Stochastic Modelling and Computational Sciences

LAPLACE Q-TRANSFORM HYPERBOLIC FUNCTIONS

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ABSTRACT
In this paper, we obtain the extorial and trigonometric functions using the Laplace q-Transform, suitable examples are inserted to illustrate the mainresults.

Keywords: Laplace q-Transform, Difference operator, Inverse difference operator, Sine function, Cosine Function, Extorial functions, q- difference operator.

1. INTRODUCTION
The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provided an easy and effective means for the solution of many problems arising in engineering. This subject originated from the operational methods applied by the English engineer Oliver-Heaviside (1850-1925) unsystematic and lacked rigour, which was placed on sound mathematical footing by Brownwich and Carson during 1916-17. It was found that Heavisides operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms.

The method of Laplace transform has the advantage of directly giving the solution of differential equations with given the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants.

2. PRELIMINARIES
This section focuses on the basic definition of the q-difference operator and hyperbolic function.

Definition 2.1. If $u(k)$ is a sequence of numbers and q is any positive integer, then we define the generalized difference operator $\Delta q\ n\ u(k)$ as

$$\Delta q\ u(k) = u(kq) - u(k) \quad (1)$$

Also, if $\Delta q\ v(k) = u(k)$ then $v(k) = \Delta_q^{-1} u(k)$.

Definition 2.2. Let $q > 0$ and $u(k), v(k)$ are real valued bounded functions. Then

$$\Delta_q^{-1}\left(u(k) - v(k)\right) = u(k)\Delta_q^{-1}v(k) - \Delta_q^{-1}\left(\Delta_q^{-1}v(kq).\Delta q\ u(k)\right) \quad (2)$$

Definition 2.3. If $\lim_{m\to\infty} \Delta_q^{-1} f(kq^m) = 0$, then

$$\sum_{r=0}^{\infty} f(kq^r) = -\Delta_q^{-1} f(k) \quad (3)$$

Definition 2.4. Let $q > 0$ and $kq^r \neq 0$, then

$$\Delta_q^{-1} e^{-sk}|_0^\infty = -\sum_{r=0}^{\infty} e^{-skq^r} \quad (4)$$

Definition 2.5. For a given function $f(k)$, the generalized Laplace transform is defined as

$$L_q[f(k)] = (q-1)\Delta_q^{-1} f(k) k e^{-sk}|_0^\infty \quad (5)$$

3. Laplace q-Transform of sine Function

In this section, we define Laplace q-transform of hyperbolic function and results using the operator Δ_q^{-1}.

Definition 3.1. Let $k \in [0,\infty)$ and $q > 0$ then

$$\sin \square k_q^{(1)} = k_q^{(1)} - \frac{k_q^{(3)}}{3!} + \frac{k_q^{(5)}}{5!} - \frac{k_q^{(7)}}{7!} + \cdots \quad (6)$$

Definition 3.2. Let $k \in [0,\infty)$ and $q > 0$ then

$$\sin \square (ak)_q^{(1)} = ak_q^{(1)} - \frac{a^3 k_q^{(3)}}{3!} + \frac{a^5 k_q^{(5)}}{5!} - \frac{a^7 k_q^{(7)}}{7!} + \cdots \quad (7)$$

Definition 3.3. Let $k \in [0,\infty)$ and $q > 0$ and $k_q^{(n)}$ be a $n^{t\square}$ power of polynomial factorial, then

$$\sin h k_q^{(n)} = k_q^{(n)} - \frac{k_q^{(3n)}}{3!} + \frac{k_q^{(5n)}}{5!} - \frac{k_q^{(7n)}}{7!} + \cdots \quad (8)$$

Definition 3.4. Let $k \in [0,\infty)$ and $q > 0$ and $k_q^{(n)}$ be a $n^{t\square}$ power of polynomial factorial, then

$$\sin \square (ak)_q^{(n)} = (ak)_q^{(n)} - \frac{(ak)_q^{(3n)}}{3!} + \frac{(ak)_q^{(5n)}}{5!} - \frac{(ak)_q^{(7n)}}{7!} + \cdots \quad (9)$$

