2\binom{n}{n-1}]+1$
……………….(2.1.1)

A_2 = The number of rank 2 convex sublattices in $S^3(B_n)$

= The number of edges in $S^3(B_n)$

= The number of edges containing 0 + number of edges with an atom at the bottom + The number of edges from the rank 2 elements+ $\cdots$ +The number of edges with a coatom of $S^3(B_n)$at the bottom.

Number of edges containing 0 is, $2+\binom{n}{1}+2+2$ ……………….(2.2)

The number of edges with an extreme atom at the bottom of the edge= $2+\binom{n}{1}+2$. There are 2 extreme atoms, this means that the total number of these edges will be equal to $2[2+\binom{n}{1}+2]$

Let x be an atom in the middle copy, then
$[x,1] \cong \{\{S^2(B_n) \text{ if } x \text{ be in an extreme copies of } S^3(B_n), S^3(B_{n-1}) \text{ if} x \text{ be in the middle copy of } S^3(B_n)\}\}$

If $[x,1] \cong S^2(B_n)$, there are $2+\binom{n}{1}+2$ edges.

There are 2 extreme atoms, this means that the total number of these edges will be equal to
$2[2+\binom{n}{1}+2]$. If $[x,1] \cong S^3(B_{n-1})$, there are $2+2+\binom{n-1}{1}+2$ edges. There are $2+\binom{n}{1}$ such atoms, since, the middle copy of $S^3(B_n)$ is of the form $S^2(B_n)$, whose middle copy is of the form $S(B_n)$, this means that the total number of these edges will be equal to $(2+\binom{n}{1})[2+2+\binom{n-1}{1}+2]$. Hence, the number of edges that have an atom at the bottom of the edge is a total of
$2[2+\binom{n}{1}+2]+2[2+\binom{n}{1}+2]+(2+\binom{n}{1})[2+2+\binom{n-1}{1}+2]$. ………..(2.3)

Now to find, the number of edges with an element of rank 2 at the bottom.

Let x be a rank 2 element in the left copy. Then,
$[x,1] \cong \{\{S(B_n) \text{ if } x \in \text{extreme copies of left copy of } S^3(B_n), S^2(B_{n-1}) \text{ if} x \in \text{middle copy of left copy } S^3(B_n)\}\}$

If $[x,1] \cong S(B_n)$, there are $\binom{n}{1}+2$ edges in both extreme copies. Totally, $2(\binom{n}{1}+2)$ edges are there. If $[x,1] \cong S^2(B_{n-1})$, the number of edges from x is $2+\binom{n-1}{1}+2$. There are $2+\binom{n}{1}$ such elements, since, the middle copy of $S^3(B_n)$ is of the form $S^2(B_n)$ whose middle copy is of the form $S(B_n)$, therefore, totally $2+\binom{n}{1}[2+\binom{n-1}{1}+2]$ edges in the middle of the left copy of $S^3(B_n)$.The number of edges in the left copy that have an element of rank 2 at the bottom is $=2[\binom{n}{1}+2]+(2+\binom{n}{1})[2+\binom{n-1}{1}+2]$. Similarly, the number of edges in the right copy that have an element of rank 2 at the bottom is therefore $=2[\binom{n}{1}+2]+(2+\binom{n}{1})[2+\binom{n-1}{1}+2]$.

Let x be a rank 2 element in the middle copy of $S^3(B_n)$.

Then,
$[x,1]\cong\{\{S^2(B_{n-1})\ if\ x\ \in extreme\ copies\ of\ \ middle\ copy\ of\ S^3(B_n),\ S^3(B_{n-2})\ if x\in$
$middle\ copy\ \ of\ middle\ copy\, S^3(B_n)\}\}$

If $[x,1]\cong\ S^2(B_{n-1})$, the number of edges from x is $2+\binom{n-1}{1}+2$. There are $2+\binom{n}{1}$ such elements in both extreme copies. Totally, $(2+\binom{n}{1})(2+\binom{n-1}{1}+2)$ edges. If $[x,1]\cong\ S^3(B_{n-2})$,,the number of edges from x is $2+2+\binom{n-2}{1}+2$. There are $2\binom{n}{1}+\binom{n}{2}$ such elements, therefore, totally $(2\binom{n}{1}+\binom{n}{2})[2+2+\binom{n-2}{1}+2]$ edges in the middle of the middle copy of $S^3(B_n)$. The number of edges in the middle copy that have an element of rank 2 at the bottom is therefore $2[(2+\binom{n}{1})(2+\binom{n-1}{1}+2)]+(2\binom{n}{1}+\binom{n}{2})[2+2+\binom{n-2}{1}+2]$ edges. Hence, the total number of edges from a rank 2 element can be expressed as follows:

$$2[2[\binom{n}{1}+2]+(2+\binom{n}{1})[2+\binom{n-1}{1}+2]]+2[(2+\binom{n}{1})(2+\binom{n-1}{1}+2)]+(2\binom{n}{1}+\binom{n}{2})[2+2+\binom{n-2}{1}+2] \quad \ldots\ldots\ldots\ldots(2.4)$$

Now to find, the number of edges with an element of rank 3 at the bottom. Let x be a rank 3 element in the extreme copies in the left copy of $S^3(B_n)$.

Then, $[x,1]\cong S(B_{n-1}), if x\in an\ extreme\ copies\ of\ leftcopy\ of\ S^3(B_n)$

$\cong S^2(B_{n-2}), if x\in middle\ copy\ of\ left\ copy\ of\ S^3(B_n)$

If $[x,1]\cong S(B_{n-1})$, the number of edges from x is $2+\binom{n-1}{1}$.There are $2+\binom{n}{1}$such x's in both extreme copies. Totally, $(2+\binom{n}{1})(2+\binom{n-1}{1}$ edges from such x's in the extreme copies of left copy.

