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THE TECHNIQUE: HOMOTOPY PERTURBATION METHOD OPERATED ON HEAT AND WAVE EQUATION

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ABSTRACT

In this section, we prefer a technique acknowledged as the Homotopy Perturbation Method (HPM) to obtain the exact solution of one-dimensional Heat and wave Equation with some Dirichlet, Neumann, and other boundary conditions to interpret the capability and credibility of this technique. The acquired results are extremely accurate. Besides, HPM affords a continuous solution in contrast to the finite difference method, whichexclusively presents discrete approximations. The comparison of final results between HPM and Others Iterative methods indicates a specific agreement between the results and introduces these new techniques as the relevant techniques where we see fewer computations and are much easier and more effective than different approximate methods. It is observed that this technique is a tool for effective mathematical analysis and also can be utilized in a wide variety of linear and nonlinear operations of science and technology. Also, getting an approximate solution that converges very quickly to an accurate solution can also be an advantage for Homotopy perturbation method.

Keywords: *Homotopy Perturbation Method (HPM); Partial differential equation (PDE); Heat equation; Wave equation;*

1 INTRODUCTION

With the rapid improvement over the past two decades of nonlinear science, there has been anincreasing interest among scientists and engineers in the analysis of nonlinear problems. The extensively utilized methods are perturbation methods. Here, we tried to implement an alternative and a reliable new tool for solving linear as well as non-linear Partial Differential Equations (PDE) which technique is referred to Homotopy Perturbation Method (HPM). Before discussing this method we need to know shortly about Homotopy and Perturbation techniques. The homotopy technique exploits the concept of homotopy from topology with the goal of obtaining convergent series solutions for nonlinear systems (Zheng 2017). It is developed through the usage of a homotopy–Maclarin series to deal with nonlinearities in the system. It has a simple form, a homotopy on a differential system is constructed using an auxiliary parameter (convergence-control parameter). The convergence–control parameter provides an effortless approach to verifying and enforcing the consolidation of a solution series. The capability of this approach to exhibit convergence of the series solution naturally in analytic and semi-analytic techniques of nonlinear partial differential equations is unusual**.** Nonlinear equations may have no exact solution, but there are analytical methods to determine approximate solutions, feasible, or known (Olsen 2010). In some sense, they provide results that are perturbations of known solutions and can be helpful for determining solutions that are 'close' to known solutions. Expansions of a small parameter are used to obtain approximate solutions, where the acknowledged solution is recovered when theparameter is zero, and the higher-order terms in the expansion include the extra statistics.

The homotopy perturbation method is a combination of perturbation and homotopy method (He 2003)**.** This technique is powerful with respect for finding solutions of nonlinear equations irrespective to the need of a linearization process. This method uses homotopy technology with an embedding parameter based on the homotopy technique p∈[0,1] is constructed, which can take full advantage of the traditional perturbation methods and homotopy techniques, When p = 0, Usually, the homotopy equation may be sufficiently simplified and permits a straightforward solution. For its effectiveness, a Duffing equation with an excessive order of nonlinearity is used; the outcome shows that the suggestedmethod's first-order approximation is universally valid even for the very high parameter, and is more accurate and also more acceptable than the other perturbation solutions.

HPM has a number of benefits over many traditional analytical methods. Like different nonlinear analytical techniques, perturbation techniques have their own restrictions. Firstly, nearly all perturbation approaches are entirely predicated on the requirement that a tiny parameter occurs in the equation (1999). This so-called small parameter assumption appreciably inhibits applications of perturbation techniques. Just as it is properly known, the vast majority of nonlinear problems do not have any tiny parameters. Secondly, the determination of small parameters appears to be an exclusive artwork requiring specialtechniques. A suitable desire of small parameters leads to the best results. However, an improper demand for modest parameter influences has negative, occasionally grave implications. Additionally, the approximate solutions that have been solved via the perturbation techniques are valid, in most cases, solely in the case of small parameters. It is apparent that all these barriers are based on the assumption of small parameters. In this paper,we will suggest new perturbation methods coupled with the homotopy technique. Traditional perturbation techniques cannot overcome the challenges presented by the proposed method, which requires no small parameters in the equations. Also, this method is easy to operate, closely related to the original equation, and reduces the complexity of the resulting equations so very rapidly we get an approximate solution that is close to the exact solution (Babolian 2009). That's why HPM has a wide range of applications for instance Nanofluid technology (Mehta 2021), factor Toda oscillator (2021), Population Balances Involving Aggregation and Breakage, and parameter expansion field. The truth is that the proposed HPM solves non- linear issues barring the usage of Adomian's polynomials (Filobello-Nino Uriel 2012) which can be viewed as a clear benefit of the approach over the decomposition (Seyma 2014).

