ISSN: 2752-3829 Vol. 3 No.1, (June, 2023)

Stochastic Modelling and Computational Sciences

STUDY OF EXTENDED Φ-CONTRACTION MAPPINGS IN PROBABILISTIC METRIC SPACE

Radha[1], Arvind Bhatt[2*] and Deepak Kumar Sharma[3]
[1,2]Department of Mathematics, School of Sciences, Uttarakhand Open University Haldwani, Nainital, Uttarakhand, India
[3]Department of Mathematics, MBPG College, Kumaun University, Nainital
radhajoshi321@gmail.com[1], arvindbhatt@uou.ac.in[2*] and sharmadeepak0111209@gmail.com[3]

ABSTRACT
We extend the concept of Φ-contraction mappings in the probabilistic metric space.

Keywords.Φ-contraction, non-expansive mappings, fixed points, probabilistic metric space.

2020 Mathematics Subject Classification. Primary 47H10; Secondary 54H25

1. INTRODUCTION
The main object of Fixed-point theory is solving the non – linear equations of the type $Tx = x$, in which the function T is defined on some abstract space X. The amazing Banach Contraction principle [1] is widely recognised as one of the most important and practical findings in contemporary mathematical analysis. It offers a useful technique for locating those fixed points and ensures the presence and uniqueness property of fixed points for specific self – maps in a whole metric space.

In 1968 Kannan [14] proved that there are maps with fixed points and discontinuity in their domain after that Meir and Keeler [16] obtained the unique fixed point for new contractive condition. In continuation of many researchers finds the fixed point and unique fixed points for different type of mappings (Boyd and Wong [6], Matkowski [15], Rhoades [34], Jungck et al. [13], Jachymski [12], R.P. Pant [19-32], Pasicki [34], Reich [35], V. Pant [30], Bhatt et al.[3] Bisht and Pant [4], Bisht and Rakocevic [5]).

Our main aim in this research paper the study of contraction criterion in the probabilistic metric space which was given by Boyd and Wong [6]. Menger [17] proposed the theory of probabilistic metric spaces in relation in physics. Sehgal [38, 39] made the first attempt in this area by starting the research of contraction mapping theorems in probabilistic metric spaces in his doctoral dissertation. Since then, a significant advancement in the development of fixed-point theorems in Mangar space [8, 9, 18] has been made by Sehgal and Bharucha-Reid [2] who obtained a generalization of the Banach Contraction Principle on a complete Menger space.

In this research paper we are finding the results in probabilistic metric space.

2. MATHEMATICAL PRELIMINARIES
Definition 2.1. [37, 38]. A distribution function (on $[-\infty, +\infty]$) is a function $F: [-\infty, +\infty] \to [0, 1]$ which is left-continuous on R, non-decreasing and $F(-\infty) = 0$, $F(+\infty) = 1$. The Heaviside function H is a distribution function defined by,

$$H(t) = \begin{cases} 0, & if t \leq 0 \\ 1, & if t > 0. \end{cases}$$

Definition 2.2. [10]. A distance distribution function $F: [-\infty, +\infty] \to [0, 1]$ is distribution function with support contained in $[0, \infty]$. The family of all distance distribution functions will be denoted by Δ^+. We denote

$$D^+ = \left\{F: F \in \Delta^+, \lim_{x\to\infty} F(x) = 1\right\}.$$

Definition 2.3. [36, 37].A probabilistic metric space in the sense of Schweizer and Sklar is an ordered pair (X, F), where X is a nonempty set and $F : X \times X \rightarrow \Delta^+$,if and only if the following conditions are satisfied ($F(x, y) = F_{x,y}$for every $x, y \in X \times X$):

(i) for every $(x, y) \in X \times X$, $F_{x,y}(0) = 0$;

(ii) for every $(x, y) \in X \times X$, $F_{x,y} = F_{y,x}$;

