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FURTHER RESULTS ON ABSOLUTELY HARMONIOUS LABELING OF GRAPHS

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ABSTRACT

An Absolutely harmonious labeling or ABHL f is an injection from the vertex set of a graph G with q edges to the set $\{0,1,2,\ldots,q-1\}$, that induces for each edge uv a label $f(u)+f(v)$ such that the set of edge label is $\{a_0, a_1\ a_2, \ldots, a_{q-1}\}$ where $a_i = q-i$ or $q+i, 0 \le i \le q-1$. A graph which admits Absolutely harmonious labeling or ABHL is called Absolutely harmonious graph or ABH graph. In this paper, we study Further results on absolutely harmonious labeling.

1. INTRODUCTION

In this paper, we consider finite and undirected graphs. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. A vertex labeling of a graph G is an assignment f of labels to the vertices that induces a label for each edge xy depending on the vertex labels. Seenivasan and Lourdusamy [3] introduced Absolutely harmonious labeling of graphs. In this paper, we study further results on absolutely harmonious labeling.

Definition 1.1.

An Absolutely harmonious labeling or ABHL f is an injection from the vertex set of a graph G with q edges to the set $\{0,1,2,\ldots,q-1\}$, that induces for each edge uv a label $f(u)+f(v)$ such that the set of edge label is $\{a_0, a_1\ a_2, \ldots, a_{q-1}\}$ where $a_i = q-i$ or $q+i, 0 \le i \le q-1$. A graph which admits Absolutely harmonious labeling or ABHL is called Absolutely harmonious graph or ABH graph.

Definition 1.2.

The Ring sum of two graphs G_1 and G_2 is a graph consisting of the vertex set $V(G_1) \cup V(G_2)$ and the edges that are either in G_1 or G_2 but not in both.It is denoted by $G_1 \oplus G_2$.

Definition 1.3.

The Cartesian product of two graphs G_1 and G_2 is the graph $\mathrm{G}_1 \times \mathrm{G}_2$ with the vertex set $\mathrm{V}_1 \times \mathrm{V}_2$ and two vertices $u = (u_1, u_2)$ and

$v = (v_1, v_2)$ are adjacent whenever $[u_1 = v_1$ and u_2 adj $v_2]$ or

$[u_2 = v_2$ and u_1 adj $v_1]$.

Definition 1.4.

The graph $L_n = P_n \times P_2$ is called a Ladder graph.

Definition 1.5.

The Triangular ladder $TL_n, n \geq 2$ is a graph obtained from the ladder $L_n = P_n \times P_2$ by adding the edges $u_i v_{i+1}$ for $1 \leq i \leq n-1$ where $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are the consecutive vertices of the two copies of the path P_n .

Definition 1.6.

A shell graph is defined as a cycle C_n with (n-3) chords sharing a common point called the apex. Shell graph is denoted as C(n,n-3).

2. MAIN RESULTS

Theorem 2.1

The Triangular Arrow ladder TAL_n is an Absolutely harmonious graph.

Proof

Let $G = TAL_n$. $Let\ V(G) = V(TAL_n) = \{u_i, v_i : 1 \leq i \leq n\} \cup \{w\}$.

and

$E(G) = E(TAL_n) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup \{wu_1\} \cup \{wv_1\}$.

Then G is of order 2n+1 and size 4n-1.

Now, Define $f: V(G) \to \{0,1,2,\cdots,4n-2\}$ as follows:

$$f(v_i) = 2i - 1; 1 \leq i \leq n,$$

$$f(w) = 4n - 2,$$

$$f(u_i) = 2i - 2; 1 \leq i \leq n.$$

Then the induced edge labels are as follows:

$$f^*(wv_1) = a_0,$$

$$f^*(wu_1) = a_1,$$

$$\begin{aligned}\{f^*(v_i v_{i+1}) : 1 \leq i \leq n-1\} &= \{4i : 1 \leq i \leq n-1\} \\ &= \{a_{4n-1-4i} : 1 \leq i \leq n-1\} \\ &= \{a_3, a_7, \ldots\ldots\ldots, a_{4n-5}\},\end{aligned}$$