1.1 Literature Survey:

In the physical world most events are occurring in a non-linear fashion. It is therefore very important to study non-linear problems in the fields of physics, engineering, and other disciplines. But to get an exact solution to the non-linear problems is not an easy task. Often finding an approximate analytic solution is harder than finding numerical solutions to non-linear problems. There are many methods that are prescribed to solve non-linear problems, such as the variational iteration method, homotopy perturbation method, etc. Non-linear problems can be solved using perturbation methods which suffer some restrictions due to the requirement of a very small parameter. If the choice of a small parameter is not suitable then it would be problematic to solve the non-linear equations. Due to these limitations, the perturbation method needs modification. The perturbation method is combined with homotopy which is a significant part in differential topology. Hence we get new method which is referred to as homotopy perturbation method. This new method was first recommended and presented by J.H. He (1999). With the help of the method, a non-linear problem is deformed continuously into a simple problem. This makes easier to solvethe problems. Also non-requirement of a small parameter in an equation is an important advantage that enables the method to provide approximate analytic solutions widely applicable to linear and non-linear problems in the applied field of science and mathematics.It is proposed that the choice of homotopy equation and initial guess should be suitable in order to produce an accurate and convergent solution (2012). It can be implemented very easily and efficiently which makes the method to be a powerful technique. Now we shall present the basic formalism of homotopy perturbation method in the next section elaborately.

There are huge application of the method, such as non-linear fuel dynamic equation (Hossein 2008), Volterrasintegro-diff erential equation (El-Shahed 2005), bifurcation of non-linear problems (2005), non-linear oscillators(2004), bifurcation of delay-differential equations(2005), non-linear wave equations (2005), boundary value problems (Cheniguel 2015) and many more. Another advantage of the method is that the solution converges rapidly and we get the required solution after a few iterations only (Nourazar 2015). For this reason, the method could be treated as a universal method to solve different kinds of linear and non-linear problems. MATLAB Simulink helps to plot the figures in terms of graphical visualization.

2 DESCRIPTION OF HOMOTOPY METHOD AND ITS SERIES SOLUTION

What is Homotopy? Let, Ω, Y be two topological areas and f, g be two continuous functions from Ω to Y i.e. $f : \Omega \rightarrow Y$, $g : \Omega \rightarrow Y$. A Homotopy from f to g is another continuous function $v : \Omega \times [0, 1] \rightarrow Y$ satisfying v(r, 0) = f(r) and v(r, 1) = g(r), for all $r \in \Omega$. The homotopy perturbation technique used to be proposed first by the Chinese researchis Ji-Huan He (1999) and was further developed and improved by him. He introduces a new parameter (embedding parameter) in $p \in [0,1]$ to take full advantage of the traditional perturbation methods. For a desirable perception of the homotopy perturbation technique, the reader found more developments in Dr**.** He's works and Liao's works would be the right reference for this improvement because this technique is pretty similar to the technique introduced by Liao (2017), recognized as the homotopy analysis method (HAM).

Now we shall present the basic formalism of the homotopy perturbation method. The method does not incorporate any calculation of a domain polynomial, any linearization or discretization, or any perturbation or transformation (SimonTerhemen 2019). Here only initial conditions are required to get a solution via the homotopy perturbation method. To provide a brief idea of the method, let us consider a non-linear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{1}$$

with the boundary condition: $B(u, \frac{\partial}{\partial n}) = 0, \quad r \in \Gamma$

Here A = general differential operator, B =boundary operator, f(r) = analytical function G = boundary of the domain Ω.

According to the Homotopy technique, A = L + N, L is linear and N is nonlinear.The equation (1) takes the form $L(u) + N(u) - f(r) = 0, \quad r \in \Omega$

We get the Homotopy function as follows

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{2}$$

where $v : \Omega[0, 1] \rightarrow R$, $p \in [0, 1]$ is embedding parameter , u_0 is the first approximation, satisfies the boundary conditions.

now, the following equation will be drawn$H(v, 0) = L(v) - L(u_0) = 0$

$H(v, 1) = L(u) + N(u) - f(r) = 0$

Now it can be written as a power series in p, as follows: $v = v_0 + pv_1 + p^2v_2 + ...$

If p =1, the best approximation is: $u = \lim_{p\rightarrow 1} v = v_0 + v_1 + v_2 + \ldots$

3 APPLICATION OF HOMOTOPY PERTURBATION METHOD

We attempt to apply this technique on two very popular examples of partial differential equations namely Heat and Wave equation with considering initial and boundary conditions. The series solution makes a sense of accuracy in terms of error minimisation and free of initial guess.

