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Stochastic Modelling and Computational Sciences

GAUSS BONNET THEOREM

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ABSTRACT

The Gauss Bonnet theorem is one of the most amazing results that relates the topological and geometric features of differential geometry to the topological properties of surfaces. This explanation paper will explain the local and global proofs of the Gauss-Bonnet theorem and discuss its consequences.

Keywords: Gaussian Curvature, Geodesic Curvature, Euler Poincare formula, triangulation of regular surfaces.

CONTENTS

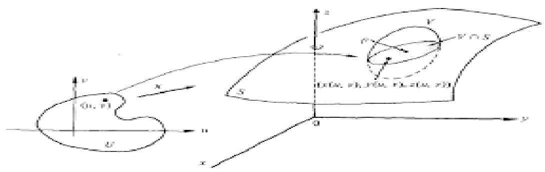
1. INTRODUCTION

The local and universal geometry of curves, surfaces, and manifolds are fascinatingly studied in the field of differential geometry. It is a widely accepted conclusion in geometry. A foundational concept in differential geometry is the Gauss Bonnet theorem. It plays a crucial role in connecting the surface's topological structure (in the sense of the topological invariant, or the Euler characteristic) and its geometrical data (in the sense of curvature).In other words, the theorem asserts that you have an integral in local geometry on one side. On the other hand, the global topological property of the manifold.

After Carl Friedrich Gauss, who worked with a special case of geodesic triangles but did not publish it until roughly 1848, Piere Ossian Bonnet expanded the theorem to a reason bounded by non-geodesic simple curves. Additionally, a surprising invariant was constructed connecting surface curvature to the idea of angle. However, recent advances in topology from the 19th and 20th centuries. And also develop into a priceless contribution to contemporary mathematics. Focus is now being placed on its use. We can highlight the fact that Fourier methods can be used to calculate the integral even when the function is a generational function, like distribution.

2. Topological and Geometrical Preliminaries.

Definition 2.1: An open subset U$\sqsubseteq \mathbb{R}^3$such that there exists a parametrization $x: U \to V \cap S$ with the properties x is infinitely differentiable, x is a homeomorphism, and for each q= $(u, v) \in U$ it holds that $\frac{\partial x}{\partial u}$,$\frac{\partial x}{\partial v}$ are linearly independent (i.e the partial differential is one -to-one) indicates that a set S$\sqsubset \mathbb{R}^3$ is **regular surface**.



Definition 2.2 A regular surface triangulated R is a finite collection T of triangles T_j , $j = 1,2 \ldots, n$ such that i) $\bigcup_{j=1}^{n} T_j$ = R. i.e. every point of a regular surface R is in at least one of the curvilinear polygons.

ii) If $T_i \cap T_j \neq \emptyset$, then T_i and T_j either share a common vertex or a common edge .i.e. Two curvilinear polygons intersect only at a common edge or a common vertex.

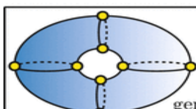
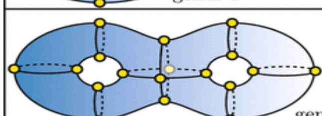
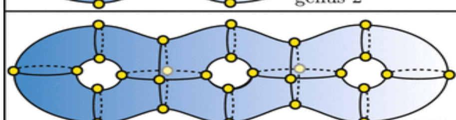
Proposition 2.3: A triangulation is admissible in any regular region of a regular surface S.

Definition 2.4: When every triangle in a certain triangulation has an orientation that is compatible, the surface is said to be orientable.

Remark 2.5: As long as it is simply connected and has precisely three corners or verities, a regular region is considered triangular. An edge of the region is any portion of the boundary between any two vertices. The term "triangulation" refers to the partition of a regular surface R into a finite number of triangular regions T_i also known as the triangulation's faces, such that whenever $T_i \cap T_j \neq \emptyset$ then either $T_i \cap T_j$ is a common vertex or a common edge of T_i and T_j.

Definition 2.6: If V, E and F denotes the number of vertices, edges, and faces respectively in the triangulation of R , then Euler Poincare formula of a surface $R \sqsubset$ S is defined by $\chi(R)$ = F – E + V.

Examples:

	V	E	F	χ
 genus 1	8	16	8	0
 genus 2	18	36	16	-2
 genus 3	28	56	24	-4
surface of genus g	$10g-2$	$20g-4$	$8g$	$2-2g$

Proposition 2.7: The Euler characteristic χ is a topological variation and does not rely on the triangulation of the region. For example, if two things have the same Euler characteristic, they are topologically comparable. However, objects having the same Euler characteristic need not be topologically identical.