If $[x,1]\cong S^2(B_{n-2})$, then the number of edges from x is $2+\binom{n-2}{1}+2$. There are $2\binom{n}{1}+\binom{n}{2}$ such elements in both extreme copies. Totally, $(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)$ edges. If $[x,1]\cong\ S^3(B_{n-2})$,,the number of edges from x is $2+2+\binom{n-2}{1}+2$. There are $2\binom{n}{1}+\binom{n}{2}$ such elements, therefore, totally $(2\binom{n}{1}+\binom{n}{2})[2+2+\binom{n-2}{1}+2]$ edges in the middle of the left copy of $S^3(B_n)$. The number of edges in the left copy that have an element of rank 3 at the bottom is therefore $2[(2+\binom{n}{1})(2+\binom{n-1}{1})]+(2\binom{n}{1}+\binom{n}{2})[2+\binom{n-2}{1}+2]$ edges. Similarly, the number of edges in the right copy that have an element of rank 3 at the bottom is therefore, $2[(2+\binom{n}{1})(2+\binom{n-1}{1})]+(2\binom{n}{1}+\binom{n}{2})[2+\binom{n-2}{1}+2]$.

Let x be a rank 3 element in the middle copy of $S^3(B_n)$.

Then,
$[x,1]\cong\{\{S^2(B_{n-2})\ if\ x\ \in extreme\ copies\ of\ \ middle\ copy\ of\ S^3(B_n),\ S^3(B_{n-3})\ if x\in$
$middle\ copy\ \ of\ middle\ copy\, S^3(B_n)\}\}$

If $[x,1] \cong S^2(B_{n-2})$, the number of edges from x is $2+\binom{n-2}{1}+2$. There are $2\binom{n}{1}+\binom{n}{2}$ such elements in both extreme copies. Totally, $(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)$ edges.

If $[x,1] \cong S^3(B_{n-3})$,,the number of edges from x is $2+2+\binom{n-3}{1}+2$. There are $2\binom{n}{2}+\binom{n}{3}$ such elements, therefore, totally $(2\binom{n}{2}+\binom{n}{3})[2+2+\binom{n-3}{1}+2]$ edges in the middle of the middle copy of $S^3(B_n)$. The number of edges in the middle copy that have an element of rank 3 at the bottom is therefore $2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)]+(2\binom{n}{2}+\binom{n}{3})[2+2+\binom{n-3}{1}+2]$ edges. Hence, the total number of edges from a rank 3 element can be expressed as follows:

$$2\{2[(2+\binom{n}{1})(2+\binom{n-1}{1})]+(2\binom{n}{1}+\binom{n}{2})[2+\binom{n-2}{1}+2]\}+2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)]+2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)]+(2\binom{n}{2}+\binom{n}{3})[2+2+\binom{n-3}{1}+2]\ldots \quad \ldots(2.5)$$

We can proceed in the same way to find the number of edges from the bottom of a coatom of $S^3(B_n)=$ the number of coatoms in $S^3(B_n)$

$$=2\{2[2\binom{n}{n-1}]\}\ldots\ldots\ldots\ldots (2.6)$$

Hence, from (2.2), (2.3), (2.4), (2.5) and (2.6) we get, the total number of edges in $S^3(B_n)$ is,

$$A_2=2+\binom{n}{1}+2+2+2[2+\binom{n}{1}+2]+2[2+\binom{n}{1}+2]+(2+\binom{n}{1})[2+2+\binom{n-1}{1}+2]+2[2[\binom{n}{1}+2]+(2+\binom{n}{1})[2+\binom{n-1}{1}+2]]+2[(2+\binom{n}{1})(2+\binom{n-1}{1}+2)]+(2\binom{n}{1}+\binom{n}{2})[2+2+\binom{n-2}{1}+2]+2\{2[(2+\binom{n}{1})(2+\binom{n-1}{1})]+(2\binom{n}{1}+\binom{n}{2})[2+\binom{n-2}{1}+2]\}+2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)]+2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)]+(2\binom{n}{2}+\binom{n}{3})[2+2+\binom{n-3}{1}+2]+\ldots+2\{2[2\binom{n}{n-1}]\}$$

$$\ldots\ldots\ldots\ldots\ \ldots\ldots (2.1.2)$$

$A_3=$ The number of 4 element convex sublattices in $S^3(B_n)$

= The number of B_2's in $S^3(B_n)$

=The number of $B_2's$ containing 0 + the number of $B_2's$ containing an atom at the bottom + ….+ the number of $B_2's$ containing a rank $n+1$ element at the bottom in $S^3(B_n)$.

The number of 4 element convex sublattices in $S^3(B_n)$ containing 0 as the bottom element is,

$$2[2+\binom{n}{1}+2]+2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2} \quad \ldots\ldots\ldots (2.7)$$

Next, we find the number of 4 element convex sublattices containing an atom as the bottom element.

Fix an atom $x\in S^3(B_n)$. If x is the bottom element of the left copy of $S^3(B_n)$, then $[x,1]\cong S^2(B_n)$. Therefore, the number of B_2's containing x at the bottom is $2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}$.Similarly, the number of $B_2's$ containing the bottom element of the right copy is $2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}$.