Lemma 3.5. *If $e^{-skq^r} \neq 0$ and $q > 0$ then the Laplace transform of hyperbolic sine function is*

$$L_q\left(\sin(k)_q^{(1)}\right) = (1-q)\left\{k \sin(k)_q^{(1)} \sum_{r=0}^{\infty} e^{-skq^r}\right.$$

$$\left. + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\left(kq^{t+1}\sin\left(kq^{t+1}\right)_q^{(1)} - kq^t \sin\left(kq^t\right)_q^{(1)}\right)\right)\right\} \quad (10)$$

Proof. Let $u(k) = \sin \square (k)_q^{(1)}$ in equation (4) we obtain,

$$L_q \sin(\square k)_q^{(1)} = q - 1\Delta_q^{-1} k \sin h (k)_q^{(1)} e^{-sk}\Big|_0^\infty$$

$$= (q-1)\Delta_q^{-1}\left(k_q^{(1)} - \frac{k_q^{(3n)}}{3!} + \frac{k_q^{(5n)}}{5!} - \frac{k_q^{(7n)}}{7!} + \cdots\right) e^{-sk}\Big|_0^\infty \quad (11)$$

Separate the terms and using the equations (3) and (4) we get,

$$\Delta_q^{-1} k\, k_q^{(1)} e^{-sk}|_0^{\infty} = \left[k\, k_q^{(1)} \Delta_q^{-1} e^{-sk} - \Delta_q^{-1}\left(\Delta_q^{-1} e^{-skq} \Delta_q k\, k_q^{(1)}\right)\right]\Big|_0^{\infty}$$

$$= k\, k_q^{(1)} \sum_{r=0}^{\infty} e^{-skq^r} - \sum_{t=0}^{\infty}\left[-\sum_{r=0}^{\infty} e^{-skq^{r+1}}\left(kq^t q(kq^t q)_q^{(1)} - kq^t (kq^t)_q^{(0)}\right)\right]$$

$$= -k\, k_q^{(1)} \sum_{r=0}^{\infty} e^{-skq^r} - \sum_{t=0}^{\infty}\left[-\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\left(kq^{(t+1)}(kq^{(t+1)})_q^{(1)} - kq^t (kq^t)_q^{(0)}\right)\right] \quad (12)$$

Also,

$$\Delta_q^{-1} \frac{k\, k_q^{(3)}}{3!} e^{-sk}\Big|_0^{\infty} = \frac{1}{3!}\left[k\, k_q^{(3)} \Delta_q^{-1} e^{-sk} - \Delta_q^{-1}\left(\Delta_q^{-1} e^{-skq} \Delta_q k\, k_q^{(3)}\right)\right]\Big|_0^{\infty}$$

we get the equation,

$$\Delta_q^{-1} \frac{k\, k_q^{(3)}}{3!} e^{-sk}\Big|_0^{\infty} = \frac{1}{3!}\Bigg[-k\, k_q^{(3)} \sum_{r=0}^{\infty} e^{-skq^r}$$

$$-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(3)} - kq^t (kq^t)_q^{(3)}\right)\right)\Bigg] \quad (13)$$

Replace $\frac{k_q^{(3)}}{3!}$ by $\frac{k_q^{(5)}}{5!}$ in equation (13) we get,

$$\Delta_q^{-1} \frac{k\, k_q^{(5)}}{5!} e^{-sk}\Big|_0^{\infty} = \frac{1}{5!}\Bigg[-k\, k_q^{(5)} \sum_{r=0}^{\infty} e^{-skq^r}$$

$$-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(5)} - kq^t (kq^t)_q^{(5)}\right)\right)\Bigg] \quad (14)$$