Definition 2.8: Given a vector w in the tangent plane, the first fundamental form of a regular surface S at a point $p=(u,v)$ is the inner product $I_p : T_p(S) \to R$ is defined at p by $I_p(w) = \langle w, w \rangle = |w|^2 \geq 0.$

Under the parameterization (u, v) , I_p can be expressed in terms of the basis { X_u, X_v}.

Let α(t) = X (u(t),v(t)) for t $\in (-\epsilon,\epsilon)$ be a curve on S such that α(0)=p and α^1(0)= w, then

I_p (w) = $<\alpha'(0),\alpha'(0)>$

$= < x_u u' + x_v v', x_u u' + x_v v' >$

$= <x_u, x_u> (u')^2 + 2<x_u, x_v> u'v' + <x_v, x_v> (v')^2$

$= E (u')^2 + 2 F u'v' + G (v')^2$

Where E = $<x_u, x_u>$, F = $<x_u, x_v>$ and G = $<x_v, x_v>$ are coefficient of the first fundamental form.

Definition 2.9: Given a parameterization of a surface x: U → S, we call that parameterization orthogonal if F = 0.

3. Gaussian Curvature and Geodesic Curvature

Definition 3.1: The deviation of a curve from the shortest arc connecting two points on a surface is measured by the geodesic curvature Kg of the curve.

Proposition 3.2: Let F = 0, then $K_g = \frac{1}{2\sqrt{EG}}\left\{G_u \frac{dv}{ds} - E_v \frac{du}{ds}\right\} + \frac{d\emptyset_i}{ds}$

Where $\emptyset_i$ is the angle between x_u and a parallel field of unit vectors in the given orientation.

Definition 3.3: A surface's intrinsic measure of curvature at a particular location is its Gaussian curvature K. Taking into accounts the surfaces maximum and minimum curvatures at a certain spot, it is determined. K is obtained by multiplying these values in a formal way.

Proposition 3.4: If F = 0, then $K = \frac{-1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}$.

Theorem 3.5: (The Gauss – Green Theorem): Consider a smooth, simple, positively oriented curve C in a plane . Let $P\ (u, v)$ and $Q\ (u, v)$ are differentiable functions in a simple region in the uv plane, the boundary of which is given by $u = u(s), v = v\ (s)$

then we have $\sum_{i=0}^{k} \int_{S_i}^{S_{i+1}} P\left(\frac{du}{ds}\right) + Q\left(\frac{dv}{ds}\right).ds = \iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) dudv.$

Where the path integral is traversed counterclockwise along with curve C.

Theorem 3.6: (Theorem of Turning Tangents): Let $\emptyset_i : [S_i, S_{i+1}] \to$ R be differentiable functions that measures the positive angle from x_u to α^1(t) at each S_i. Then we have

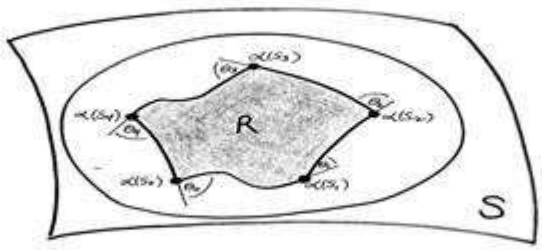
$$\sum_{i=0}^{k}[\emptyset_i(S_{i+1}) - [\emptyset_i(S_i)]] + \sum_{i=0}^{k} \emptyset_i = \pm 2\pi.$$

i.e Based on the orientation of α , the sign of 2π is either positive or negative.

4. The Local Gauss – Bonnet Theorem

Theorem -4.1: Let $x : U \to S$ be an orthogonal parametrization (i.e F = 0) of an oriented surface S ,where U ⊏ R^2 is homeomorphic to an open disk and x is compatible with the orientation of S . Let $R \sqsubset x(U)$be a simple region of S and α be such that $\partial R = \alpha(I)$,$\alpha: I \to$S .Assume α is positively oriented and parameterized piecewise by arc length s_i and let $\{\alpha(s_i)\}_{i=0}^{k}$ and $\{\theta_i\}_{i=0}^{k}$ be the vertices and external angles at that vertices α , respectively. Then

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} K_g\ (s).ds + \int_R K.d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi$$