If x is in the middle copy of $S^3(B_n)$, then,

$[x,1]\cong\{\{S^2(B_n)\ if\ x\in$ $extreme\ copies\ of\ \ middle\ copy\ of\ S^3(B_n),\ S^3(B_{n-1})\ \ if x\ middle\ copy\ \ of\ middle\ copy\ S^3(B_n)\}\}$

If $[x,1]\cong S^2(B_n)$,there are $2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}$ B_2's in both extreme copies. Totally, $2\{2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}\}$ such B_2's . If $[x,1]\cong S^3(B_{n-1})$, then the number of B_2's containing x is $2[2+\binom{n-1}{1}+2]+2[\binom{n-1}{1}+2]+2\binom{n-1}{1}+\binom{n-1}{2}$. There are $2+\binom{n}{1}$ such elements, therefore, the total number of B_2's containing all the atoms at the bottom in the middle of the middle copy is $2\{2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}\}+(2+\binom{n}{1})\{2[2+\binom{n-1}{1}+2]+2[\binom{n-1}{1}+2]+2\binom{n-1}{1}+\binom{n-1}{2}\}$.

Therefore, the number of B_2's containing all the atoms of $S^3(B_n)$ is, $2[2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}]+2\{2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}\}+(2+\binom{n}{1})\{2[2+\binom{n-1}{1}+2]+2[\binom{n-1}{1}+2]+2\binom{n-1}{1}+\binom{n-1}{2}\}$ ....(2.8)

Next, fix an element x of rank 2 in $S^3(B_n)$

If x is in the left copy of $S^3(B_n)$.

Then, $[x,1]\cong S(B_n), if x\in an\ extreme\ copies\ of\ leftcopy\ of\ S^3(B_n)$

$\cong S^2(B_{n-1}), if x\in middle\ copy\ of\ left\ copy\ of\ S^3(B_n)$

If $[x,1]\cong S(B_n)$, the number of B_2's from x is $2\binom{n}{1}+\binom{n}{2}$. There are 2 such extreme copies. Totally, $2(2\binom{n}{1}+\binom{n}{2})$ such B_2's in the extreme copies of left copy.

If $[x,1]\cong S^2(B_{n-1})$, then the number of B_2's from x is $2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}$. There are $2+\binom{n}{1}$ such elements x of rank 2in the middle of the left copy. Therefore, the total number of B_2's containing a rank 2 element at the bottom in the left copy is , $2(2\binom{n}{1}+\binom{n}{2})(2+\binom{n}{1})[2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}]$. Similarly, we have the same number in the right copy. Therefore, the total number of B_2's containing a rank 2 element at the bottom in the extreme copies $=2(2\binom{n}{1}+\binom{n}{2})(2+\binom{n}{1})[2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}]$.

If x is in the middle copy of $S^3(B_n)$, then

$[x,1]\cong S^2(B_{n-1}), if x\in an\ extreme\ copies\ of\ middle\ copy\ of\ S^3(B_n)$

$\cong S^3(B_{n-2}), if x\in middle\ copy\ of\ middle\ copy\ of\ S^3(B_n)$

If $[x,1]\cong S^2(B_{n-1})$, there are $2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}$ B_2's with x at the bottom. There are $2+\binom{n}{1}$ such x's. Totally, $2+\binom{n}{1}\{2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}\}$ B_2's in the extreme copies of the middle copy.

If $[x,1]\cong S^3(B_{n-2})$, then the number of B_2's containing x is $2[2+\binom{n-2}{1}+2]+2[\binom{n-2}{1}+2]+2\binom{n-2}{1}+\binom{n-2}{2}$. There are $2\binom{n}{1}+\binom{n}{2}$ such elements x of rank 2 in the middle of the middle copy. Therefore, the total number of B_2's containing a rank 2 element at the bottom in the middle of the middle copy is $(2\binom{n}{1}+\binom{n}{2})[2[2+\binom{n-2}{1}+2]+2[\binom{n-2}{1}+2]+2\binom{n-2}{1}+\binom{n-2}{2}]$. Therefore, the number of B_2's in the middle copy containing all the elements of rank 2 in the middle copy is, $2\{(2+\binom{n}{1})\{2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}\}\}+(2\binom{n}{1}+\binom{n}{2})[2[2+\binom{n-2}{1}+2]+2[\binom{n-2}{1}+2]+2\binom{n-2}{1}+\binom{n-2}{2}]$. Therefore, the total number of B_2's containing all the rank 2 elements in $S^3(B_n)$ is,

$$2\{2\{2\binom{n}{1}+\binom{n}{2}\}+(2+\binom{n}{1})[2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}]\}+2\{(2+\binom{n}{1})[2[\binom{n-1}{1}+2]+2\binom{n-1}{1}+\binom{n-1}{2}]\}+(2\binom{n}{1}+\binom{n}{2})[2[2+\binom{n-1}{1}+2]+2[\binom{n-2}{1}+2]+2\binom{n-2}{1}+\binom{n-2}{2}] \quad ....(2.9)$$

In the same manner, the total number of B_2's containing all the rank 3 elements in $S^3(B_n)$ is,

$$2\{2\{(2+\binom{n}{1})[2\binom{n-1}{1}+\binom{n-1}{2}]\}+(2\binom{n}{1}+\binom{n}{2})[2(\binom{n-2}{1}+2)+2\binom{n-2}{1}+\binom{n-2}{2}]\}+2\{(2\binom{n}{1}+\binom{n}{2})[2[2+\binom{n-2}{1}]+2\binom{n-2}{1}+\binom{n-2}{2}]\}+(2\binom{n}{2}+\binom{n}{3})[2[2+\binom{n-3}{1}+2]+2[\binom{n-3}{1}+2]+2\binom{n-3}{1}+\binom{n-3}{2}] \quad ..(2.10)$$

