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ON COMBINATORIAL AND COMPUTATIONAL PROBLEMS IN GEOMETRY WITH SPECIAL REFERENCE TO POLYTOPES

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ABSTRACT

We start with basics in Discrete Geometry such as polytope, polyhedral. Its combinatorial property and covered some results on simplicial polytopes

Keywords: Convex polytopes, Simplicial polytopes, ranked atomic lattices, face lattice, Posets, Simplicial Complex

1 INTRODUCTION

Since ancient times, convex polytopes have been studied as fundamental geometrical entities, their relevance to a wide range of other mathematical topics, including algebraic geometry, algebraic topology and linear and combinatorial optimization, today serves to enhance the beauty of their theory. Moreover, give a quick summary of "what polytopes look like" and "how they behave", with numerous concrete examples and briefly stated some main results.

Combinatorial properties faces(vertices, edges,…..facets) of polytopes and their relations, with special considerations of the classes of low-dimensional polytopes and the polytopes 'with few vertices' .Geometric properties such as volume and surface area, mixed volumes and quer-mass integrals with precide formulas studied for the situations of regular simplices, cubes and cross-polytopes.

In this section we use some Definations and results

Definition 1.1: A polytope is considered as a smallest convex polygon with all the points (i.e convex hull) taken from a finite set, a subset

þ=Conv(S) ⊆$Ŕ^d$ for some finite set S⊆$Ŕ^d$

Example 1.2: Any type of triangle, point, covered closed line segment, empty set ({ }) and convex polygon(in some $Ŕ^n$)

Definition 1.3: For any S ⊆$Ŕ^d$ the Convex hull of S is a conv(S) = ∩{ K ⊆$Ŕ^d$: K convex, S⊆K⊆ $Ŕ^d$ }

Definition 1.4: Face of a conv(þ) is subset of the form F = {a∈ þ:e^ta=α̇} , e^ta ≤ α̇ is valid for þ (i.e it is true for all a∈þ)

We know that ∅ and þ themselves are faces , the trivial faces. Remaining all are non-trivial faces.

Definition 1.5: A poset is a partially ordered set Ȣ with binary relation "≤" is reflexive a ≤ a, ∀ a ∈ Ȣ (i.e every element is related to itself), symmetric (a ≤ b & b ≤ a ⟹ a = b, Transitive (a ≤ b & b ≤ d⟹ a ≤ d).The posets we consider are countable . The poset is written as (Ȣ,≤)

Subposet contains interval which is a poset (Ȣ,≤)

[a,b] = ({ d∈Ȣ : a is less than or equal to d & d is less than or equal to b }, ≤) for all a,b∈Ȣ with a ≤ b

If there is unique minimal elt (Ō̈) and a unique maximal elt Ï in a þ then it is said to be bounded

Here ř : Ȣ→ N becomes rank function of Ȣ as a poset and has unique minimal element Ȍ, and with maximal chains from Ȍ to having equipped with equal length called the rank of the element ř(a) then such poset is called as graded G. .

So finally ř(Ȣ)=ř(Ï) is the form to be considered for the rank of the poset which is graded with Ï

Definition 1.6: Lattice is a Poset with any two elements a & b , having infimum (unique minimal) upper bound i.e avb (called the join of a & b) and supremum(unique maximal) lower bound i.e a∧b (called the meet of a & b)

Theorem 1.7

The face poset (Ꞙ, ⊂) is a finite graded G lattice of a convex polytope , given by Ł = Ł(þ) with rank ř(Ł(þ))= dim (þ) +1

Proof : Clearly with minimal element Ȍ =∅ and maximal element Ï= þ may exist face lattice is finite bounded poset , also as F∧ F^I=F∩ F^I one of the biggest face which is contained in F and F^I

Thus Ł(þ) can be said as ranked atomic lattice with set of faces of polytope(þ). Thus P is face lattice of þ

For every x∈Ł the Lattice(Ł) is ranked if x has the same size called the rank of x Thus an atom has rank Ï and Ł is atomic due to each and every join being an atom.