OR

$\sum_{edges} \int K_g(s).ds + \iint_R K + \sum_{vertices} \theta_i = 2\pi$

Proof: Let $u = u(s), v = v(s)$ be the expression of α under the parameterization x .By (proposition 3.2) Kg=$\frac{1}{2\sqrt{EG}}\left\{G_u \frac{dv}{ds} - E_v \frac{du}{ds}\right\}+\frac{d\emptyset_i}{ds}$ where $\emptyset_i(s)$is the differentiable function that measures the positive angle from x to $\alpha'(s)$ in each internal $[s_i, s_{i+1}]$. Now integrate the above expression, sum up values for each $[s_i, s_{i+1}]$.

i.e. $\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} K_g(s).ds = \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \left\{\frac{G_u}{2\sqrt{EG}}.\frac{dv}{ds} - \frac{E_v}{2\sqrt{EG}}.\frac{du}{ds}\right\}.ds + \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\emptyset_i}{ds}.ds$ ....(1)

Using Gauss –Green Theorem in the uv -plane on the right hand side first term of equation (1) we get $P = -\frac{E_v}{2\sqrt{EG}}$ and $Q = \frac{G_u}{2\sqrt{EG}}$ then it follows that

$\iint_{X^{-1}(R)} \left\{\frac{G_u}{2\sqrt{EG}}.\frac{dv}{ds} - \frac{E_v}{2\sqrt{EG}}.\frac{du}{ds}\right\} du.dv$ ....(2)

From proposition 3.4 K = $\frac{-1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}$ therefore above equation (2) becomes $-\iint_{X^{-1}(R)} K\sqrt{EG}\,du.dv = -\iint_{X^{-1}(R)} K.d\sigma$ ....(3)

Now consider equation (1) [right side second term] $\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\emptyset_i}{ds}.ds$

$$= \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} 1.d\emptyset_i$$

$= \sum_{i=0}^{k} (\varphi_i)_{s_i}^{s_{i+1}}$

$=\sum_{i=1}^{k} \varphi_i(s_{i+1}) - \varphi_i(s_i)$

By the theorem 3.6 of turning tangent we can write as

$\sum_{i=0}^{k}[\emptyset_i(S_{i+1}) - [\emptyset_i(S_i)]] = 2\pi - \sum_{i=0}^{k} \theta_i$ ....(4)

By substituting equation (3) and (4) values in equation (1) we get

$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} K_g(s).ds = -\iint_{X^{-1}(R)} K.d\sigma + 2\pi - \sum_{i=0}^{k} \theta_i$

Therefore $\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} K_g(s).ds + \iint_{X^{-1}(R)} K.d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi.$

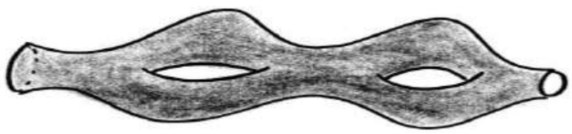
Remark: We can acquire the positive sign 2π if we choose α to be positive.

Global Gauss Bonnet theorem

We generalize the local Gauss-Bonnet theorem in each triangle of our triangulation for the provided surface where the triangle may not be simply connected.

Theorem 4.2: Let $R \sqsubset S$ be a regular region of an oriented surface and let $C_1, C_2 \ldots\ldots C_n$ be the closed simple, piecewise regular curves which form the boundary ∂R of R .Suppose that each C_i is positively oriented and $\theta_1 \ldots\ldots \theta_n$ be the set of all external angles of the curves $C_1, C_2 \ldots\ldots C_n$ then $\sum_{i=0}^{n} \int_{C_i} K_g(s).ds + \iint_R K.d\sigma + \sum_{i=1}^{k} \theta_i = 2\pi\chi(R)$.

Where $K_g(s)$ is the geodesic curvature of the sides of α and K is the Gaussian curvature of the surface (S), $\chi(R)$ is Euler Characteristic and the integral over C_i i.e the sum of integrals in every regular arc of C_i .



A surface with Euler characteristic $\chi = -4$

OR

$$\sum_{i=0}^{n} \int_{C_i} K_g(s).ds + \iint_R K.d\sigma + \sum_{i=1}^{k} \theta_i = 2\pi(2-2g)$$

OR

$$\sum_{i=0}^{n} \int_{C_i} K_g(s).ds + \iint_R K.d\sigma + \sum_{i=1}^{k} \theta_i = 4\pi(1-g).$$

OR

Let $R \sqsubset S$ be a regular region with exterior angles θ_i then $\int_{\Omega} K + \int_{\partial\Omega} K_g + \sum_{i=1}^{k} \theta_i = 2\pi\chi(R)$