(iii) $F_{x,y} = 1$, for every $t > 0 \Leftrightarrow x = y$;

(iv) for every $(x, y, z) \in X \times X \times X$ and for every $t_1, t_2 > 0$,

$F_{x,y}(t_1) = 1, \; F_{y,z}(t_2) = 1, \Longrightarrow F_{x,z}(t_1 + t_2) = 1.$

For each x and y in X and for each real number $t \geq 0, F_{x,y}(t)$ is to be thought of as the probability that the distance between x and y is less than t. Indeed, if (X, d) is a metric space, then the distribution function $F_{x,y}(t)$ defined by the relation $F_{x,y}(t) = H(t - d(x, y))$ induces a probabilistic metric space.

Definition 2.6 [10]. A t – norm is a function T: [0, 1]$\times [0,1] \longrightarrow [0,1]$ satisfying the following conditions:

(i) $T(a, 1) = a, T(0, 0) = 0$

(ii) $T(a, b) = T(b, a)$

(iii) $T(c, d) \geq T(a, b)$ for c $\geq a, d \geq b$

(iv) $T(T(a, b), c) = T(a, T(b, c))$ for all a, b, c in [0, 1].

Definition 2.7 [10]. A Menger probabilistic metric space (X, F, T) is an ordered triad, where T is a t – norm, and (X, F) is probabilistic metric space satisfying the following condition:

$F_{xz}(t_1 + t_2) \geq T(F_{xy}(t_1), F_{yz}(t_2))$ for all x, y, z in X and $t_1, t_2 \geq 0$.

Definition 2.8 [10]. Let (X, F) be a probabilistic metric space. The $(\in, \lambda)$ – topology in (X, F) is generated by the family of neighbourhoods

$U = \{(U_v(\epsilon, \lambda): (v, \epsilon, \lambda \in X \times R^+ \times (0,1))\}$,

Where $(U_v(\epsilon, \lambda) = \{u: u \in X, F_{u,v}(\epsilon) > 1 - \lambda\}$.

If a t-norm T is such that $sup_{x<1} T(x, x) = 1$ then (X, F, T) is with the (ϵ, λ) topology, a metrizable topological space.

Definition 2.9 [10]. Let (X, F) be a probabilistic metric space. A sequence $\{x_n\}$ in (X, F) is said to converge a point $x \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N(ϵ, λ)such that

$F_{x_n, x}(\epsilon) > 1 - \lambda$ for all n $\geq$ N(ϵ, λ).

Definition 2.10 [10]. Let (X, F) be a probabilistic metric space. A sequence $\{x_n\}$ in (X, F) is said to be a Cauchy sequence point if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N(ϵ, λ) such that

$F_{x_n, x_m}(\epsilon) > 1 - \lambda$ for all n, m $\geq$ N(ϵ, λ).

Definition 2.11 [10].A probabilistic metric space (X, F) with continuous t – norm is said to be complete if every Cauchy sequence in X converge to a point in X.

Definition 2.4 [7, 11]. If T is a self-mapping of a set X then a point x in X is called an eventually fixed point of T if there exists a natural number N such that

$T^{n+1}(x) = T^{n}(x)$ for $n \geq N$.

If $T(x) = x$ then x is called a fixed point of T. A point x in X is called a periodic point of period n if $T^{n}x = x$. The least positive integer n for which $T^{n}x = x$ is called the prime period of x.

Definition 2.5 [6,7] The set $\{x \in X: Tx = x\}$ is called the fixed point set of the mapping $T: X \rightarrow X$.

3. MAIN RESULTS

In this section we are finding the results in probabilistic metric space.

