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NEW FINDINGS OF OPEN SETS IN A TOPO SPACE

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ABSTRACT

The main aim of this paper is to define $\widehat{D_\alpha}$-open sets and $\widehat{D_\alpha}$-interior in topological spaces and obtain certain characterizations of these sets. Furthermore, we have discussed about the concept of $\widehat{D_\alpha}$-derived, $\widehat{D_\alpha}$-border, $\widehat{D_\alpha}$-frontier and $\widehat{D_\alpha}$-exterior of a set using the concept of $\widehat{D_\alpha}$-open sets are introduced.

INTRODUCTION

In this section, first we define $\widehat{D_\alpha}$-open sets and $\widehat{D_\alpha}$-interior in topological spaces and obtain certain characterizations of these sets. M. Caldas and J. Dontchev [8] introduced $\wedge_s$as semi kernel by using semi open sets.

Using this concepts, $\widehat{D_\alpha}$-kernal has been defined. M. Caldas, S. Jafari and T. Noiri [6] introduced and studied the topological properties of *g*-derived, *g*-border, *g*-frontier and *g*-exterior of a set using the concept of *g*-open sets. By the same technique the concept of $\widehat{D_\alpha}$-derived, $\widehat{D_\alpha}$-border, $\widehat{D_\alpha}$-frontier and $\widehat{D_\alpha}$-exterior of a set using the concept of $\widehat{D_\alpha}$-open sets are introduced.

2. PRELIMINARIES

Definition 2.1:

1) A generalized pre-regular closed set (briefly *gpr*-closed) [1] if $pcl(A) \subset U$ whenever $A \subset U$ and U is regular open in (X, τ).
2) ω-closed set [3] (= b*g*-closed [5]) if $cl(A) \subset U$ whenever $A \subset U$ and U is semi-open in (X, τ).
3) $\widehat{D_\alpha}$-closed set [4] if $\alpha cl(A) \subset U$ whenever $A \subset U$ and U is $\hat{D}$-open in (X, τ).

Preliminaries:2.2

1) Every $\widehat{D_\alpha}$-closed set is gspr- closed (resp. gpr-open, gps-open).
2) Every closed set is $\widehat{D_\alpha}$- closed
3) Every ω- closed set is $\widehat{D_\alpha}$- closed

Definition 2.3:[6] A proper nonempty open subset U of X is said to be a minimal open set if any open set contained in U is φ or U.

Definition 2.4:[6] A proper nonempty open subset U of X is said to be a maximal open set if any open set containing U is X or U.

Definition 2.5:[2] $b(A) = A - int(A)$ is said to be the border of A.

Definition 2.6: [2] $Fr(A) = cl(A) - int(A)$ is said to be the frontier of A.

Definition 2.7:[2] $Ext(A) = int(A^c)$ is said to be the exterior of A.

3 ON $\widehat{D_A}$ OPEN SETS

Definition 3.1. A subset A in (X,τ) is called $\widehat{D_\alpha}$-open in (X,τ) if A^c is $\widehat{D_\alpha}$-closed in (X,τ). We denote the family of all $\widehat{D_\alpha}$-open sets in X by $\widehat{D_\alpha}o(X)$.

Proposition 3.2. Every $\widehat{D_\alpha}$-open set is gspr-open (resp.gpr-open,gps-open).

Proposition 3.3. Every open set is $\widehat{D_\alpha}$-open

Proposition 3.4. Every ω-open set is $\widehat{D_\alpha}$-open

Proposition 3.5. If A and B are $\widehat{D_\alpha}$-open sets, then A∪B and A∩B is $\widehat{D_\alpha}$-open sets.

Theorem 3.6. A subset A of a topological space (X,τ) is said to be $\widehat{D_\alpha}$-open if and only if F $\subset\alpha$int(A) whenever F ⊂A and F is $\widehat{D}$-closed in (X,τ).

Proof.
Suppose A is $\widehat{D_\alpha}$-open in X and F ⊂A, where F is $\widehat{D}$-closed in (X,τ).

Then $A^c \subset F^c$, where F^c is $\widehat{D}$-open in X.

Hence, we get $\alpha cl(A^c) \subset F^c$ implies$(\alpha int(A))^c \subset F^c$.

Thus we have F $\subset \alpha$int(A).

Conversely, Suppose that $A^c \subset U$ and U is $\widehat{D}$-open in (X,τ). Then $U^c \subset A$ and $U^c$is $\widehat{D}$-closed and by hypothesis $U^c \subset \alpha int(A)$ implies $(\alpha int(A))^c \subset U$.

Hence$\alpha cl(A^c) \subset U$ implies that $A^c$is $\widehat{D_\alpha}$-closed in (X,τ).

Therefore, A is $\widehat{D_\alpha}$-open in (X,τ).

Proposition 3.7. If αint(A) ⊂B ⊂A and if A is $\widehat{D_\alpha}$-open, then B is $\widehat{D_\alpha}$-open.

Proof.
Suppose αint(A) ⊂B ⊂A and A is $\widehat{D_\alpha}$-open.

Then $A^c \subset B^c \subset \alpha cl(A^c)$.

Since $A^c$is $\widehat{D_\alpha}$-closed, $B^c$is $\widehat{D_\alpha}$-closed. Hence, B is $\widehat{D_\alpha}$-open.

Proposition 3.8. If a set A is $\widehat{D_\alpha}$-closed, then αcl(A) − A is $\widehat{D_\alpha}$-open.

Proof. Suppose A is $\widehat{D_\alpha}$-closed.

Let F $\subset\alpha$cl(A) − A where F is $\widehat{D}$-closed.

F = φ.