2 Combinatorially Equivalent:

þ & þ' are equivalent(Combinatorially) , if Ł & Ł' (face lattices) are isomorphic posets also as there exist a Bijection f from Ł to Ł' ∋ a ≤ b is true in þ iff f(a) ≤f(b) is true in þ'

Lemma 2.1: polytopes þ, þ' are affinely Isomorphic then they are equivalent (Combinatorially) where as the converse is not true

The collection of all the flat surfaces called the faces of polyhedron þ ordered by an inclusion is a Lattice.

A meet -semilattice forms a poset (þ) with meet operation and addition of maximal element, so each finite meet semilattice becomes lattice

Face lattice of þ is isomorphic (Combinatorially) to polyhedral complex (which is meet semilattice where lower interval is isomorphic) .Such lattice can be called the face lattice or B(þ) i.e boundary complex of þ . Hence any two R & S polytopes are equivalent (Combinatorially) as their face lattices (P) are isomorphic to each other.

$[F, G]$ is isomorphic P(face lattice) of some Q polytope called Quotient polytope of þ as $F \subset G$ then they are two different faces of a & þ

And $Q =$ þ$/F$ if $G =$ þ

Lemma 2.2: (Face lattice (F)of a simplex)

Face lattice is isomorphic(i.e their face posets (Ꞙ, ⊂)) are isomorphic to the poset all subsets of a k-elt set, ordered by inclusion known as Boolean lattice which is complimented distributive or Boolean Algebra B_k (with rank k where X_{k-1} is a k-1 dim simplex with vertices k)

Theorem 2.3: Face lattices of þ and intervals of face lattices of þ are same

Proof: Let H⊆ F be faces of þ.Then [H, F] is face lattice of þ with dim(F) - dim(H) -1 &

[H, F] =({ I∈ Ł(þ) such that H ⊆ I⊆ F },⊆) with dimn (F) – dimn (H) -1 is face lattice of þ i.e if H=∅ then [H, F] = Ł(þ)

If F= þ & H={▯} ia a vertex then [H,F]= $Ł(p/v)$

3 Simple and Simplicial Polytopes

þ is simplicial and simple if all its facets are simplices and vertex figures are simplices respectively

Definition 3.1: Star of Ƒ in a simplicial complex K

Let K be simplicial complex with face Ƒ of K

Then st(Ƒ,K) star of Ƒ in K is a simplicial complex containing all faces

For K with ▯ the vertex then st(▯,K) is cone over ▯ in Ƒ

Lemma 3.2: Let K be the shellable (S) simplicial complex then any order of K restricted is a shelling star (υ, K) which imbibes a shelling order link of (υ,K) where K is the shellable simplicial complex

B(þ) known as Boundary complex of þ and polytope (faces) are simplicial complices.

Lemma 3.3: Let K=K(B(þ)) where B(þ) is called the boundary complex of þ

Then h_m(K$/V$) ≤ h_m (K) ,$for\ every\ \upsilon\epsilon$vert(K) and for all m≤1 iff þ is (l+1) – neighbourly

Every face link of simplicial complex (K) is itself simplicial complex

If Ƒ is face of þ with [Ƒ,þ] in Ł(þ) is the face lattice denoted by þ/Ƒ(of a simplicial polytope whose facets are all simplices) is also simplicial polytope

B(þ) of þ/Ƒ and the face Ƒ in B(þ) have a link

A simplicial d-sphere is K such that |K| is homeomorphic to S^d (a sphere with d- dimension)

So B(þ) d- polytope (simplicial) is a simplicial (d-1) sphere however the d-1 sphere cannot be simplicial d-polytope.