Proceeding like this, we find the number of B_2's containing all the rank $n+1$ element at the bottom in $S^3(B_n)=$ the number of rank $n+1$ elements in $S^3(B_n)=2\{2[2\binom{n}{n-2}+\binom{n}{n-1}]+2\binom{n}{n-1}\}+2[2\binom{n}{n-1}]$ ....(2.11)

Hence, using (2.7),(2.8),(2.9), (2.10) and (2.11) we get the total number of 4 element convex sublattices in $S^3(B_n)$ is

$A_3 = 2[2+\binom{n}{1}+2] + 2[\binom{n}{1}+2] + 2\binom{n}{1} + \binom{n}{2} + 2[2[\binom{n}{1}+2] +$
$2\binom{n}{1}+\binom{n}{2}] + 2\{2[\binom{n}{1}+2] + 2\binom{n}{1}+\binom{n}{2}\} + (2+\binom{n}{1})\{2[2+\binom{n-1}{1}+2] + 2[\binom{n-1}{1}+2] + 2\binom{n-1}{1}+\binom{n-1}{2}\} +$
$2\{2\{2\binom{n}{1}+\binom{n}{2}\} + (2+\binom{n}{1})[2((\binom{n-1}{1}+2) + 2\binom{n-1}{1}+\binom{n-1}{2}]\} + 2\{(2+\binom{n}{1})[2[\binom{n-1}{1}+2] + 2\binom{n-1}{1}+\binom{n-1}{2}]\} +$
$(2\binom{n}{1}+\binom{n}{2})[2[2+\binom{n-1}{1}+2] + 2[\binom{n-2}{1}+2] + 2\binom{n-2}{1}+\binom{n-2}{2}] + 2\{2\{(2+\binom{n}{1})[2\binom{n-1}{1}+\binom{n-1}{2}]\} + (2\binom{n}{1} +$
$\binom{n}{2})[2((\binom{n-2}{1}+2) + 2\binom{n-2}{1}+\binom{n-2}{2}]\} + 2\{(2\binom{n}{1}+\binom{n}{2})[2[2+\binom{n-2}{1}] + 2\binom{n-2}{1}+\binom{n-2}{2}]\} + (2\binom{n}{2}+\binom{n}{3})[2[2+\binom{n-3}{1} +$
$2] + 2[\binom{n-3}{1}+2] + 2\binom{n-3}{1}+\binom{n-3}{2}] + \ldots + 2\{2[2\binom{n}{n-2}+\binom{n}{n-1}] + 2\binom{n}{n-1}\} + 2[2\binom{n}{n-1}]$ …..(2.1.3)

Proceeding like this, we find that $A_4, A_5, \ldots. A_{n+3}$

$A_4 = 2[2((\binom{n}{1}+2) + 2\binom{n}{1}+\binom{n}{2}] + 2[2\binom{n}{1}+\binom{n}{2}] + 2\binom{n}{2}+\binom{n}{3} + 2\{2[2\binom{n}{1}+\binom{n}{2}] + 2\binom{n}{2}+\binom{n}{3}\} + 2\{2[2\binom{n}{1}+\binom{n}{2}] +$
$2\binom{n}{2}+\binom{n}{3}\} + (2+\binom{n}{1})[2[2((\binom{n-1}{1}+2) + 2\binom{n-1}{1}+\binom{n-1}{2}] + 2[2\binom{n-1}{1}+\binom{n-1}{2}] + 2\binom{n-1}{2}+\binom{n-1}{3}] + 2\{2\{2\binom{n}{2}+\binom{n}{3}\} +$
$(2+\binom{n}{1})[2(2\binom{n-1}{1}+\binom{n-1}{2}) + 2\binom{n-1}{2}+\binom{n-1}{3}]\} + 2\{(2+\binom{n}{1})\{2[2\binom{n-1}{1}+\binom{n-1}{2}] + 2\binom{n-1}{2}+\binom{n-1}{3}\}\} + (2\binom{n}{1} +$
$\binom{n}{2})[2[2[\binom{n-2}{2}+2] + 2\binom{n-2}{1}+\binom{n-2}{2}] + 2[2\binom{n-2}{1}+\binom{n-2}{2}] + 2\binom{n-2}{2}+\binom{n-2}{3}] + \ldots + 2\{2[2\binom{n}{n-3}+\binom{n}{n-2}] + 2\binom{n}{n-2} +$
$\binom{n}{n-1}\} + 2[2\binom{n}{n-2}+\binom{n}{n-1}] + 2\binom{n}{n-1}$ (2.1.4)

In the same manner, A_{n+1} =The number of convex sublattices of rank n in $S^3(B_n)$

$2\{2(2\binom{n}{n-3}+\binom{n}{n-2}) + 2\binom{n}{n-2}+\binom{n}{n-1}\} + 2[2\binom{n}{n-2}+\binom{n}{n-1}] + 2\binom{n}{n-1} + 2\{2(2\binom{n}{n-2}+\binom{n}{n-1}) + 2\binom{n}{n-1}\} + 2\{2[2\binom{n}{n-2} +$
$\binom{n}{n-1}] + 2\binom{n}{n-1}\} + (2+\binom{n}{1})\{2[2(2\binom{n-1}{n-2}) + 2\binom{n-1}{n-2} + 2\binom{n-1}{n-2}] + 2[2\binom{n-1}{n-2}]\} + 2\{2\{2\binom{n}{n-1}\} + (2+\binom{n}{1})\{2(2\binom{n-1}{n-2})\}\} +$
$= 2\{(2+\binom{n}{1})\{2[2\binom{n-1}{n-2}]\} + (2\binom{n}{1}+\binom{n}{2})\{2[2[2\binom{n-2}{n-3}]]\} + 2\{2[\binom{n}{1}+2] + 2\binom{n}{1}+\binom{n}{2}\} + 2[2\binom{n}{1}+\binom{n}{2}] + 2\binom{n}{2}+\binom{n}{3}$. …..(2.1.5)