Therefore F $\subset \alpha$int(αcl(A) − A) and by Theorem 2.3.7,

αcl(A) − A is$\widehat{D_\alpha}$- open.

Remark 3.9. The converse of proposition 3.8 is not true by the following example.

Example 3.10. Let X= {*a, b, c, d, e*} and τ = { *φ*, {*a*}, {*a, b*}, {*c, d*}, {*a, c, d*}, {*a, b, c, d*},*X*} be defined. The set A = {a, d}, αcl(A) − A = {a, b, d} − {a, d} = {b} is $\widehat{D_\alpha}$-open but A is not $\widehat{D_\alpha}$ -closed.

Proposition 3.11. Let A be a subset of a topological space X. For any x ∈X, x ∈$\widehat{D_\alpha}$cl(A) if and only if U ∩ A≠ φ for every $\widehat{D_\alpha}$-open set U containing x.

Proof.

Necessity: Suppose that x ∈$\widehat{D_\alpha}$cl(A). Let U be a $\widehat{D_\alpha}$-open set containing x such that A ∩ U = φ and so A $\subset U^c$.

But $U^c$is $\widehat{D_\alpha}$-closed and hence $\widehat{D_\alpha}$cl(A) $\subset U^c$.

Since x $\notin U^c$ we obtain x∉ $\widehat{D_\alpha}$cl(A) which is a contradiction.

Therefore, U∩A≠ φ for every $\widehat{D_\alpha}$-open set U containing x.

Sufficiency: Suppose that every $\widehat{D_\alpha}$-open set U of X containing x such that

U ∩ A≠ φ.

If x ∉ $\widehat{D_\alpha}$cl(A) then there exist a $\widehat{D_\alpha}$-closed set F of X such that A ⊂F

and x ∉F.

Therefore, x ∈F^c and F^c is a $\widehat{D_\alpha}$-open set containing x. But F^c∩ A = φ.

which is contradiction to the hypothesis.

Therefore, x ∈$\widehat{D_\alpha}$cl(A).

Definition 3.12. For any A ⊂X, $\widehat{D_\alpha}$int(A) is defined as the union of all $\widehat{D_\alpha}$-open sets contained in A. That is, $\widehat{D_\alpha}$int(A) = ∪{U : U ⊂A and U ∈$\widehat{D_\alpha}$o(τ)}.

Proposition 3.13. Let A be a subset of a space (X,τ), then the following are true.

(i) ($\widehat{D_\alpha}$int(A))^c= $\widehat{D_\alpha}$cl(A^c)

(ii) $\widehat{D_\alpha}$int(A) =($\widehat{D_\alpha}$cl(A^c))^c

(iii) $\widehat{D_\alpha}$cl(A) =($\widehat{D_\alpha}$int(A^c))^c

Proof. (i) Let x ∈($\widehat{D_\alpha}$ int(A))^c.

Then x ∉$\widehat{D_\alpha}$int(A). That is, every $\widehat{D_\alpha}$-open set U containing x is such that U ⊄A.

Thus every $\widehat{D_\alpha}$-open set U containing x is such that U ∩A^c≠ φ.

By proposition 2.3.12, x ∈$\widehat{D_\alpha}$cl(A^c) and therefore, ($\widehat{D_\alpha}$ int(A))^c⊂$\widehat{D_\alpha}$cl(A^c).

Conversely, let x ∈$\widehat{D_\alpha}$cl(A^c).

Then by proposition 3.11, every $\widehat{D_\alpha}$-open set U containing x is such that U ∩A^c≠ φ.

By definition 3.12, x ∉$\widehat{D_\alpha}$int(A).

Hence x∈($\widehat{D_\alpha}$int(A))^{c}and so $\widehat{D_\alpha}$cl(A^c) ⊂($\widehat{D_\alpha}$int(A))^c.

Thus ($\widehat{D_\alpha}$int(A))^c= $\widehat{D_\alpha}$cl(A^c).

(ii) Follows by taking complements in (i).

(iii) Follows by replacing A by A^cin (i).

4.ON MINIMAL AND MAXIMAL $\widehat{D_\alpha}$-OPEN SETS

Definition 4.1: Let (X,τ) be a topological space.A non-empty $\widehat{D_\alpha}$- open set (A) of (X,τ) is said to be a **minimal $\widehat{D_\alpha}$-open set** if any $\widehat{D_\alpha}$- open set which is contained in (A) is $\tilde{\phi}$ or (A).

Lemma 4.2:

(1) Let (A) be a minimal $\widehat{D_\alpha}$- open set and (F) be a $\widehat{D_\alpha}$- open set. Then $(A) \cap (F) = \tilde{\phi}$ or $(A) \subset (F)$.

(2) Let (A) and (B) be minimal $\widehat{D_\alpha}$- open sets. Then $(A) \cap (B) = \tilde{\phi}$ or $(A) = (B)$.

Proof:

(1) Let (F) be a $\widehat{D_\alpha}$- open set such that $(A) \cap (F) \neq \tilde{\phi}$. Since $(A)$is a minimal $\widehat{D_\alpha}$- open set and $(A) \cap (F) \subset (A)$, we have $(A) \cap (F) = (A)$. Therefore,$(A) \subset (F)$.

(2) If $(A) \cap (B) \neq \tilde{\phi}$, then we see that $(A) \subset (B)$ and $(B) \subset (A)$ by (1). Therefore, $(A) = (B)$.

Proposition 4.3: Let (A) be a minimal $\widetilde{D_\alpha}$- open set. If (x) is an element of (A), then $(A) \subset (F)$ for any $\widetilde{D_\alpha}$- neighbourhood (F) of (x).