$$\begin{aligned}\{f^*(u_i u_{i+1}) : 1 \leq i \leq n-1\} &= \{4i - 2 : 1 \leq i \leq n-1\} \\ &= \{a_{4n+1-4i} : 1 \leq i \leq n-1\} \\ &= \{a_5, a_9, \ldots\ldots\ldots, a_{4n-3}\},\end{aligned}$$

$$\{f^*(u_i v_{i+1}) : 1 \leq i \leq n-1\} = \{4i - 1 : 1 \leq i \leq n-1\}$$

$$= \{a_{4n-4i}: 1 \leq i \leq n-1\}$$
$$= \{a_4, a_8, \dots\dots\dots, a_{4n-4}\},$$

$$\{f^*(u_i v_i): 1 \leq i \leq n\} = \{4i-3: 1 \leq i \leq n\}$$
$$= \{a_{4n-4i+2}: 1 \leq i \leq n\}$$
$$= \{a_2, a_6, \dots\dots\dots, a_{4n-2}\}.$$

From the above, $a_0, a_1, a_2, \dots a_{q-1}$ where $a_i = q - i \; (or) \; q + i;$

$0 \leq i \leq q-1$ are the arranged edge labels.

Therefore f is an absolutely harmonious labeling of Triangular Arrow Ladder TAL_n and hence the Triangular Arrow Ladder TAL_n is an Absolutely harmonious graph.

Theorem 2.2

The Triangular Double Arrow ladder $TDAL_n$ is an Absolutely harmonious graph.

Proof

Let $G = TDAL_n$.

Let $V(G) = V(TDAL_n) = \{u_i, v_i: 1 \leq i \leq n\} \cup \{w\} \cup \{z\}$.

and

$E(G) = E(TDAL_n) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_{i+1}: 1 \leq i \leq n-1\} \cup \{u_i v_i: 1 \leq i \leq n\} \cup \{wu_1\} \cup \{wv_1\} \cup \{zu_n\} \cup \{zv_n\}$.

Then G is of order 2n+2 and size 4n+1.

Now, Define $f: V(G) \to \{0,1,2,\cdots,4n\}$ as follows:

$$f(v_i) = 2i-1; 1 \leq i \leq n,$$

$$f(w) = 4n,$$

$$f(z) = 2n+5,$$

$$f(u_i) = 2i-2; 1 \leq i \leq n.$$

Then the induced edge labels are as follows:

$$f^*(wv_1) = a_0, \; f^*(wu_1) = a_1,$$

$$f^*(zu_n) = a_2, \; f^*(zv_n) = a_3,$$

$$\{f^*(v_i v_{i+1}): 1 \leq i \leq n-1\} = \{4i: 1 \leq i \leq n-1\}$$
$$= \{a_{4n+1-4i}: 1 \leq i \leq n-1\}$$
$$= \{a_5, a_9, \dots\dots\dots, a_{4n-3}\},$$
$$\{f^*(u_i u_{i+1}): 1 \leq i \leq n-1\} = \{4i-2: 1 \leq i \leq n-1\}$$

$$= \{a_{4n+3-4i}: 1 \le i \le n-1\}$$
$$= \{a_7, a_{11}, \dots\dots\dots, a_{4n-1}\},$$

$$\{f^*(u_i v_{i+1}): 1 \le i \le n-1\} = \{4i-1: 1 \le i \le n-1\}$$
$$= \{a_{4n+2-4i}: 1 \le i \le n-1\}$$
$$= \{a_6, a_{10}, \dots\dots\dots, a_{4n-2}\},$$
$$\{f^*(u_i v_i): 1 \le i \le n\} = \{4i-3: 1 \le i \le n\}$$
$$= \{a_{4n-4i+4}: 1 \le i \le n\}$$
$$= \{a_4, a_8, \dots\dots\dots, a_{4n}\}.$$

From the above,

$a_0, a_1, a_2, \dots a_{q-1}$ where $a_i = q - i\ (or)\ q + i; 0 \le i \le q-1$ are the arranged edge labels.