Applying the process mentioned above, we get

$$\Delta_q^{-1} \frac{k\, k_q^{(7)}}{7!} e^{-sk}\Big|_0^{\infty} = \frac{1}{7!}\Bigg[-k\, k_q^{(7)} \sum_{r=0}^{\infty} e^{-skq^r}$$

$$-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(7)} - kq^t (kq^t)_q^{(7)}\right)\right)\Bigg] \quad (15)$$

Substituting equation (12) to (15) in (11) we get,

$$L_q\left(sin(k)_q^{(1)}\right) = (q-1)\left\{\left[-k\,k_q^{(1)}\sum_{r=0}^{\infty} e^{-skq^r}\right.\right.$$

$$\left.-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(1)} - kq^t(kq^t)_q^{(1)}\right)\right)\right]$$

$$-\frac{1}{3!}\left[-k\,k_q^{(3)}\sum_{r=0}^{\infty} e^{-skq^r}\right.$$

$$\left.-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(3)} - kq^t(kq^t)_q^{(3)}\right)\right)\right]$$

$$+\frac{1}{5!}\left[-k\,k_q^{(5)}\sum_{r=0}^{\infty} e^{-skq^r}\right.$$

$$\left.-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(5)} - kq^t(kq^t)_q^{(5)}\right)\right)\right]$$

$$-\frac{1}{7!}\left[-k\,k_q^{(3)}\sum_{r=0}^{\infty} e^{-skq^r}\right.$$

$$\left.\left.-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(7)} - kq^t(kq^t)_q^{(7)}\right)\right)\right] + \cdots\right\}$$

$$= (1-q)\left\{\left(kk_q^{(1)} - \frac{kk_q^{(3)}}{3!} + \frac{kk_q^{(5)}}{5!} - \frac{kk_q^{(7)}}{7!} + \cdots\right)\sum_{r=0}^{\infty} e^{-skq^r}\right.$$

$$+\sum_{t=0}^{\infty}\left[\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\, kq^{t+1}\left[\frac{(kq^{t+1})_q^{(1)}}{1!} - \frac{(kq^{t+1})_q^{(3)}}{3!} + \frac{(kq^{t+1})_q^{(5)}}{5!}\right.\right.$$

$$\left.-\frac{(kq^{t+1})_q^{(7)}}{7!} + \cdots\right] - kq^t\left[\frac{(kq^t)_q^{(1)}}{1!} - \frac{(kq^t)_q^{(3)}}{3!} + \frac{(kq^t)_q^{(5)}}{5!}\right.$$

$$\left.\left.\left.-\frac{(kq^{t+1})_q^{(7)}}{7!} + \cdots\right]\right]\right\}$$

which yields the proof.

Theorem: 3.6 *If* $e^{-skq^{t+r+1}} \neq 0$ *and* $q > 0$ *then the Laplace q-transform of sine function is*

$$L_q\left(\sin(ak)_q^{(1)}\right) = (1-q)\left\{k\sin(ak)_q^{(1)}\sum_{r=0}^{\infty} e^{-skq^r}\right.$$
$$\left.+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\, kq^{t+1}\sin\square(akq^{t+1})_q^{(1)} - kq^t\sin\mathrm{h}(akq^t)_q^{(1)}\right)\right\} \qquad (16)$$

Proof. *Replacing* $\sin\ hk_q^{(1)} = \sin h(ak)_q^{(1)}$ *in the previous Lemma 3.5, we get the proof.*

Theorem: 3.7 *If* $e^{-skq^r} \neq 0$ *and* $q > 0$ *then the Laplace q-transform of* n^{th} *power of sine function is*

$$L_q\left(\sin\ h(k)_q^{(1)}\right) = (1-q)\left\{k\sin hk_q^{(1)}\sum_{r=0}^{\infty} e^{-skq^r}\right.$$
$$\left.+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\, kq^{t+1}\sin\square(kq^{t+1})_q^{(n)} - kq^t\sin(akq^t)_q^{(n)}\right)\right\} \qquad (17)$$