Proof: Let (F) be a $\widetilde{D_\alpha}$- neighbourhood of (x) such that $(A) \not\subset (F)$.

Then $(A) \cap (F)$ is a $\widetilde{D_\alpha}$- open set such that $(A) \cap (F) \subsetneq (A)$ and $(A) \cap (F) \neq \tilde{\phi}$.

This controverts our assumption that (A) is a minimal $\widetilde{D_\alpha}$- open set.

Proposition 4.4: Let (A) be a minimal $\widetilde{D_\alpha}$- open set. Then $(A) = \cap\{(F) \mid (F)$ is a $\widetilde{D_\alpha}$- neighborhood of $(x)\}$ for any element (x) of (A).

Proof: By Proposition 4.3 and the fact that (A) is a $\widetilde{D_\alpha}$- neighbourhood of (x), we have $(A) \subset \cap\{(F) \mid (F)$ is a $\widetilde{D_\alpha}$- neighbourhood of $(x)\} \subset (A)$.

Therefore, the argument proceeds.

Theorem 4.5: Let (A) be a non-empty $\widetilde{D_\alpha}$- open set. Then the here under three conditions are equivalent:

(1) (A) is a minimal $\widetilde{D_\alpha}$- open set.

(2) $(A) \tilde{\subseteq} Cl(B)$ for any non-empty subset (B) of (A).

(3) $Cl(A) = Cl(B)$ for any non-empty subset (B) of (A).

Proof:
(1)⇒(2) Let (B) be any nonempty subset of (A).

By Proposition 4.3, for any element (x) of (A) and any $\widetilde{D_\alpha}$- neighbourhood (B) of (x), we have $(B) = (A) \cap (B) \subset (C) \cap (B)$.

Then, we have $(C) \cap (B) \neq \tilde{\phi}$ and hence (x) is an element of $Cl(B)$. It follows that $(A) \subset Cl(B)$.

(2)⇒(3) For any non-empty subset (B) of (A), we have $Cl(B) \subset Cl(A)$.

On the other hand, by (2), we see $Cl(A) \subset Cl(Cl(B)) = Cl(B)$.

Therefore we have $Cl(A) = Cl(B)$ for any non-empty subset (B) of (A).

(3)⇒(1) Suppose that (A) is not a minimal $\widetilde{D_\alpha}$- open set.

Then there exists a non-empty open set (F) such that $(F) \subsetneq (A)$ and hence there exists an element $y \tilde{\in} (A)$ such that $y \tilde{\notin} (F)$.

Then we have $Cl(\{y\}) \subset (F)^C$, the complement of (F).

It proceeds that $Cl(\{y\}) \neq Cl(A)$.

Definition 4.6: Let (X, τ) be a topological space. A proper non-empty $\widetilde{D_\alpha}$- open subset (A) of (X, τ) is said to be a **maximal $\widetilde{D_\alpha}$- open set** if any $\widetilde{D_\alpha}$- open set which contains (A) is $\tilde{X}$ or (A).

Lemma 4.7:
(1) Let (A) be a maximal $\widetilde{D_\alpha}$- open set and (F) a $\widetilde{D_\alpha}$- open set. Then, $(A) \cup (F) = \tilde{X}$ or $(F) \subset (A)$.

(2) Let (A) and (B) be maximal $\widetilde{D_\alpha}$-open sets. Then, $(A) \cup (B) = \tilde{X}$ or $(A) = (B)$.

Proof:
(1) Let (F) be a $\widetilde{D_\alpha}$- open set such that $(A) \cup (F) \neq \tilde{X}$. Since (A) is a maximal $\widetilde{D_\alpha}$- open set and $(A) \subset (A) \cup (F)$, we have $(A) \cup (F) = (A)$. Therefore, $(F) \subset (A)$.

(2) If $(A) \cup (B) \neq \tilde{X}$, then $(A) \subset (B)$ and $(B) \subset (A)$ by (1). Therefore, $(A) = (B)$.

Proposition 4.8: Let (A) be a maximal $\widetilde{D_\alpha}$- open set. If (x) is an element of (A), then for any $\widetilde{D_\alpha}$- neighbourhood (F) of (x), $(F) \cup (A) = \tilde{X}$ or $(F) \subset (A)$.

Proof: By Lemma 4.7(1), the argument proceeds.

Theorem 4.9: Let $(A), (B)$ and (C) be $\widetilde{D_\alpha}$- maximal open sets such that $(A) \neq (B)$. If $(A) \cap (B) \subset (C)$, then $(A) = (C)$ or $(B) = (C)$.

Proof: We see that $(A) \cap (C) = (A) \cap \big((C) \cap X\big)$

$= (A) \cap \big((C) \cap ((A) \cup (B))\big)$ (by Lemma 4.7(2))

$= (A) \cap (((C) \cap (A)) \cup ((C) \cap (B)))$

$= ((A) \cap (C)) \cup ((C) \cap (A) \cap (B))$

$= ((A) \cap (C)) \cup ((A) \cap (B))$ (since $(A) \cap (B) \subset (C)$)

$= (A) \cap \big((C) \cap (B)\big)$

If $(C) \neq (B)$, then $(C) \cup (B) = \tilde{X}$, and hence $(A) \cap (C) = (A)$; namely, $(A) \subset (C)$. Since (A) and $(C)$are $\widetilde{D_\alpha}$- maximal open sets, we have $(A) = (C)$.

Theorem 4.10: Let $(A), (B)$ and (C) be $\widetilde{D_\alpha}$ maximal open sets, which are different from each other. Then, $(A) \cap (B) \not\subset (A) \cap (C)$.