Therefore f is an absolutely harmonious labeling of Triangular Double Arrow Ladder $TDAL_n$ and hence the Triangular Double Arrow Ladder $TDAL_n$ is an Absolutely harmonious graph.

Theorem 2.3

The graph $C(5,2) \oplus S_n$ is an Absolutely harmonious graph for any positive integer n.

Proof

Let $G = C(5,2) \oplus S_n$ be obtained by attaching to the center of S_n,

(the center of star graph with n vertices are merged with the graph $3C_3$).

Let $V(G) = \{u_i : 1 \le i \le n\} \cup \{v_i : 0 \le i \le 4\}$ and $u = v_0$ be the apex vertex of S_n and the edge set is given by

$$E(G) = \{v_i v_0: 1 \le i \le 4\} \cup \{v_0 u_i: 1 \le i \le n\} \{u_i v_{i+1}: 1 \le i \le 3\}$$

Then G is of order n+5 and size n+7.

Now, Define $f: V(G) \to \{0,1,2,\cdots, n+6\}$ as follows:

Let v_0, v_1, v_2, v_3, v_4 and u_1 are respectively labeled as

0, n+6, 1, 2, 4 and 5.

$$f(u_i) = 5 + i; 2 \le i \le n,$$

Then the induced edge labels are as follows:

$$f^*(v_0 v_1) = n + 6 = a_1,$$

$$f^*(v_1 v_2) = n + 7 = a_0,$$

$$f^*(v_0 v_2) = 1 = a_{n+6},$$

$f^*(v_0v_3) = 2 = a_{n+5}$,

$f^*(v_0v_4) = 4 = a_{n+3}$,

$f^*(v_2v_3) = 3 = a_{n+4}$,

$f^*(v_3v_4) = 6 = a_{n+1}$,

$f^*(v_0u_1) = 5 = a_{n+2}$,

$$\{f^*(v_0u_i): 2 \leq i \leq n\} = \{5 + i : 2 \leq i \leq n\}$$
$$= \{a_i : 2 \leq i \leq n\}$$
$$= \{a_2, a_3, \ldots\ldots\ldots, a_n\}.$$

Clearly the edge labels can be arranged as

$a_0, a_1, a_2, \ldots a_{q-1}$ where $a_i = q - i\ (or)\ q + i; 0 \leq i \leq q - 1$.

Therefore f is an absolutely harmonious labeling of $C(5,2) \bigoplus S_n$ graph for any positive integer n and hence it is an Absolutely harmonious graph.

Definition 2.4

The graph $K_4 \bigoplus S_n$ obtained by attaching to the center of S_n (star graph with n vertices) to a vertex of the complete graph K_4.

Theorem 2.5

The connected simple graph $K_4 \bigoplus S_n$ is an Absolutely harmonious graph for any positive integer n.

Proof

Let $G = K_4 \bigoplus S_n$ be a connected simple graph.

Let $V(G) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq 4\}$ and $u_0 = v_1$ be the apex vertex of G and the edge set is given by

$$E(G) = \{v_iv_{i+1}: 1 \leq i \leq 3\} \cup \{v_1u_i: 1 \leq i \leq n\} \cup \{v_1 v_3\} \cup \{v_2 v_4\}$$

Then G is of order n+4 and size n+6.

Now, Define $f: V(G) \to \{0,1,2,\cdots, n+5\}$ as follows:

Let v_1, v_2, v_3, v_4 and u_1 are respectively labeled as 0, 3, n+5, 1 and 2.