Theorem 3.1. Let (X, F) be a probabilistic metric space and $T: X \rightarrow X$ such that for each x, y in X with $x \neq Tx \text{ or } y \neq Ty$ we have

$F_{Tx,\ Ty}(t) \geq \emptyset[F_{x,\ y}(t)]$ .... (1)

Where $\emptyset: \mathbb{R}_{+} \cup \{0\} \longrightarrow \mathbb{R}_{+} \cup \{0\}$ is such that $\emptyset(t) < t$ for $t > 0$. If T is upper semi continuous from the right or if $\emptyset$ is non-decreasing and $\lim_{n\rightarrow\infty} \emptyset^{n}(t) = 0$, $t > 0$, then T has a fixed point. T has a unique fixed point $\Leftrightarrow$ (i) is satisfied for each $x \neq y$ in X.

Proof. We can say when $x = Tx$ and $y = Ty$ and using (i), then

$F_{Tx,\ Ty}(t) = [F_{x,\ y}(t)]$.

We can say that T is continuous and $F_{Tx,\ Ty}(t) \geq \emptyset[F_{x,\ y}(t)]$ for each x, y in X.Let y_0 be any point in X and $\{y_n\}$ be the sequence defined by

$y_n = Ty_{n-1}$,

that is $y_n = T^{n}y_0$. If $y_{n+1} = y_n$ for some n, then y_n is a fixed point of T and theorem holds. Therefore, assume that $y_{n+1} \neq y_n$ for each n $\geq$ 0.

Given an integer p ≥ 1, let $k_n = F_{y_n,\ y_{n+p}}(t)$. Then using (i), for each n ≥ 1 and p ≥ 1

We have $k_n = F_{y_n,\ y_{n+p}}(t)$

$= F_{Ty_{n-1},\ Ty_{n+p-1}}(t)$

$\geq \emptyset\left[F_{y_{n-1},\ y_{n+p-1}}(t)\right] = \emptyset(k_{n-1})$

$\geq \emptyset^{2}\left[F_{y_{n-2},\ y_{n+p-2}}(t)\right] \geq \emptyset^{2}(k_{n-2})\ldots\ldots\geq \emptyset^{n}[F_{y_0,\ y_p}(t)] = \emptyset^{n}(k_0)$

Therefore $k_n \geq \emptyset^{n}(k_0)$.

Since $\{k_n\}$is strictly increasing in $\mathbb{R}_{+}$,

there exist L ≤ 1 such that

$\lim_{n\rightarrow\infty} k_n = L = \lim_{n\rightarrow\infty} \emptyset(k_n)$. (2)

Now assume that T satisfies Matkowski condition [10], that is, $\emptyset$is nonincreasing and

$\lim_{n\to\infty}\emptyset^n(t) = 1$ for each t > 0.

Then, $\lim_{n\to\infty}k_n = \lim_{n\to\infty}\emptyset^n(k_0) = 1.$

This implies that $\{y_n\}$is a Cauchy sequence.

Next assume that T satisfies Boyd and Wond [6] condition, that is, $\emptyset$is upper semi continuous from the right.

If L < 1 then we get

$\lim_{n\to\infty}\sup\emptyset(k_n) \geq \emptyset(L) > L,$

Which contradicts (2) since k_n< L.

Hence, $\lim_{n\to\infty}k_n = \lim_{n\to\infty}F_{y_n,\ y_{n+p}}(t)$ = 1and $\{y_n\}$ is a Cauchy sequence.

Since X is complete, there exists z in X such that $\lim_{n\to\infty}y_n$ = z and $\lim_{n\to\infty}Ty_n$ = z. Continuity of T implies $\lim_{n\to\infty}Ty_n$ = Tz,that is, z = Tz and z is a fixed point of T.

Further, let u be any point in X.