$A_{n+2} = 2\{2(2\binom{n}{n-2}+\binom{n}{n-1}) + 2\binom{n}{n-1}\} + 2[2\binom{n}{n-1}] + 2\{2[2\binom{n}{n-1}]\} + 2\{2[2\binom{n}{n-1}]\} + (2+\binom{n}{1})[2\{2[2\binom{n-1}{n-2}]\}] + 2[2 +$
$\binom{n}{1}] + 2] + 2[\binom{n}{1}) + 2\binom{n}{1}+\binom{n}{2} \ldots$
…….(2.1.6)

$A_{n+3} = 2\{2[2\binom{n}{n-1}]\} + 2 + \binom{n}{1} + 2 + 2.$ …………(2.1.6)

Case(i): Suppose that n is odd. Therefore, $n+4$ is odd.

$A_1 - A_2 + A_3 - \ldots - A_{n+1} + A_{n+2} - A_{n+3} = 1 + 2 + \binom{n}{1} + 2 + 2 + 2[2+\binom{n}{1}+2] + 2[\binom{n}{1}+2] + 2\binom{n}{1}+\binom{n}{2} +$
$22[\binom{n}{1}+2] + 2\binom{n}{1}+\binom{n}{2} + 2[2\binom{n}{1}+\binom{n}{2}] + 2\binom{n}{2}+\binom{n}{3} + 22[2\binom{n}{1}+\binom{n}{2}] + 2\binom{n}{2}+\binom{n}{3} + 2[2\binom{n}{2}+\binom{n}{3}] + 2\binom{n}{3}+\binom{n}{4} +$
$\cdots + 22[2\binom{n}{n-3}+\binom{n}{n-2}] + 2\binom{n}{n-2}+\binom{n}{n-1} + 2[2\binom{n}{n-2}+\binom{n}{n-1}] + 2\binom{n}{n-1} + 22[2\binom{n}{n-2}+\binom{n}{n-1}] + 2\binom{n}{n-1} + 2[2\binom{n}{n-1}] +$
$22[2\binom{n}{n-1}] + 1 - 2 + \binom{n}{1} + 2 + 2 + 2[2+\binom{n}{1}+2] + 2[2+\binom{n}{1}+2] + (2+\binom{n}{1})[2+2+\binom{n-1}{1}+2] + 2[2[\binom{n}{1}+2] +$
$(2+\binom{n}{1})[2+\binom{n-1}{1}+2]] + 2[(2+\binom{n}{1})(2+\binom{n-1}{1}+2)] + (2\binom{n}{1}+\binom{n}{2})[2+2+\binom{n-2}{1}+2] + 2\{2[(2+\binom{n}{1})(2+$
$\binom{n-1}{1})] + (2\binom{n}{1}+\binom{n}{2})[2+\binom{n-2}{1}+2]\} + 2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)] + 2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)] + (2\binom{n}{2} +$
$\binom{n}{3})[2+2+\binom{n-3}{1}+2] + \ldots + 2\{2[2\binom{n}{n-1}]\} + 2[2+\binom{n}{1}+2] + 2[\binom{n}{1}+2] +$
$2\binom{n}{1}+\binom{n}{2} + 2[2[\binom{n}{1}+2] +$
$2\binom{n}{1}+\binom{n}{2}] + 2\{2[\binom{n}{1}+2] + 2\binom{n}{1}+\binom{n}{2}\} + (2+\binom{n}{1})\{2[2+\binom{n-1}{1}+2] + 2[\binom{n-1}{1}+2] + 2\binom{n-1}{1}+\binom{n-1}{2}\} +$
$2\{2\{2\binom{n}{1}+\binom{n}{2}\} + (2+\binom{n}{1})[2((\binom{n-1}{1}+2) + 2\binom{n-1}{1}+\binom{n-1}{2}]\} + 2\{(2+\binom{n}{1})[2[\binom{n-1}{1}+2] + 2\binom{n-1}{1}+\binom{n-1}{2}]\} +$
$(2\binom{n}{1}+\binom{n}{2})[2[2+\binom{n-1}{1}+2] + 2[\binom{n-2}{1}+2] + 2\binom{n-2}{1}+\binom{n-2}{2}] + 2\{2\{(2+\binom{n}{1})[2\binom{n-1}{1}+\binom{n-1}{2}]\} + (2\binom{n}{1} +$
$\binom{n}{2})[2((\binom{n-2}{1}+2) + 2\binom{n-2}{1}+\binom{n-2}{2}]\} + 2\{(2\binom{n}{1}+\binom{n}{2})[2[2+\binom{n-2}{1}] + 2\binom{n-2}{1}+\binom{n-2}{2}]\} + (2\binom{n}{2}+\binom{n}{3})[2[2+\binom{n-3}{1} +$
$2] + 2[\binom{n-3}{1}+2] + 2\binom{n-3}{1}+\binom{n-3}{2}] + \ldots + 2\{2[2\binom{n}{n-2}+\binom{n}{n-1}] + 2\binom{n}{n-1}\} + 2[2\binom{n}{n-1}]$