Proof: If $(A) \cap (B) \subset (A) \cap (C)$, then we see that $((A) \cap (B)) \cup ((B) \cap (C)) \subset ((A) \cap (C)) \cup ((B) \cap (C))$.

Hence$(B) \cap \big((A) \cup (C)\big) \subset \big((A) \cup (B)\big) \cap (C)$.

Since $\big((A) \cup (C)\big) = \tilde{X} = \big((A) \cup (B)\big)$, we have $(B) \subset (C)$. It proceeds that $(B) = (C)$, which contradicts our assumption.

Proposition 4.11: Let (A)be a $\widetilde{D_\alpha}$- maximal open set and (x) an element of (A). Then, $(A) = \cup \{(F) | (F)$ is an $\widetilde{D_\alpha}$- neighbourhood of (x) such that $(F) \cup (A) \neq \tilde{X}\}$.

Proof: By Proposition 4.8 and the fact that (A) is an $\widetilde{D_\alpha}$- neighbourhood of (x), we have $(A) \subset \tilde{\cup} \{(F) | (F)$ is a $\widetilde{D_\alpha}$- neighbourhood of (x) such that $(F) \cup (A) \neq \tilde{X}\} \subset (A)$.

Therefore, the argument proceeds.

Proposition 4.12: Let (X, τ) be a topological space. If (A) be a proper $\widetilde{D_\alpha}$- maximal open subset of $\tilde{X}$ then $(A)^C$ is a $\widetilde{D_\alpha}$- minimal closed set.

Proof: Suppose $(A)^C$ is not a $\widetilde{D_\alpha}$- minimal closed set.

Then there exists a $\widetilde{D_\alpha}$- closed set (B) such that $\tilde{\phi} \neq (B) \subset (A)^C$.

Hence $(A) \subset (B)^C \subset \tilde{X}$.

This means that (A) is not $\widetilde{D_\alpha}$- maximal which is contradicting that(A) is $\widetilde{D_\alpha}$- maximal.

Proposition 4.13: Let (X, τ) be a topological space. If (A) be a proper $\widetilde{D_\alpha}$- minimal open subset of $\tilde{X}$ then $(A)^C$ is a $\widetilde{D_\alpha}$- maximal closed set.

Proof: Suppose $(A)^C$ is not a $\widetilde{D_\alpha}$- maximal closed set.

Then there exists a $\widetilde{D_\alpha}$- closed set (B) such that $(A)^C \subset (B) \subset \tilde{X}$.

Hence $\tilde{\phi} \neq (B)^C \subset (A)$.

This means that (A) is not $\widehat{D_\alpha}$- minimal which is contradicting that (A) is $\widehat{D_\alpha}$- minimal.

5.THEORETICAL APPLICATION OF $\widehat{D_\alpha}$ OPEN SETS

Definition 5.1. For any A ⊂X, $\widehat{D_\alpha}$ker(A) is defined as the intersection of all

$\widehat{D_\alpha}$-open sets containing A. In symbol, $\widehat{D_\alpha}$ker(A) = ∩{U : A ⊂U and U ∈$\widehat{D_\alpha}$o(τ)}.

Example 5.2. Let X = {a, b, c} and τ= {φ, {a},X}. Here $\widehat{D_\alpha}$o(τ) = {φ, {a}, {a, b},

{a, c},X}. If A = {b, c} then $\widehat{D_\alpha}$ker(A) = X. If B = {a}, then $\widehat{D_\alpha}$ker(B) = {a}.

Definition 5.3. A subset A of a topological space X is an U-set if $\widehat{D_\alpha}$ker(A) = A.

Example 5.4. By Example 3.10, {a}, {a, b} and {a, c} are U-sets and the set

{b, c} is not an U-set, because $\widehat{D_\alpha}$ker({b, c}) = X.

Lemma 5.5. For subsets A,B and A_α(α∈Λ) of a topological space X, the

following are hold.

1. A ⊂$\widehat{D_\alpha}$ker(A).
2. If A ⊂B, then $\widehat{D_\alpha}$ker(A) ⊂$\widehat{D_\alpha}$ker(B).
3. If $\widehat{D_\alpha}$ker($\widehat{D_\alpha}$ker(A)) = $\widehat{D_\alpha}$ker(A).
4. If A is $\widehat{D_\alpha}$-open, then A = $\widehat{D_\alpha}$ker(A).

5$\widehat{D_\alpha}$ker(∪{$A_\alpha/(\alpha$ ∈Λ)}) ⊃∪{$\widehat{D_\alpha}$ker$(A_\alpha)/(\alpha$ ∈Λ)}.

6. $\widehat{D_\alpha}$ker(⋂{$A_\alpha/(\alpha$ ∈Λ)}) ⊂ ⋂{$\widehat{D_\alpha}$ker$(A_\alpha)/(\alpha$ ∈Λ)}.

Proof. 1. Clearly follows from Definition5.1.

2. Suppose x ∉ $\widehat{D_\alpha}$ker(B). Then there exists a subset U ∈$\widehat{D_\alpha}$o(τ) such that U ⊃B with x ∉U. Since A ⊂B, x ∉ $\widehat{D_\alpha}$ker(A). Thus $\widehat{D_\alpha}$ker(A) ⊂$\widehat{D_\alpha}$ker(B).
3. Follows from (1) and Definition5.1.
4. Since A ∈$\widehat{D_\alpha}$o(τ), we have $\widehat{D_\alpha}$ker(A) ⊂A. By (1), A ⊂$\widehat{D_\alpha}$ker(A). Therefore, A = $\widehat{D_\alpha}$ker(A).
5. For each α ∈Λ, $\widehat{D_\alpha}$ker(A_α) ⊂$\widehat{D_\alpha}$ker($\cup_{\alpha\in\Lambda} A_\alpha$). Therefore, we obtain $\widehat{D_\alpha}$ker(∪{$A_\alpha/(\alpha$ ∈Λ)}) ⊃∪{$\widehat{D_\alpha}$ker$(A_\alpha)/(\alpha$ ∈Λ)}.
6. Suppose that x ∉ ⋂{$\widehat{D_\alpha}$ker$(A_\alpha)/(\alpha$ ∈Λ)}vthen there exists an α_0 ∈Λsuch that x ∉ $\widehat{D_\alpha}$ker(A_{α_0}) and there exist an $\widehat{D_\alpha}$-open set U such that x ∉U and A(α_0) ⊂U.