Since $T^n y_0 \neq T^{n-1}y_0$ foe each n, using (i) we get

$$F_{T^n u,\ T^n y_0}(t) \geq \emptyset\left(F_{T^{n-1}u,\ T^{n-1}y_0}(t)\right)$$

$$\geq \emptyset^2(F_{T^{n-2}u,\ T^{n-2}y_0}(t)) \geq \ldots\ldots\ldots\ldots \geq \emptyset^n(F_{u,\ y_0}(t))$$

Our assumption on $\emptyset$ imply that

$\lim_{n\to\infty}F_{T^n u,\ T^n y_0}(t)$ = 0, that is, $\lim_{n\to\infty}T^n u = z.$

Thus, if there exists a point y_0 such that

$T^{n+1}y_0 \neq T^n y_0$ for each n,

then for each u in X the sequence of iterates $\{T^n u\}$ converges to z and z is the unique fixed point of T. Therefore,$T^{n+1}y_0 \neq T^n y_0$, n ≥ 0, for somey_0 implies uniqueness of the fixed point.

Now, assume the condition (i) is satisfied for all x, y in X. Then T can have only one fixed point. Conversely, suppose that T has a unique fixed point.

Then for distinct x, y we have x $\neq$fx or y $\neq$ fy which implies that condition (i) holds for each x $\neq$ y it may be noted that a mapping T satisfying this theorem cannot possess periodic point of prime period $\geq$2.

If possible, suppose T satisfies Theorem 2.1 and x is periodic point of T with prime period 2, that is, $T^2x = x$but T$x \neq x$.

Then using (1) we get

$$F_{Tx,\ T^2y_0}(t) \geq \emptyset[F_{x,\ Ty}(t)] > F_{x,\ Ty}(t) = F_{T^2x,\ Tx}(t).$$

This is a contradiction.

It follows similarly that T cannot have periodic points of prime period < 2.

This completes the proof of the theorem.

Example 3.2. Let (X, F) be a probabilistic metric space, where X = [1, ∞) and

$$F_{x,y}(t) = \begin{cases} e^{-\frac{|x-y|}{t}}, & if\ t > 0 \\ 0, & if\ t = 0 \end{cases}.$$

Where T: X$\longrightarrow X$be a signum function Tx = sgn x defined as

Tx = -1 if $x < 0, T0 = 0, Tx = 1$ if $x > 0.$

Then Tx = 1 for each x and 1 is the unique fixed point of T. T satisfies condition (i) with $\emptyset(t) = \frac{1}{2}t$. Here if $x \neq 1$ then Tx = T^2x and x is an eventually fixed point.

REFERENCES

[1] Banach, S.: Sur les operations Dans les ensembles Abstraits et leur application aux equations integrals. Fund. Math., 133-181(1922).

[2] Bharucha-Reid, A.T.: Fixed point theorems in Probabilistic analysis, Bull.Amer. Math. Soc. 82(1976) 641-657.

[3] Bhatt, A., Chandra, H.: Common fixed points for JH-operators and occasionally weakly g-biased pairs under relaxed condition on probabilistic metric space, Journal of Function Spaces and Applications. Article ID 846315, volume 2013.

[4] Bisht,R. K., Pant,R. P.:A remark on discontinuity at fixed point, J. Math. Anal. Appl. 445-2(2017), 1239-1242.

[5] Bisht R. K., Rakocevic, V.: Generalized Meir-Keeler type contractions and discontinuity at fixed point. Fixed Point Theory, 19(1) (2018), 57-64.

[6] Boyd, D.W., Wong, J. S.: On nonlinear contractions, Proc. Amer. Math. Soc. 20(1969) 458-464.

[7] Devaney, R. L.: An introduction to chaotic dynamical systems, Benjamin/Cummings Publishing Co., California, 1986.

[8] Dorel, M.: Altering distances in probabilistic Menger spaces, Nonlinear Analysis 71(2009)2734 - 2738.

[9] Egbert, R.J.: Products and quotients of probabilistic metric spaces, Pacific J.Math. 24(1968) 437-455.

[10] Hadzic, O., Pap, E.: Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Academic Publishers, 2001.

[11] Holmgren, R. A.: A first course in discrete dynamical systems, Springer- Verlag, New York, 1994.

[12] Jachymski, J.: Common fixed-point theorems for some families of maps, Indian J. Pure Appl. Math. 25 – 9(1994), 925-937.