3.1 Solving Heat Equation by Homotopy Perturbation Method

We consider the one-dimensional Heat Equation as follows. It states that the sum of the 2nd order partial derivatives of U (the temperature parameter), with respect to the Cartesian coordinates, equals zero. Heat Equation (One Dimensional) describes the distribution of heat in a given space over time.

$$\frac{ðu}{ðt} - k\,\frac{ð^2}{ðx^2} = 0$$

for $0 \leq x \leq t$, with $t > 0$

Heat equation has the initial boundary condition as u(x, 0) = g(x) = sin(2πx)

Heat equation has the boundary condition as u(0, t) = u(1, t) = 0

Now applying the convex Homotopy method we get

$$\frac{ð(u_0+p\ u_1+p^2u_2\dots)}{\delta t} - kp\frac{ð^2(u_0+p\ u_1+p^2u_2\dots)}{\delta x^2} = 0$$

This new technique helps us to solve the one-dimensional Heat equation by separating the linear and non-linear part for the convex Homotopy method and formatting the expression of Heat equation in the said manner, now we comparing the coefficient of terms with equal powers of p and we get,

$$(0) : \frac{ðu_0}{ðt} = 0 ;$$

$$(1) : \frac{ðu_1}{ðt} - k\frac{ð^2u_0}{ðx^2} = 0 ;$$

$$(2) : \frac{ðu2}{ðt} - k\frac{ð^2u_1}{ðx^2} = 0 ; \dots \text{ and so on.}$$

Now with the help of integration, we get the following set of solutions

u_0 = sin(2πx) ;

u_1 = sin(2πx) - $4\pi^2$kt [sin(2πx)] ;

u_2 = sin(2πx) - $4\pi^2$kt [sin(2πx)] + $8\pi^4k^2t^2$ [sin(2πx)] ; …,and so on.

So the series is u(x, t) = $\lim_{p\to 1} (x, t)$

$= \lim_{p\to 1}(u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots)$

$= u_0 + u_1 + u_2 \dots$

Substituting the results of u_0, $u_{1,2}$, … , and so on we get,

u(x, t) = sin(2πx) [1 - ($4\pi^2$kt)+ $\frac{1}{2}$ ($4\pi^2$kt)^2 $- \cdots$] = sin(2πx) $\exp^{-4\pi^2 kt}$

Now the graphical representation of u(x,t) = sin(2πx) $\exp^{-4\pi^2 kt}$

for $0 \leq x \leq 1, 0 \leq t \leq 0.4$ $at\ k = 0.05$

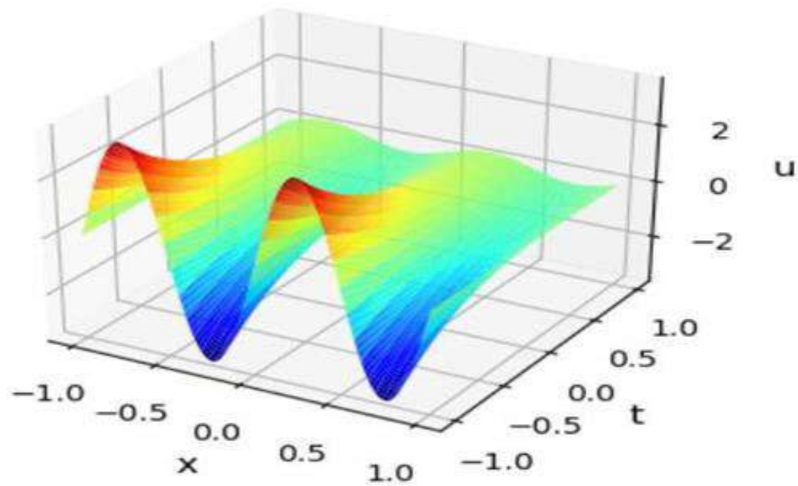

Fig. 1. 3D Plot for the Series Solution of Heat Equation using Homotopy Perturbation Method.

3.2 Solving Wave Equation by Homotopy Perturbation Method

Let's consider the one-dimensional Wave Equation with u as the temperature in the followingmanner

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

with the initial condition u(x, 0) = sin x

and boundary condition $u_t(x, 0) = u(0, t) = u(\pi, t) = 0$

Now, to solve the wave equation we apply Homotopy perturbation method by separating the linear and non-linear part for the same equation. So the equational part looks as follows,

$$H(u, p) = (1-p)\left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2}\right] + \left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right]$$

Now substituting $u = u_0 + pu_1 + p^2 u_2 + p^3 u_3$ .... in the above equation, we get

$$H(u, p) = (1-p)\left[\frac{\partial^2(u_0+p\,u_1+p^2u_2+\cdots)}{\delta t^2} - \frac{\partial^2 u_0}{\partial x^2}\right] + p\left[\frac{\partial^2(u_0+p\,u_1+p^2u_2+\cdots)}{\delta t^2} - \frac{\partial^2(u_0+p\,u_1+p^2u_2+\cdots)}{\delta x^2}\right]$$

As like above (for heat equation) comparing the equal powers of p we get,

p^0 : $\frac{\partial^2 u_0}{\partial x^2} = 0$;

p^1 : $\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} = 0$;

p^2: $\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} = 0$; and so on..