$f(u_i) = 3 + i; 2 \leq i \leq n,$

Then the induced edge labels are as follows:

$f^*(v_2v_3) = n + 8 = a_2,$

$f^*(v_1v_3) = n + 5 = a_1$,

$f^*(v_3v_4) = n + 6 = a_0$,

$f^*(v_1v_4) = 1 = a_{n+5}$,

$f^*(v_1u_1) = 2 = a_{n+4}$,

$f^*(v_1u_2) = 3 = a_{n+3}$,

$f^*(v_2v_4) = 4 = a_{n+2}$,

$$\{f^*(v_1u_i): 2 \leq i \leq n\} = \{3 + i : 2 \leq i \leq n\}$$
$$= \{a_{n+3-i} : 2 \leq i \leq n\}$$
$$= \{a_3, a_4, \ldots\ldots\ldots, a_{n+1}\}.$$

Clearly the edge labels can be arranged as

$a_0, a_1, a_2, \ldots a_{q-1}$ where $a_i = q - i\ (or)\ q + i; 0 \leq i \leq q - 1$.

Therefore f is an absolutely harmonious labeling of $K_4 \oplus S_n$ graph for any positive integer n and hence it is an Absolutely harmonious graph.

Theorem 2.6

The Diamond star graph $DA \oplus S_n$ is an Absolutely harmonious graph.

Proof

Let $G = DA \oplus S_n$.

Let the vertices of the Diamond DA be $w_1, w_2, w_3, w_4, w_5, w_6, w_7$ and the vertices of the star be $v, v_1, v_2, v_3, v_4, \ldots\ldots, v_n$. Also, $v = w_1$ is the apex vertex of the star and $E(G) = \{w_iw_{i+1}: 1 \leq i \leq 6\} \cup \{w_1 w_6\} \cup \{w_1 w_7\} \cup \{w_3 w_7\} \cup \{w_4 w_7\}$ is the edge set of the diamond, the edge set of star is given by $\{w_1 v_i: 1 \leq i \leq n\}$.

Then G is of order n+7 and size n+10.

Now, Define $f: V(G) \to \{0,1,2,\cdots,n+9\}$ as follows:

The vertices $w_1, w_2, w_3, w_4, w_5, w_6, w_7$ are respectively labeled as 1, 5, 6, 3, 4, 0 and 2. Also,the vertices $v_1, v_2, v_3, v_4, \ldots\ldots, v_n$ are respectively labeled as 9,11,12,13,………,n+9.

Then the induced edge labels are as follows:

$\{f^*(w_iw_{i+1}): 1 \leq i \leq 6\}$ are respectively labeled as $a_{n+4}, a_{n-1}, a_{n+1}, a_{n+3}, a_{n+6}$ and a_{n+8}.

$f^*(w_1w_6) = 1 = a_{n+9}$,

$f^*(w_1w_7) = 3 = a_{n+7}$,

$f^*(w_3w_7) = 8 = a_{n+2}$,

$f^*(w_4w_7) = 5 = a_{n+5}$,

$f^*(w_1v_1) = 10 = a_n$,

$$\{f^*(w_1u_i): 2 \leq i \leq n\} = \{10 + i : 2 \leq i \leq n\}$$
$$= \{a_{n-i} : 2 \leq i \leq n\}$$
$$= \{a_0, a_1, \ldots\ldots\ldots, a_{n-2}\}.$$

Clearly the edge labels can be arranged as

$a_0, a_1, a_2, \ldots a_{q-1}$ where $a_i = q - i\ (or)\ q + i; 0 \leq i \leq q - 1$.

Therefore f is an absolutely harmonious labeling of $DA \bigoplus S_n$ and hence it is an Absolutely harmonious graph.

Theorem 2.7

$2S_n \Delta P_2$ is a simple connected graph such that any one pendent vertex of S_n is joined by an edge with any one pendent vertex of another copy of S_n .Then two stars merged with a chord is an absolutely harmonious graph.

Proof

Let G=$2S_n \Delta P_2$.

Let $v_1, v_2, v_3, v_4, \ldots\ldots\ldots, v_n$ be the vertices of the first copy of star S_n and v_0 be its apex vertex.