[13] Jungck, G., Pathak, H.K.: Fixed points via biased maps, Proc. Amer. Math. Soc. 123(1995) 2049-2060.

[14] Kannan, R.: Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968) 71-76.

[15] Matkowski, J.: Integral solutions of functional equations, Diss.Math.127 (1975), 1-68.

[16] Meir, A., Keeler, E.:A theorem on contraction mappings. J. math. Anal. Appl. 28, 326 – 329 (1969).

[17] Menger, K.: Statistical metrices, Nat. Acad.Sci.USA. 28(1942) 535-537.

[18] Mishra, S.N.: Common fixed points of compatible mappings in PM- spaces, Math. Japon. 36(1991) 283-289.

[19] Pant, R. P.: Common fixed points of noncommitting mappings, J. Math. Anal. Appl. 188-2(1994), 436-440.

[20] Pant, R.P.: Common fixed points of four maps, Bull. Calcutta Math.Soc. 90(1998) 281-286.

[21] Pant, R. P.: Common fixed-point theorems for contractive maps, J Math. Anal.Appl. 226 (1998), 251-258.

[22] Pant, R. P.: Discontinuity and fixed points, J. Math. Anal. Appl., 240(1999)284-289.

[23] Pant, A., Pant R. P.: and Joshi, M. C.: Caristi type and Meir-Keeler type fixed point theorems, Filomat 33-12(2019), 3711 – 3721.

[24] Pant, R. P., Özgür, N. Y., Tas, N.: Discontinuity at fixed points with applications, Bull. Belgian Math. Soc. - Simon Stevin 26 - 4 (2019), 571-589.

[25] Pant, R. P., Özgür, N. Y., Tas, N.: On Discontinuity Problem at Fixed Point, Bull. Malays. Math. Sci. Soc43 (2020), 499-517.

[26] Pant, R. P.: A common fixed-point theorem under a new condition, Indian J. Pure Appl. Math.30-2(1999), 147-152.

[27] Pant, R.P., Pant, V.: Common fixed points under strict contractive conditions, J.Math.Anal.Appl.248 (2000) 327-332.

[28] Pant, R. P.: Noncompatible mappings and common fixed points, Soochow J. Math. 26(1) (2000), 29-35.

[29] Pant, R. P.: A new common fixed-point principle, Soochow J. Math. 27- 3(2001), 287-297.

[30] Pant, V.: Remarks on discontinuity at fixed points, J. Indian Math. Soc. 69- (1-4) (2002), 173-175.

[31] Pant, A., Pant, R. P.: Fixed points and continuity of contractive maps, Filomat 31:11(2017), 3501-3506.

[32] Pant, R. P., Ozgur, N., Tas, N., Pant, A., Joshi, M. C.: New results on discontinuity at fixed points, J. Fixed Point Theory Appl. 22(2020), 1-14.

[33] Pasicki, L.: Boyd and Wong idea extended, Fixed Point Theory and Applications (2016) 2016: 63.

[34] Rhoades, B. E.: Contractive definitions and continuity, Contemporary Mathematics(Amer. Math. Soc.) 72(1988), 233-245.

[35] Reich, S.: Fixed points of contractive functions, Boll. Un. Mat. Ital. 5(1972), 26-42.

[36] Schweizer, B. and Skalar, A.: Probabilistic metric spaces, Pacific J. of Math.10 (1960) 313 - 324.

[37] Schweizer, B., Sklar, A.: Statistical metric spaces, North Holland Amsterdam, (1983).

[38] Sehgal, V.M.: Some fixed-point theorems in functional analysis and probability Ph.D. Dissertation, Wayne State Univ. Michigan (1966).

[39] Sehgal, V.M., Bharucha-Reid, A.T.: Fixed points of contraction mappings on Probabilistic metric spaces, Math. Systems Theo. 06(1972) 97-102.