-

$$2[2((\tbinom{n}{1}+2)+2\tbinom{n}{1}+\tbinom{n}{2}]+2[2\tbinom{n}{1}+\tbinom{n}{2}]+2\tbinom{n}{2}+\tbinom{n}{3}+2\{2[2\tbinom{n}{1}+\tbinom{n}{2}]+2\tbinom{n}{2}+\tbinom{n}{3}\}+2\{2[2\tbinom{n}{1}+\tbinom{n}{2}]+2\tbinom{n}{2}+\tbinom{n}{3}\}+(2+\tbinom{n}{1})[2[2((\tbinom{n-1}{1}+2)+2\tbinom{n-1}{1}+\tbinom{n-1}{2}]+2[2\tbinom{n-1}{1}+\tbinom{n-1}{2}]+2\tbinom{n-1}{2}+\tbinom{n-1}{3}]+2\{2[2\tbinom{n}{2}+\tbinom{n}{3}\}+(2+\tbinom{n}{1})[2(2\tbinom{n-1}{1}+\tbinom{n-1}{2}))+2\tbinom{n-1}{2}+\tbinom{n-1}{3}]]\}+2\{(2+\tbinom{n}{1})\{2[2\tbinom{n-1}{1}+\tbinom{n-1}{2}]+2\tbinom{n-1}{2}+\tbinom{n-1}{3}\}\}+(2\tbinom{n}{1}+\tbinom{n}{2})[2[2[\tbinom{n-2}{2}+2]+2\tbinom{n-2}{1}+\tbinom{n-2}{2}]+2[2\tbinom{n-2}{1}+\tbinom{n-2}{2}]+2\tbinom{n-2}{2}+\tbinom{n-2}{3}]+\dots+2\{2[2\tbinom{n}{n-3}+\tbinom{n}{n-2}]+2\tbinom{n}{n-2}+\tbinom{n}{n-1}\}+2[2\tbinom{n}{n-2}+\tbinom{n}{n-1}]+2\tbinom{n}{n-1}$$

$+\dots-$

$$2\{2(2\tbinom{n}{n-3}+\tbinom{n}{n-2})+2\tbinom{n}{n-2}+\tbinom{n}{n-1}\}+2[2\tbinom{n}{n-2}+\tbinom{n}{n-1}]+2\tbinom{n}{n-1}+2\{2(2\tbinom{n}{n-2}+\tbinom{n}{n-1})+2\tbinom{n}{n-1}\}+2\{2[2\tbinom{n}{n-2}+\tbinom{n}{n-1}]+2\tbinom{n}{n-1}\}+(2+\tbinom{n}{1})\{2[2(2\tbinom{n-1}{n-2})+2\tbinom{n-1}{n-2}+2\tbinom{n-1}{n-2}]+2[2\tbinom{n-1}{n-2}]\}+2\{2\{2\tbinom{n}{n-1}\}+(2+\tbinom{n}{1})\{2(2\tbinom{n-1}{n-2})\}\}+2\{(2+\tbinom{n}{1})\{2[2\tbinom{n-1}{n-2}]\}+(2\tbinom{n}{1}+\tbinom{n}{2})\{2[2[2\tbinom{n-2}{n-3}]]\}+2\{2[\tbinom{n}{1}+2]+2\tbinom{n}{1}+\tbinom{n}{2}\}+2[2\tbinom{n}{1}+\tbinom{n}{2}]+2\tbinom{n}{2}+\tbinom{n}{3}+2\{2(2\tbinom{n}{n-2}+\tbinom{n}{n-1})+2\tbinom{n}{n-1}\}+2[2\tbinom{n}{n-1}]+2\{2[2\tbinom{n}{n-1}]\}+2\{2[2\tbinom{n}{n-1}]\}+(2+\tbinom{n}{1})[2\{2[2\tbinom{n-1}{n-2}]\}]+2[2+\tbinom{n}{1}]+2]+2[\tbinom{n}{1}+2\tbinom{n}{1}+\tbinom{n}{2}$$

$$-2\{2[2\tbinom{n}{n-1}]\}+2+\tbinom{n}{1}+2+2$$

$$=0.$$

Case(ii): Suppose that n is even. Therefore, $n+4$ is even.

$$A_1-A_2+A_3-\dots+A_{n+1}-A_{n+2}+A_{n+3}=1+2+\tbinom{n}{1}+2+2+2[2+\tbinom{n}{1}+2]+2[\tbinom{n}{1}+2]+2\tbinom{n}{1}+\tbinom{n}{2}+22[\tbinom{n}{1}+2]+2\tbinom{n}{1}+\tbinom{n}{2}+2[2\tbinom{n}{1}+\tbinom{n}{2}]+2\tbinom{n}{2}+\tbinom{n}{3}+22[2\tbinom{n}{1}+\tbinom{n}{2}]+2\tbinom{n}{2}+\tbinom{n}{3}+2[2\tbinom{n}{2}+\tbinom{n}{3}]+2\tbinom{n}{3}+\tbinom{n}{4}+\dots+22[2\tbinom{n}{n-3}+\tbinom{n}{n-2}]+2\tbinom{n}{n-2}+\tbinom{n}{n-1}+2[2\tbinom{n}{n-2}+\tbinom{n}{n-1}]+2\tbinom{n}{n-1}+22[2\tbinom{n}{n-2}+\tbinom{n}{n-1}]+2\tbinom{n}{n-1}+2[2\tbinom{n}{n-1}]+22[2\tbinom{n}{n-1}]+1-2+\tbinom{n}{1}+2+2+2[2+\tbinom{n}{1}+2]+2[2+\tbinom{n}{1}+2]+(2+\tbinom{n}{1})[2+2+\tbinom{n-1}{1}+2]+2[2[\tbinom{n}{1}+2]+(2+\tbinom{n}{1})[2+\tbinom{n-1}{1}+2]]+2[(2+\tbinom{n}{1})(2+\tbinom{n-1}{1}+2)]+(2\tbinom{n}{1}+\tbinom{n}{2})[2+2+\tbinom{n-2}{1}+2]+2\{2[(2+\tbinom{n}{1})(2+\tbinom{n-1}{1})]+(2\tbinom{n}{1}+\tbinom{n}{2})[2+\tbinom{n-2}{1}+2]\}+2[(2\tbinom{n}{1}+\tbinom{n}{2})(2+\tbinom{n-2}{1}+2)]+2[(2\tbinom{n}{1}+\tbinom{n}{2})(2+\tbinom{n-2}{1}+2)]+(2\tbinom{n}{2}+\tbinom{n}{3})[2+2+\tbinom{n-3}{1}+2]+\dots+2\{2[2\tbinom{n}{n-1}]\}+2[2+\tbinom{n}{1}+2]+2[\tbinom{n}{1}+2]+$$