Now with the help of integration, we get the following set of solutions

$u_0 = \sin x$, $u_1 = -\frac{\sin x}{2} t^2$, $u_2 = \frac{\sin x}{24} t^4 \cdots$, and so on.

So the series is u(x, t) = $\lim_{p\to 1} (x, t)$

= $\lim_{p\to 1}(u_0 + pu_1 + p^2 u_2 + p^3 u_3 + ...)$

= $u_0 + u_1 + u_2 ...$

Substituting the results of u_0, $u_{1,2}$, … , and so on we get,

$$u(x, t) = \sin x \left[1 - \frac{-t^2}{2} + \frac{t^4}{24} ...\right] ; u(x, t) = \sin x \cos t$$

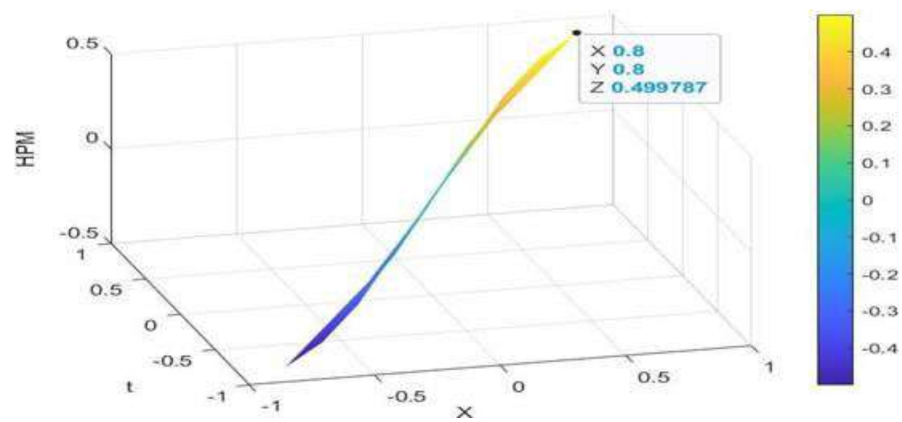

Fig. 2. 3D Plot for the Series Solution of Wave Equation using Homotopy Perturbation Method.

The advantages of the Homotopy Perturbation Method has been shown in the above graphicalrepresentation. This is relatively a new technic and easy to handle for solving linear and non- linear partial differential equation. It is simple method compare to another iterative method. For solving this method we get nearest value of exact solution than another method. Error is more less than other method.

4 CONCLUDING REMARKS

In this paper, we employ a new technique based on approximate solution, which use the homotopy perturbation method for the solution of the heat and wave equation. A direct approach is used without transforming, discretizing, or assuming restrictive assumptions. Itis clear that this method is a very powerful and effective technique. Several partial differential equations can be solved using the algorithm and the results demonstrate its reliability. The main strength of HPM is in reality its speedy convergence; however, the form of initial approximation and initial-boundary conditions is very vital for the effectivity of the HPM (Fern 2020).

In the bodily world, most occasions are taking place in a non-linear fashion. It is consequently very essential to learn about non-linear issues in the fields of physics, engineering, and different disciplines. But to get a precise answer to non-linear troubles is now not a convenient task. Often finding an approximate analytic answer is tougher than finding a numerical answer to non-linear problems. Many strategies are prescribed to resolve non-linear problems, such as the variational generation method, homotopy perturbation method, etc. The HPM used to discover the answer into a framework of partial differential equations. The method is used directly, barring the use of linearization, transformation,discretization, or limiting assumptions. It is feasible to conclude that the HPM is extraordinarily effective and environment-friendly for discovering analytical options for an extensive variety of boundary price problems. It is a semi-analytic method.

In physical issues, the strategy gives an extra sensible sequence of options that converge pretty quickly. It is well worth noting that the strategy can minimize the quantity of computing labour in contrast to normal strategies whilst keeping suitable numerical precision.

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6 FUTURE SCOPE OF STUDY

This perturbation technique (HPM) presented in the study converges to the exact solution does not incorporate any calculation of adomian polynomial, any linearization or discretization, any perturbation or transformation which could also improve the accuracy of our model. This research can be extended to dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid. The challenge with such models is that the resulting system of LaneEmden equation will be non-linear and mostly analytical. This work can also be extended by further research to see what additional elements may influence our method towards accuracy. Anew technique in an advanced way has been adopted in the name of New Homotopy Perturbation Method can also be amplified in the field of series of PDE to solve.

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