Proof. *Replace* $\sin\ hk_q^{(1)}$ *by* $\sin hk_q^{(2)}$ *in equation (10), and using (8)and (12) we arrive.*

$$L_q\left(\sin\mathrm{h}(k)_q^{(2)}\right) = (1-q)\left\{k\sin\ h(k)_q^{(2)}\sum_{r=0}^{\infty} e^{-skq^r}\right.$$
$$\left.+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\, kq^{t+1}\sin\ \square(kq^{t+1})_q^{(2)} - kq^t\sin\square(kq^t)_q^{(2)}\right)\right\} \qquad (18)$$

Replace $\sin\ hk_q^{(1)}$ *by* $\sin h(k)_q^{(3)}$ *in equation (10), and using (8) and (12) we arrive*

$$L_q\left(\sin h(k)_q^{(3)}\right) = (1-q)\left\{k\sin h(k)_q^{(3)}\sum_{r=0}^{\infty} e^{-skq^r}\right.$$
$$\left.+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\, kq^{t+1}\sin\square(kq^{t+1})_q^{(3)} - kq^t\sin\square(kq^t)_q^{(3)}\right)\right\} \qquad (19)$$

Replace $\sin\ hk_q^{(1)}$ *by* $\sin h(k)_q^{(4)}$ *in equation (10), and using (8) and (12), we arrive.*

$$L_q\left(\sin\mathrm{h}(k)_q^{(4)}\right) = (1-q)\left\{k\sin h(k)_q^{(4)}\sum_{r=0}^{\infty} e^{-skq^r}\right.$$
$$\left.+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\, kq^{t+1}\sin(akq^{t+1})_q^{(4)} - kq^t\sin(akq^t)_q^{(4)}\right)\right\} \qquad (20)$$

By repeating the process n times, we get

$$L_q\left(\sin h(k)_q^{(n)}\right) = (1-q)\left\{k \sin h(k)_q^{(n)} \sum_{r=0}^{\infty} e^{-skq^r}\right.$$

$$\left. + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\, kq^{t+1} \sin \Box(kq^{t+1})_q^{(n)} - kq^t \sin \Box(kq^t)_q^{(n)}\right)\right\}$$

Corollary: 3.8 *If* $e^{-skq^{t+r+1}} \neq 0$ *and* $q > 0$ *then the Laplace q-transform of sine function is*

$$L_q\left(\sin\ h(k)_q^{(n)}\right) = (1-q)\left\{k \sin\ h(k)_q^{(n)} \sum_{r=0}^{\infty} e^{-skq^r}\right.$$

$$\left. + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\, kq^{t+1} \sin\ \Box(kq^{t+1})_q^{(n)} - kq^t \sin\ \Box(akq^t)_q^{(n)}\right)\right\} \qquad (21)$$

Proof. The proof follows from taking $\sin \mathrm{h}\, k_q^{(n)} = \sin\ \Box(kq^t)_q^{(n)}$ *in equation (18), we get the proof of the corollary.*

Example: 3.9. Let $n = 5, q = \frac{1}{3}$ and $a = 4$ in equation (18), we have,

$$L_q\left(\sin \mathrm{h}(4k)_q^{(2)}\right) = (1-q)\left\{k \sin\ h(k)_{\frac{1}{3}}^{(5)} \sum_{r=0}^{\infty} e^{-skq\left(\frac{1}{3}\right)^r}\right.$$

$$\left. + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\, k\left(\frac{1}{3}\right)^{t+1} \sin\ \Box(4kq^{t+1})_q^{(5)} - k\frac{1^t}{3} \sin \Box(4kq^t)_{\frac{1}{3}}^{(5)}\right)\right\}$$

CONCLUSION

In this paper we have developed the discrete Laplace q-transform for sine function. The given example shows the values of Laplace q-transform for sine function.

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