$$2\tbinom{n}{1}+\tbinom{n}{2}+2[2[\tbinom{n}{1}+2]+$$

$$2\tbinom{n}{1}+\tbinom{n}{2}]+2\{2[\tbinom{n}{1}+2]+2\tbinom{n}{1}+\tbinom{n}{2}\}+(2+\tbinom{n}{1})\{2[2+\tbinom{n-1}{1}+2]+2[\tbinom{n-1}{1}+2]+2\tbinom{n-1}{1}+\tbinom{n-1}{2}\}+2\{2\{2\tbinom{n}{1}+\tbinom{n}{2}\}+(2+\tbinom{n}{1})[2((\tbinom{n-1}{1}+2)+2\tbinom{n-1}{1}+\tbinom{n-1}{2}]\}+2\{(2+\tbinom{n}{1})[2[\tbinom{n-1}{1}+2]+2\tbinom{n-1}{1}+\tbinom{n-1}{2}]\}+(2\tbinom{n}{1}+\tbinom{n}{2})[2[2+\tbinom{n-1}{1}+2]+2[\tbinom{n-2}{1}+2]+2\tbinom{n-2}{1}+\tbinom{n-2}{2}]+2\{2\{(2+\tbinom{n}{1})[2\tbinom{n-1}{1}+\tbinom{n-1}{2}]\}+(2\tbinom{n}{1}+\tbinom{n}{2})[2((\tbinom{n-2}{1}+2)+2\tbinom{n-2}{1}+\tbinom{n-2}{2}]\}+2\{(2\tbinom{n}{1}+\tbinom{n}{2})[2[2+\tbinom{n-2}{1}]+2\tbinom{n-2}{1}+\tbinom{n-2}{2}]\}+(2\tbinom{n}{2}+\tbinom{n}{3})[2[2+\tbinom{n-3}{1}+2]+2[\tbinom{n-3}{1}+2]+2\tbinom{n-3}{1}+\tbinom{n-3}{2}]+\dots+2\{2[2\tbinom{n}{n-2}+\tbinom{n}{n-1}]+2\tbinom{n}{n-1}\}+2[2\tbinom{n}{n-1}]$$

$-$

$$2[2((\tbinom{n}{1}+2)+2\tbinom{n}{1}+\tbinom{n}{2}]+2[2\tbinom{n}{1}+\tbinom{n}{2}]+2\tbinom{n}{2}+\tbinom{n}{3}+2\{2[2\tbinom{n}{1}+\tbinom{n}{2}]+2\tbinom{n}{2}+\tbinom{n}{3}\}+2\{2[2\tbinom{n}{1}+\tbinom{n}{2}]+2\tbinom{n}{2}+\tbinom{n}{3}\}+(2+\tbinom{n}{1})[2[2((\tbinom{n-1}{1}+2)+2\tbinom{n-1}{1}+\tbinom{n-1}{2}]+2[2\tbinom{n-1}{1}+\tbinom{n-1}{2}]+2\tbinom{n-1}{2}+\tbinom{n-1}{3}]+2\{2[2\tbinom{n}{2}+\tbinom{n}{3}\}+(2+\tbinom{n}{1})[2(2\tbinom{n-1}{1}+\tbinom{n-1}{2}))+2\tbinom{n-1}{2}+\tbinom{n-1}{3}]]\}+2\{(2+\tbinom{n}{1})\{2[2\tbinom{n-1}{1}+\tbinom{n-1}{2}]+2\tbinom{n-1}{2}+\tbinom{n-1}{3}\}\}+(2\tbinom{n}{1}+\tbinom{n}{2})[2[2[\tbinom{n-2}{2}+2]+2\tbinom{n-2}{1}+\tbinom{n-2}{2}]+2[2\tbinom{n-2}{1}+\tbinom{n-2}{2}]+2\tbinom{n-2}{2}+\tbinom{n-2}{3}]+\dots+2\{2[2\tbinom{n}{n-3}+\tbinom{n}{n-2}]+2\tbinom{n}{n-2}+\tbinom{n}{n-1}\}+2[2\tbinom{n}{n-2}+\tbinom{n}{n-1}]+2\tbinom{n}{n-1}$$