We have $\cap_{\alpha\in\Lambda} A_\alpha \subset A_{\alpha_0}$⊂U and x ∉U. Therefore, x ∉ $\widehat{D_\alpha}$ker(⋂{$A_\alpha/(\alpha$ ∈Λ)}).

Hence, $\widehat{D_\alpha}$ker(⋂{$A_\alpha/(\alpha$ ∈Λ)}) ⊂ ⋂{$\widehat{D_\alpha}$ker$(A_\alpha)/(\alpha$ ∈Λ)}.

Remark 5.6. In (6) of Lemma5.5, the equality does not necessarily hold as shown by the following example.

Example 5.7. Let X = {a, b, c, d} and τ= {φ, {c}, {d}, {a, c}, {c, d}, {a, c, d},X}.

Here $\widehat{D_\alpha}$o(τ) = {φ, {b, c},X}. Let P = {a, b}, Q = {b, c}. Here $\widehat{D_\alpha}$ker(P ∩ Q) = {b} and

$\widehat{D_\alpha}$ker(P) ∩ $\widehat{D_\alpha}$ker(Q) = {a, b, c} ∩ {b, c} = {b, c}.

Remark 5.8. From Lemma5.5, it is clear that $\widehat{D_\alpha}$ker(A) is a U-set and every open set is a U-set.

Lemma 5.9. Let $A_\alpha/(\alpha$ ∈Λ) be a subset of a topological space X. If $A_\alpha$is an U-set, then $\cap_{\alpha\in\Lambda} A_\alpha$ is an U-set.

Proof. $\widehat{D_\alpha}$ker($\cap_{\alpha\in\Lambda} A_\alpha$) ⊂$\cap_{\alpha\in\Lambda}$ $\widehat{D_\alpha}$ker(A_α) by Lemma 2.3.19.

Since each $A_\alpha$is anU-set, we get $\widehat{D_\alpha}$ker($\cap_{\alpha \in \Lambda} A_\alpha$) ⊂($\cap_{\alpha \in \Lambda} A_\alpha$).

Again by Lemma5.5, ($\cap_{\alpha \in \Lambda} A_\alpha$) ⊂$\widehat{D_\alpha}$ker($\cap_{\alpha \in \Lambda} A_\alpha$).

Thus $\widehat{D_\alpha}$ker($\cap_{\alpha \in \Lambda} A_\alpha$) = ($\cap_{\alpha \in \Lambda} A_\alpha$) implies ($\cap_{\alpha \in \Lambda} A_\alpha$) is an U-set.

Definition5.10. A subset A of a topological space X is said to be U-closed if

A = L ∩ F where L is anU-set and F is a closed set of X.

Remark 5.11 It is clear that every U-set and closed sets are U-closed.

Theorem 5.12. For a subset A of a topological space X, the following conditions are equivalent:

1. A is U-closed.
2. A = L ∩ cl(A) where L is anU-set.
3. A = $\widehat{D_\alpha}$ker(A) ∩ cl(A).

Proof. (1) =⇒(2) : Let A = L ∩ F where L is a U-set and F is a closed set. Since

A ⊂F we have cl(A) ⊂F and A ⊂L ∩ cl(A) ⊂L ∩ F = A. Therefore, we obtain

L ∩ cl(A) = A.

(2) =⇒(3) : Let A = L ∩ cl(A) where L is an U-set. Since A ⊂L we have

$\widehat{D_\alpha}$ker(A) ⊂$\widehat{D_\alpha}$ker(L) = L and hence A ⊂$\widehat{D_\alpha}$ker(A) ∩ cl(A) ⊂L ∩ cl(A) = A.

Therefore, we obtain A = $\widehat{D_\alpha}$ker(A) ∩ cl(A).

(3) =⇒(1) : Since $\widehat{D_\alpha}$ker(A) is a U-set, therefore, A is U-closed.

Definition5.13. Let A be subset of a space X. A point x ∈X is said to be a

$\widehat{D_\alpha}$-limit point of A if for each $\widehat{D_\alpha}$-open set U containing x, U ∩ (A−{x}) $\neq \varphi$. The set

of all $\widehat{D_\alpha}$-limit points of A is called a $\widehat{D_\alpha}$derived set of A and is denoted by $D_{\widehat{D_\alpha}}$(A).

Theorem 5.14. For subset A, B of a space X, the following statements hold:

(i) If $D_{\widehat{D_\alpha}}$(A) ⊂D(A), where D(A) is the derived set of A.

(ii) If A ⊂B, then $D_{\widehat{D_\alpha}}$(A) ⊂$D_{\widehat{D_\alpha}}$(B).

(iii) If $D_{\widehat{D_\alpha}}$(A) ∪$D_{\widehat{D_\alpha}}$(B) ⊂$D_{\widehat{D_\alpha}}$(A ∪B) and $D_{\widehat{D_\alpha}}$(A ∩ B) ⊂$D_{\widehat{D_\alpha}}$(A) ∩ $D_{\widehat{D_\alpha}}$(B).