Proof: Consider a triangulation J of the regular region R where each triangle T_j is surrounded by orthogonal parameterizations that are appropriate for an orientated surface S. The propositions 6.2 and 6.3 result in triangulation (see [1] and [2]). Also take note that the Euler characteristic applies to the case where S compact linked surface without boundary and R is all of S, regardless of the triangulation method that is chosen. i.e $\iint_R K.d\sigma = 2\pi\chi(R)$. Now, we are merely applying each triangulation's T_j face to the Local Gauss-Bonnet theorem and adding up all of the faces of a triangulation of R, we get the following result.

$$\sum_{i=0}^{n} \int_{C_i} K_g(s).ds + \iint_R K.d\sigma + \sum_{j,k=1}^{F,3} \theta_{j,k} = 2\pi F$$

Where F is the number of faces in triangulation and the interior angles of the triangles, can be expressed as $\beta_{j,k} = \pi - \theta_{j,k}$. In general $\sum_{j=1}^{F} \sum_{k=1}^{3} \theta_{j,k} = \sum_{j=1}^{F} \sum_{k=1}^{3} (\pi - \beta_{j,k})$

$= 3\pi F - \sum_{j=1}^{F} \sum_{k=1}^{3} \beta_{j,k}$

Now we introduce some notations to classify the vertices V and the edges E of the triangulation J.

V_{ext} = Total number of external vertices of J

V_{int} = Total number of internal vertices of J

E_{ext} =Total number of external edges of J

E_{int} = Total number of internal edges of J

and $E = E_{ext} + E_{int}$ And $V = V_{ext} + V_{int}$.

Since the C_i are closed, hence $V_{ext} = E_{ext}$ and thus faces and edges is given by $3F = 2E_{int} + E_{ext}$, this implies $\sum_{j=1}^{F} \sum_{k=1}^{3} \theta_{j,k} = 2\pi E_{int} + \pi E_{ext} - \sum_{j=1}^{F} \sum_{k=1}^{3} \beta_{j,k}$

The vertices must belong to either some triangulation T_j or a C_i so

V_{extc} =Total number of external vertices of C_i.

V_{extt} =Total number of external vertices of J.

That implies V_{ext}=$V_{extc} + V_{extt}$ and consider the sum of the angles around each internal vertex is 2π.

i.e. $\sum_{j=1}^{F} \sum_{k=1}^{3} \theta_{j,k} = 2\pi E_{int} + \pi E_{ext} - 2\pi V_{int} - \pi V_{extt} - \sum(\pi - \theta_l)$.

By adding πE_{ext} and πE_{ext} to the above equation, we get

$$\sum_{j=1}^{F} \sum_{k=1}^{3} \theta_{j,k} = 2\pi E_{int} + 2\pi E_{ext} - 2\pi V_{int} - \pi V_{ext} - \pi V_{extt} - \pi V_{extc} + \sum_{l=1}^{q} \theta_l$$

$= 2\pi E - 2\pi V + \sum_{l=1}^{q} \theta_l$

Now, combine all the above equations we get

$$\sum_{i=0}^{n} \int_{C_i} K_g(s).ds + \iint_R K.d\sigma + \sum_{i=1}^{k} \theta_i = 2\pi(F - E + V)$$

$$= 2\pi\chi(R)$$

5. APPLICATIONS:

Theorem-(i): Let S be an orientable compact surface then $\iint_S K.d\sigma = 2\pi\chi(S) = 4\pi(1 - g)$.

Theorem-(ii): A basic closed planar curve's overall geodesic curvature is 2π.

Theorem-(iii): A surface that is homeomorphic to the torus must always have a Gaussian curvature of zero.

Theorem-(iv): The index is unaffected by the parametrization x method chosen.

Proposition: The number $V - E + F$ does not depend on the triangulation, but only on the topology of the surface S implies $V - E + F = 2 - 2g$.

Remark: All conceivable embeddings of a surface of genus have the same total curvature, which implies that the total curvature depends only on the topological feature, genus. This is a highly counterintuitive conclusion.

Proposition: Let $R \sqsubset S$ be a regular region of an orientable compact surface, then $\int_R K.d\sigma = 2\pi\chi(R)$.

In Conclusion: Under local isometrics, the Gaussian curvature is invariant, but the integral of the curvature over a compact surface is even more so; it only depends on the contour of the surface. A regular (simple) region homeomorphic to an open disk has the local theorem (version) as its Euler characteristic 1, which is present in all regular (simple) regions. Consequently, the global theorem in each triangle of our surface, adding them so that we can divide corners and edges into internal and external parts to evaluate the angles individually. We are also interested in creating connections between a few other areas, such as micro local analysis, inverse problems, and spectral problems.

6. ACKNOWLEDGEMENTS

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