$+\dots+$

$$2\{2(2\tbinom{n}{n-3}+\tbinom{n}{n-2})+2\tbinom{n}{n-2}+\tbinom{n}{n-1}\}+2[2\tbinom{n}{n-2}+\tbinom{n}{n-1}]+2\tbinom{n}{n-1}+2\{2(2\tbinom{n}{n-2}+\tbinom{n}{n-1})+2\tbinom{n}{n-1}\}+2\{2[2\tbinom{n}{n-2}+\tbinom{n}{n-1}]+2\tbinom{n}{n-1}\}+(2+\tbinom{n}{1})\{2[2(2\tbinom{n-1}{n-2})+2\tbinom{n-1}{n-2}+2\tbinom{n-1}{n-2}]+2[2\tbinom{n-1}{n-2}]\}+2\{2\{2\tbinom{n}{n-1}\}+(2+\tbinom{n}{1})\{2(2\tbinom{n-1}{n-2})\}\}+2\{(2+\tbinom{n}{1})\{2[2\tbinom{n-1}{n-2}]\}+(2\tbinom{n}{1}+\tbinom{n}{2})\{2[2[2\tbinom{n-2}{n-3}]]\}+2\{2[\tbinom{n}{1}+2]+2\tbinom{n}{1}+\tbinom{n}{2}\}+2[2\tbinom{n}{1}+\tbinom{n}{2}]+2\tbinom{n}{2}+\tbinom{n}{3}+2\{2(2\tbinom{n}{n-2}+\tbinom{n}{n-1})+2\tbinom{n}{n-1}\}+2[2\tbinom{n}{n-1}]+2\{2[2\tbinom{n}{n-1}]\}+2\{2[2\tbinom{n}{n-1}]\}+(2+\tbinom{n}{1})[2\{2[2\tbinom{n-1}{n-2}]\}]-2[2+\tbinom{n}{1}]+2]+2[\tbinom{n}{1}+2\tbinom{n}{1}+\tbinom{n}{2}+$$

$$2\{2[2\tbinom{n}{n-1}]\}+2+\tbinom{n}{1}+2+2$$

$$=2.$$

Hence the interval $[\emptyset, S^3(B_n)]$ has the same number of elements of odd and even rank.

Though in the above theorem we have proved that $CS[S^3(B_n)]$ is Eulerian, it is neither Simplicial nor dual simplicial.

$CS[S^3(B_n)]$is not dual simplicial since, the upper interval $[\{1\}, S^3(B_n)]$ in $CS[S^3(B_n)]$contains $8\binom{n}{n-1}$ number of atoms which is greater than $n+3$, the rank of $[\{1\}, S^3(B_n)]$,implying that $[\{1\}, S^3(B_n)]$is not Boolean.

$CS[S^3(B_n)]$is not simplicial since, the lower interval $[\emptyset, S^3(B_n)]$ where l_1 is the left extreme atom of $S^3(B_n)$ contains $3^3.2^n - 26$ number of atoms by Lemma 2.1, which cannot be equal to $n+3$, the rank of $[\emptyset, [l_1, 1]]$, implying that $[\emptyset, [l_1, 1]]$ is not Boolean.

Conclusions

In this paper, we have proved that $CS[S^3(B_n)]$ is an Eulerian lattice under the set inclusion relation which is neither simplicial nor dual simplicial, if $n > 1$. We strongly believe that the result proved in this paper, can be extended to more general Eulerian lattices and any other general lattices.

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REFERENCES

[1] Chen C. K., Koh K. M., *On the lattice of convex sublattices of a finite lattice*, Nanta Math., 5 (1972), 92-95.

[2] Gratzer G., *General Lattice Theory*, Birkhauser Verlag, Basel, 1978.

[3] Koh K. M., *On the lattice of convex sublattices of a finite lattice*, Nanta Math., 5 (1972), 18-37.

[4] Lavanya S., Parameshwara Bhatta S., *A new approach to the lattice of convex sublattices of a lattice*, Algebra Univ., 35 (1996), 63-71.

[5] Paffenholz A., *Constructions for Posets, Lattices and Polytopes, Doctoral Dissertation*, School of Mathematics and Natural Sciences, Technical University of Berlin,(2005).

[6] Ramana Murty P. V., *On the lattice of convex sublattices of a lattice* ,Southeast Asian Bulletin of Mathematics, 26 (2002), 51-55.

[7] Rota G. C., *On the foundations of Combinatorial theory I, Theory of Mobius functions*, Z. Wahrschainlichkeitstheorie, 2 (1964), 340-368.

[8] Sheeba Merlin and Vethamanickam. A., *On the Lattice of Convex Sublattices of S(B_n) and S(C_n)*, European journal of pure and applied Mathematics., Vol. 10, No. 4, 2017, 916-928.

[9] Stanley R.P., *Some aspects of groups acting on finite posets*, J. Combinatoria theory, A. 32 (1982), 131-161.

[10] Stanley R.P., *A survey of Eulerian posets*, *Polytops: abstract, convex and computational*, Kluwer Acad. Publi., Dordrecht, (1994), 301-333.

[11] Stanley R.P., *Enumerative Combinatorics*, Woodsworth and Brooks, Cole, Vol 1, 1986.

[12] Santhi V. K., *Topics in Commutative Algebra*, Ph. D thesis, Madurai Kamaraj University, 1992.

[13] Vethamanickam A., *Topics in Universal Algebra*, Ph. D thesis, Madurai Kamaraj University, 1994.

[14] Vethamanickam A., Subbarayan R., *Some simple extensions of Eulerian lattices*, Acta Math. Univ., Comenianae, 79(1) (2010), 47-54.

[15] Vethamanickam A., Subbarayan R., *On the lattice of convex sublattices,* Elixir Dis.Math., Comenianae, 50 (2012), 10471-10474.