Let $w_1, w_2, w_3, w_4, \ldots\ldots\ldots, w_n$ be the vertices of second copy of star S_n and w_0 be its apex vertex.

The edge set of G is given by

$$E(G) = \{v_0v_i: 1 \leq i \leq n\} \cup \{w_0w_i: 1 \leq i \leq n\} \cup \{v_n w_1\}$$

Then G is of order 2n+2 and size 2n+1.

Now, Define $f: V(G) \to \{0,1,2,\cdots,2n\}$ as follows:

$$f(v_0) = 0,$$
$$f(v_i) = i;\ 1 \leq i \leq n,$$
$$f(w_0) = 0,$$
$$f(w_i) = n + i; 1 \leq i \leq n.$$

Then the induced edge labels are as follows:

$$f^*(v_nw_1) = a_0,$$
$$\{f^*(v_0v_i): 1 \leq i \leq n\} = \{a_{2n+1+i}: 1 \leq i \leq n\}$$
$$= \{a_{n+1}, a_{n+2}, \ldots\ldots\ldots, a_{2n}\}.$$
$$\{f^*(w_0w_i): 1 \leq i \leq n\} = \{a_{n+1-i}: 1 \leq i \leq n\}$$
$$= \{a_1, a_2, \ldots\ldots\ldots, a_n\}.$$

From the above,it is clear that

$a_0, a_1, a_2, \ldots a_{q-1}$ where $a_i = q - i\ (or)\ q + i; 0 \leq i \leq q - 1$.

Therefore f is an absolutely harmonious labeling of $2S_n \Delta P_2$ and hence it is an Absolutely harmonious graph.

Theorem 2.8

$2S_n \Delta 2P_2$ is a simple connected graph such that any two pendent vertices of S_n are joined by two edges with any two pendent vertices of another copy of S_n .Then two stars merged with two chords is an absolutely harmonious graph.

Proof

Let G=$2S_n \Delta 2P_2$.

Let $v_1, v_2, v_3, v_4, \ldots\ldots, v_n$ be the vertices of the first copy of star S_n and v_0 be its apex vertex.

Let $w_1, w_2, w_3, w_4, \ldots\ldots, w_n$ be the vertices of second copy of star S_n and w_0 be its apex vertex.

The edge set of G is given by

$$E(G) = \{v_0v_i : 1 \leq i \leq n\} \cup \{w_0w_i : 1 \leq i \leq n\} \cup \{v_n w_1\} \cup \{v_1 w_n\}$$

Then G is of order 2n+2 and size 2n+2.

Now, Define $f : V(G) \to \{0,1,2,\cdots,2n+1\}$ as follows:

$$f(v_0) = 0,\ f(v_1) = 2,\ f(v_2) = 1,$$

$$f(v_i) = i;\ 3 \leq i \leq n,$$

$$f(w_0) = 0,$$

$$f(w_i) = n + i; 1 \leq i \leq n.$$

Then the induced edge labels are as follows:

$$f^*(v_nw_1) = a_1,\ f^*(v_1w_n) = a_0,$$

$$\{f^*(v_0v_i) : 1 \leq i \leq n\} = \{a_{2n+2-i} : 1 \leq i \leq n\}$$
$$= \{a_{n+2}, a_{n+3}, \ldots\ldots\ldots, a_{2n+1}\}.$$

$$\{f^*(w_0w_i) : 1 \leq i \leq n\} = \{a_{n+2-i} : 1 \leq i \leq n\}$$
$$= \{a_2, a_3, \ldots\ldots\ldots, a_{n+1}\}.$$

From the above,it is clear that

$a_0, a_1, a_2, \ldots a_{q-1}$ where $a_i = q - i\ (or)\ q + i; 0 \leq i \leq q - 1$.

Therefore f is an absolutely harmonious labeling of $2S_n \Delta 2P_2$ and hence it is an Absolutely harmonious graph.

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