(iv) If $D_{\widehat{D_\alpha}}$($D_{\widehat{D_\alpha}}$(A)) − A ⊂$D_{\widehat{D_\alpha}}$(A).

(v) If $D_{\widehat{D_\alpha}}$(A ∪$D_{\widehat{D_\alpha}}$(A)) ⊂A ∪$D_{\widehat{D_\alpha}}$(A).

Proof.

(i) Since every open set is $\widehat{D_\alpha}$-open, $D_{\widehat{D_\alpha}}$(A) ⊂$\widehat{D_\alpha}$(A).

(ii) Follows by definition 5.13.

(iii) Follows by (ii).

(iv) If x ∈($D_{\widehat{D_\alpha}}$($D_{\widehat{D_\alpha}}$(A))− A) and U is $\widehat{D_\alpha}$-open set containing x,

then U ∩ ($D_{\widehat{D_\alpha}}$(A) −{x}) $\neq \varphi$.

Let y ∈U ∩ ($D_{\widehat{D_\alpha}}$(A) − {x}). Since y ∈$D_{\widehat{D_\alpha}}$(A) and y ∈U, U ∩ (A −{y}) $\neq \varphi$.

Let z $\in$U $\cap$ (A − {y}). Then z $\neq$x for z $\in$A and x $\notin$A. Hence,

U $\cap$ (A − {x}) $\neq \varphi$. Therefore, x $\in D_{\widehat{D}_\alpha}$(A).

(v) Let x $\in D_{\widehat{D}_\alpha}$(A $\cup D_{\widehat{D}_\alpha}$(A)). If x $\in$A, the result is obvious.

So let x $\in D_{\widehat{D}_\alpha}$(A $\cup D_{\widehat{D}_\alpha}$(A))−A, then for a $\widehat{D}_\alpha$-open set U containing x such that

U $\cap$((A$\cup D_{\widehat{D}_\alpha}$(A))−{x}) $\neq \varphi$.

Thus U $\cap$ (A − {x}) $\neq \varphi$or U $\cap$ ($D_{\widehat{D}_\alpha}$(A) − {x}) $\neq \varphi$. Now, it follows

similarly from (iv) that U $\cap$(A−{x}) $\neq \varphi$.

Hence, x $\in D_{\widehat{D}_\alpha}$(A).

Therefore, in any case $D_{\widehat{D}_\alpha}$(A $\cup D_{\widehat{D}_\alpha}$(A)) $\subset$A $\cup D_{\widehat{D}_\alpha}$(A).

Remark 5.15. In general the converse of (i) is not true.

Example 5.16. Let X = {a, b, c} and τ= {φ, {a, b},X}. Then $\widehat{D}_\alpha$o(τ) = P(X) − {c}. Take A = {a, b}, then $D_{\widehat{D}_\alpha}$(A) = {c} and D(A) = X. Hence, D(A) $\not\subset D_{\widehat{D}_\alpha}$(A).

Theorem 5.17. For any subset A of a space X, $\widehat{D}_\alpha$cl(A) = A $\cup D_{\widehat{D}_\alpha}$(A).

Proof. Since $D_{\widehat{D}_\alpha}$(A) $\subset\widehat{D}_\alpha$cl(A), A $\cup D_{\widehat{D}_\alpha}$(A) $\subset\widehat{D}_\alpha$cl(A).

On the other hand, let x $\in\widehat{D}_\alpha$cl(A). If x $\in$A, then the proof is complete.

If x $\notin$A, each $\widehat{D}_\alpha$-open set U containingx intersects A at a point distinct from x, so x $\in D_{\widehat{D}_\alpha}$(A). Thus $\widehat{D}_\alpha$cl(A) $\subset$A $\cup D_{\widehat{D}_\alpha}$(A) which completes the proof.

Definition 5.18. $b_{\widehat{D}_\alpha}$(A) = A − $\widehat{D}_\alpha$int(A) is said to be the $\widehat{D}_\alpha$border of A.

Theorem 5.19. For a subset A of a space X, the following statements hold:

1. $b_{\widehat{D}_\alpha}$(A) $\subset$b(A) where b(A) denotes the border of A.
2. A = $\widehat{D}_\alpha$int(A) $\cup b_{\widehat{D}_\alpha}$(A).
3. $\widehat{D}_\alpha$int(A) $\cap$ $b_{\widehat{D}_\alpha}$(A) = φ.
4. If A is $\widehat{D}_\alpha$-open, then $b_{\widehat{D}_\alpha}$(A) = φ.
5. $\widehat{D}_\alpha$int($b_{\widehat{D}_\alpha}$(A)) = φ.
6. $b_{\widehat{D}_\alpha}$($b_{\widehat{D}_\alpha}$(A)) = $b_{\widehat{D}_\alpha}$(A).
7. $b_{\widehat{D}_\alpha}$(A) = A $\cap$ $\widehat{D}_\alpha$cl(A^c).

Proof. 1), 2) and 3) clearly follows.

4) If A is $\widehat{D}_\alpha$-open, then A = $\widehat{D}_\alpha$int(A). Therefore, $b_{\widehat{D}_\alpha}$(A) = φ.

5) If x $\in\widehat{D}_\alpha$int($b_{\widehat{D}_\alpha}$(A)), then x $\in b_{\widehat{D}_\alpha}$(A). On the otherhand, since $b_{\widehat{D}_\alpha}$(A) $\subset$A,

x $\in\widehat{D}_\alpha$int($b_{\widehat{D}_\alpha}$(A)) $\subset\widehat{D}_\alpha$int(A). Hence, x $\in\widehat{D}_\alpha$int(A) $\cap$ $b_{\widehat{D}_\alpha}$(A) which contradicts (3).