As a consequence of the above details covered many results on simplicial polytopes continue to arbitrary simplicial spheres

One such consequence is

A lattice Ł (ranked atomic) can be called as relatively complimented if in Ł every interval is atomic. Consider a >b be two elements in Ł which are atomic of rank two and b is not covered by a then we know that there should exist atleast two elements of Ł between a & b strictly. So we prove that if Ł is of finite length (with smallest element Ő and greatest element Ï) then the Ł with no 3- elements interval is relatively complemented too and if ∀ a∈Ł the set C_a={b∈Ł such that b covers a} satisfies ∨$\mathfrak{C}_a$=1, then Ł we can say as complemented

A bounded Ł in which every interval [a,b] is reltively complimented is also a complimented lattice .

we discuss here lattice of finite length with some of its compliments and intervals considering the property join of elements (greatest element)

Here we consider closed interval lattices with length 2

Interval [a,b] ={ d∈Ł ∋ ,a≤ d& d ≤ b } Here [a,b] in lattice Ł with length finite (∨-regular) is atomic if b is (v)of atoms and dually, [a,b] a lattice Ł of length finite (∧- regular) is said to be coatomic if every element is a (∧)of a set coatoms. Thus interval [a,b] of finite length and Ł a complimented lattice having Ő (smallest number) and Ï (greatest number)

Thus [Ő,Ï] is ⋁-regular where b=1 is called upper and denoted by (Ừ) and a=Ő is called lower denoted by (Ὼ)

The hexagon lattice is the example which shows that [Ő,Ï] this may be the only interval of a which Ł(complimented) is ⋁-regular

Theorem 3.4

Let lattice (Ł) of uniform neighbourhood such that all Ừ upper intervals are ⋁-regular so Ł is complimented.

Proof: Let us apply mathematical induction on its length Ł i.e if Ł has length two (ŉ=2)

Assume all lattices Ł with ŉ<2 (length) and also l(Ł) = ŉ≥ 3. That any element a∈{3,4,5,6….} then there must be an atoms S≰a. we see all atoms of Ł are here below 'a' hence [0,1] is ⋁ - regular.

It implies that Ï≤a. Let us choose such atoms then all upper intervals of the sub lattice [s,1] are ⋁-regular and l([s,1])<ŉ. Hence, by induction assumption [s,1] is complimented

We know that since s≰a and a is an atom s⋀a =Ő

Now let c denote for s⋁a in [s,1] then we know that s⋁a must have

C ⋁ a = (C ⋁ s) ⋁ a = C ⋁(s ⋁ a) = Ï

C ⋀ a = C ⋀ ((s ⋁ a) ⋀ a = (C ⋀(s ⋁ a)) ⋀ a =s⋀a =Ő

Therefore C is compliment to a in Ł. It is clear that Ừ cannot be replaced by family of Ὼ, if so then Ł is not complimented.

Theorem 3.5:

Let lattice (Ł) of uniform neighbourhood then the equivalent cases arise

i) Lattice Ł is relatively complimented such that every interval is relatively complimented

ii) All of the (Ừ) are ⋁-regular

iii) All (Ὼ) are ⋀-regular

iv) Ł does not have interval of 3- element

Proof: It is clearly seen from theorem 1 (i)⇒(iv) & (ii)⟹(i) & (iv)⟹(ii)

Assume (iv) is true in Ł because intervals with length 2 are v-regular intervals

Let us continue with mathematical induction to the left part of interval [a,b] in Ł

Assume that length ŉ, ŉ≥3 & interval of length less than ŉ are ⋁-regular

Let a=a_1<a_2<……a_{n+1} =b be maxima chain in [a,b] . By (iv) there must be some d≠a_n,such that a_{n-1}<d<b since b covers a_n & a_n covers a_{n-1} we get d⋁a_n =b.

Now let $Å_b$ of the interval [a,b], $Å_d$ of [a,d] and $Å_{an}$ of [a,a_n] . So ($Å_d$U $Å_{an}$ ⊆$Å_b$. By the induction assumption d=⋁$Å_d$ & a_n =⋁$Å_{an}$. ∴ b≥⋁ $Å_b$≥⋁($Å_d$∪ $Å_{an}$) =(⋁ $Å_d$)⋁(⋁$Å_{an}$)=d⋁a_n=b

So [a,b] is ⋁-regular. Since (i) &(iv) are its own self dual ,equivalence of (iii) with some of other conditions is also true

CONCLUSION

Lattice Ł(ranked atomic) is relatively complimented if in Ł every interval is atomic, enough to consider every interval with rank 2 being atomic or if a>b in Ł for two elements a & b and b is not covered by a, then there exist atleast elts two strictly between a & b of Ł

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