Thus $\widehat{D}_\alpha$int($b_{\widehat{D}_\alpha}$(A)) = φ.

6) Follows by (5).

7) $b_{\widehat{D}_\alpha}(A) = A - \widehat{D}_\alpha int(A) = A - (\widehat{D}_\alpha cl(A^c))^c = A \cap \widehat{D}_\alpha cl(A^c)$.

Remark 5.20. In general the converse of (1) is not true.

Example 5.21. Let X = {a, b, c, d} and τ= {φ, {c}, {d}, {a, c}, {c, d}, {a, c, d},X}.

Here $\widehat{D}_\alpha o(\tau)$ = {φ, {c}, {d}, {a, c}, {c, d}, {b, c}{b, d}, {a, b, c}, {a, c, d}, {b, c, d},X}.

If A = {a, b, c} then $b_{\widehat{D}_\alpha}(A)$ = {a, b, c} − {a, b, c} = φ, b(A) = {a, b, c} − {a, c} = {b}.

Hence, $b(A) \not\subset b_{\widehat{D}_\alpha}(A)$.

Definition 5.22. $Fr\widehat{D}_\alpha(A) = \widehat{D}_\alpha cl(A) - \widehat{D}_\alpha int(A)$ is said to be the $\widehat{D}_\alpha$-frontier of A.

Theorem 5.23. For a subset A of a space X, the following statements are hold:

1. $Fr\widehat{D}_\alpha(A) \subset Fr(A)$ where Fr(A) denotes the frontier of A.
2. $\widehat{D}_\alpha cl(A) = \widehat{D}_\alpha int(A) \cup Fr\widehat{D}_\alpha(A)$.
3. $\widehat{D}_\alpha int(A) \cap Fr\widehat{D}_\alpha(A) = \varphi$.
4. $b_{\widehat{D}_\alpha}(A) \subset Fr\widehat{D}_\alpha(A)$.
5. $Fr\widehat{D}_\alpha(A) = b_{\widehat{D}_\alpha}(A) \cup D_{\widehat{D}_\alpha}(A)$.
6. If A is $\widehat{D}_\alpha$-open, then $D_{\widehat{D}_\alpha}(A) = Fr\widehat{D}_\alpha(A)$.
7. $Fr\widehat{D}_\alpha(A) = \widehat{D}_\alpha cl(A) \cap \widehat{D}_\alpha cl(A^c)$.
8. $Fr\widehat{D}_\alpha(A) = Fr\widehat{D}_\alpha(A^c)$.
9. $Fr\widehat{D}_\alpha((\widehat{D}_\alpha int(A)) \subset Fr\widehat{D}_\alpha(A)$.
10. $Fr\widehat{D}_\alpha(\widehat{D}_\alpha cl(A)) \subset Fr\widehat{D}_\alpha(A)$.

Proof. 1. Since every open set is $\widehat{D}_\alpha$-open we get the proof.

2. $\widehat{D}_\alpha int(A) \cup Fr\widehat{D}_\alpha(A) = \widehat{D}_\alpha int(A) \cup (\widehat{D}_\alpha cl(A) - \widehat{D}_\alpha int(A)) = \widehat{D}_\alpha cl(A)$.

3. $\widehat{D}_\alpha int(A) \cap Fr\widehat{D}_\alpha(A) = \widehat{D}_\alpha int(A) \cap (\widehat{D}_\alpha cl(A) - \widehat{D}_\alpha int(A)) = \varphi$.

4. Clearly follows from Definitions 5.22

5. Since $\widehat{D}_\alpha int(A) \cup Fr\widehat{D}_\alpha(A) = \widehat{D}_\alpha int(A) \cup b_{\widehat{D}_\alpha}(A) \cup D_{\widehat{D}_\alpha}(A)$ we get $Fr\widehat{D}_\alpha(A) = b_{\widehat{D}_\alpha}(A) \cup D_{\widehat{D}_\alpha}(A)$.

6. If A is $\widehat{D}_\alpha$-open, then $b_{\widehat{D}_\alpha}((A) = \varphi$ then by (5), $D_{\widehat{D}_\alpha}(A) = Fr\widehat{D}_\alpha(A)$.

7. $Fr\widehat{D}_\alpha(A) = \widehat{D}_\alpha cl(A) - \widehat{D}_\alpha int(A) = \widehat{D}_\alpha cl(A) - (\widehat{D}_\alpha cl(A^c))^c = \widehat{D}_\alpha cl(A) \cap \widehat{D}_\alpha cl(A^c)$.

8. Follows by (7)

9. Clearly follow

10. $Fr\widehat{D}_\alpha(\widehat{D}_\alpha cl(A)) = \widehat{D}_\alpha cl(\widehat{D}_\alpha cl(A)) - \widehat{D}_\alpha int(\widehat{D}_\alpha cl(A)) \subset \widehat{D}_\alpha cl(A) - \widehat{D}_\alpha int(A) = Fr\widehat{D}_\alpha(A)$

Remark 5.24. In general the converse of (1) is not true.

Example 5.25. Let X and φ be defined as Example 5.21. Take A = {d} then

$Fr\widehat{D}_\alpha(A) = \varphi$, Fr(A) = {b}. Thus $Fr(A) \subset Fr\widehat{D}_\alpha(A)$.

Definition 5.26. $\widehat{D}_\alpha Ext(A) = \widehat{D}_\alpha int(A^c)$ is said to be the $\widehat{D}_\alpha$-exterior of A.

Theorem 5.27. For a subset A of a space X, the following statement are hold:

1. $Ext(A) \subset \widehat{D}_{\alpha} Ext(A)$, where Ext(A) denotes the exterior of A.

2. $\widehat{D}_{\alpha} Ext(A) = \widehat{D}_{\alpha} int(A^c) = (\widehat{D}_{\alpha} cl(A))^c$.

3. $\widehat{D}_{\alpha} Ext(\widehat{D}_{\alpha} Ext(A)) = \widehat{D}_{\alpha} int(\widehat{D}_{\alpha} cl(A))$.

4. If $A \subset B$ then $\widehat{D}_{\alpha} Ext(A) \supset \widehat{D}_{\alpha} Ext(B)$.

5. $\widehat{D}_{\alpha} Ext(A \cup B) \subset \widehat{D}_{\alpha} Ext(A) \cup \widehat{D}_{\alpha} Ext(B)$.

6. $\widehat{D}_{\alpha} Ext(A \cap B) \supset \widehat{D}_{\alpha} Ext(A) \cap \widehat{D}_{\alpha} Ext(B)$.

7. $\widehat{D}_{\alpha} Ext(X) = \varphi$.

8. $\widehat{D}_{\alpha} Ext(\varphi) = X$.

9. $\widehat{D}_{\alpha} int(A) \subset \widehat{D}_{\alpha} Ext(\widehat{D}_{\alpha} Ext(A))$.

Proof. (1) & (2) Clearly follows from Definition5.26.

(3) $\widehat{D}_{\alpha} Ext(\widehat{D}_{\alpha} Ext(A)) = \widehat{D}_{\alpha} Ext(\widehat{D}_{\alpha} int(A^c) = \widehat{D}_{\alpha} Ext(\widehat{D}_{\alpha} cl(A))^c = \widehat{D}_{\alpha} int(\widehat{D}_{\alpha} cl(A))$.

(4) If $A \subset B$, then $A^c \supset B^c$. Hence $\widehat{D}_{\alpha} int(A^c) \supset \widehat{D}_{\alpha} int(B^c)$ and so $\widehat{D}_{\alpha} Ext(A) \supset \widehat{D}_{\alpha} Ext(B)$.

(5) If $A \subset A \cup B$, then $(A \cup B)^c \subset A^c$. Hence $\widehat{D}_{\alpha} int((A \cup B)^c) \subset \widehat{D}_{\alpha} int(A^c)$ and so $\widehat{D}_{\alpha} Ext(A \cup B) \subset \widehat{D}_{\alpha} Ext(A)$. If $B \subset A \cup B$, then $(A \cup B)^c \subset B^c$. Hence $\widehat{D}_{\alpha} int((A \cup B)^c) \subset \widehat{D}_{\alpha} int(B^c)$ and so $\widehat{D}_{\alpha} Ext(A \cup B) \subset \widehat{D}_{\alpha} Ext(B)$. Therefore $\widehat{D}_{\alpha} Ext(A \cup B) \subset \widehat{D}_{\alpha} Ext(A) \cup \widehat{D}_{\alpha} Ext(B)$.

(6) If $A \cap B \subset A$, then $A^c \subset (A \cap B)^c$. Hence $\widehat{D}_{\alpha} int(A^c) \subset \widehat{D}_{\alpha} int((A \cap B)^c)$ and so $\widehat{D}_{\alpha} Ext(A) \subset \widehat{D}_{\alpha} Ext(A \cap B)$. If $A \cap B \subset B$, then $B^c \subset (A \cap B)^c$. Hence $\widehat{D}_{\alpha} int(B^c) \subset \widehat{D}_{\alpha} int((A \cap B)^c)$ and so $\widehat{D}_{\alpha} Ext(B) \subset \widehat{D}_{\alpha} Ext(A \cap B)$. Therefore $\widehat{D}_{\alpha} Ext(A) \cap \widehat{D}_{\alpha} Ext(B) \subset \widehat{D}_{\alpha} Ext(A \cap B)$.

(7) & (8) Follows from Definition5.26.

(9) $\widehat{D}_{\alpha} int(A) \subset \widehat{D}_{\alpha} int(\widehat{D}_{\alpha} cl(A)) = \widehat{D}_{\alpha} int(\widehat{D}_{\alpha} int(A^c))^c = \widehat{D}_{\alpha} int(\widehat{D}_{\alpha} Ext(A))^c = \widehat{D}_{\alpha} Ext(\widehat{D}_{\alpha} Ext(A))$.

Remark5.28. Converse of (1), (5) and (6) in theorem 5.27 need not be true.

Example 5.29. Let X = {a, b, c} and $\tau = \{\varphi, \{a, b\}, X\}$. Then $\widehat{D}_{\alpha} o(\tau) = P(X) - \{c\}$. If A = {a}, B = {b}, C = {c}, then $\widehat{D}_{\alpha} Ext(A) = \{b, c\}$, $Ext(A) = \varphi$, $\widehat{D}_{\alpha} Ext(B) = \{a, c\}$, $\widehat{D}_{\alpha} Ext(C) = \{a, b\}$, $\widehat{D}_{\alpha} Ext(A \cup B) = \varphi$, $\widehat{D}_{\alpha} Ext(A \cap C) = X$. Then $\widehat{D}_{\alpha} Ext(A) \not\subset Ext(A)$, $\widehat{D}_{\alpha} Ext(A \cap C) \neq \widehat{D}_{\alpha} Ext(A) \cap \widehat{D}_{\alpha} Ext(C)$ and $\widehat{D}_{\alpha} Ext(A \cup B) \neq \widehat{D}_{\alpha} Ext(A) \cup \widehat{D}_{\alpha} Ext(B)$.

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