SOME RESULTS ON INVARIANT SUBMANIFOLDS OF (LCS)N-MANIFOLDS

Sagarika Nandy
Assistant Professor of Mathematics, Deshabandhu Mahavidyalaya, Chittaranjan, Paschim Bardhaman, West Bengal, India
dbmsagarika@gmail.com

ABSTRACT
This paper investigates invariant submanifolds within (LCS)$_n$-manifolds, subject to the conditions: $C(X,Y).h = 0$ *and* $C(X,Y).\overline{\nabla}h = 0$; $\bar{C}(X,Y).h = 0$ *and* $\bar{C}(X,Y).\overline{\nabla}h = 0$; $B(X,Y).h = 0$ *and* $B(X,Y).\overline{\nabla}h = 0$, *alongside the constraint* $P(\xi,X)S = 0$ *where* $P, S, C, \bar{C}, B$ *and* h *denote the Projective curvature tensor, Ricci tensor, Conformal curvature tensor, Conharmonic curvature tensor, C-Bochner curvature tensor, and the second fundamental form, respectively.*

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INTRODUCTION

The concept of Lorentzian concircular structure manifolds, denoted as (LCS)$_n$-manifolds, was introduced by Shaikh [1] in 2003, extending the framework of LP-Sasakian manifolds initially presented by Matsumoto [26] and further developed by Mihai and Rosca [23]. Subsequent investigations into the properties of (LCS)$_n$-manifolds have been conducted by numerous researchers, including Atçeken [31], Narain and Yadav [19], Prakasha [18], Shaikh [2], Shaikh et al. [9, 10], Shaikh and Binh [8], and Sreenivasa et al. [20], among others.

The study of submanifold geometry holds significant implications in various branches of applied mathematics [38] and theoretical physics [38], particularly in the context of geodesic submanifolds which play a crucial role in the theory of relativity [27]. Bejancu and Papaghuic [5] introduced the concept of invariant submanifolds, which has since been extensively investigated by several researchers [14, 15, 34, 33, 21, 22, 32, 24, 25, 6, 29].

Özgüar and Murathan [16] have examined semiparallel and 2-semiparallel invariant submanifolds of LP-Sasakian manifolds. In their work, A. A. Shaikh et al. [11] analyzed invariant submanifolds of odd-dimensional (LCS)n-manifolds, while Siddesha and Bagewadi explored invariant submanifolds of (k, μ)-Contact manifolds [35].

This paper explores invariant submanifolds of (LCS)$_n$-manifolds of odd dimension that satisfy the condition $P(\xi,X)S = 0$, where P and S denote the Projective curvature tensor and Ricci tensor, respectively. The organization of the paper is as follows:

Section 2: Provides fundamental conditions concerning (LCS)$_n$-manifolds.

Section 3: Presents properties related to invariant submanifolds of (LCS)$_n$-manifolds.

Sections 4, 5, and 6: Discuss totally geodesic manifolds of invariant submanifolds of (LCS)$_n$-manifolds under the conditions:

- $C(X,Y).h = 0$ and $C(X,Y).\overline{\nabla}h = 0$
- $\bar{C}(X,Y).h = 0$ and $\bar{C}(X,Y).\overline{\nabla}h = 0$
- $B(X,Y).h = 0$ and $B(X,Y).\overline{\nabla}h = 0$.

Here, $C, \bar{C}, B$ and h represent the Conformal curvature tensor, Conharmonic curvature tensor, C-Bochner curvature tensor, and the second fundamental form, respectively.

The Weyl Conformal curvature tensor C (n > 3) is given by

(1.1) $$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\begin{bmatrix} S(Y,Z)X - S(X,Z)Y + \\ g(Y,Z)QX - g(X,Z)QY \end{bmatrix} + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]$$

The Conharmonic curvature tensor $\bar{C}$ is given by

(1.2) $$\bar{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY],$$

where R is the Riemann curvature tensor, S is the Ricci tensor of type (0, 2), Q is the Ricci operator and r is the scalar curvature.

D. E. Blair [17] explored the geometric significance of the Bochner curvature tensor. Utilizing the fibration method introduced by Boothby-Wang [37], M. Matsumoto and G. Chuman [36] developed the C-Bochner curvature tensor, derived from the original Bochner curvature tensor. The *C*-Bochner curvature tensor is given by

(1.3) $$B(X,Y),Z = R(X,Y)Z + \frac{1}{n+3}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX + S(\phi X,Z)\phi Y - g(\phi Y,Z)Q\phi X - S(\phi Y,Z)\phi X + g(\phi X,Z)Q\phi Y + 2S(\phi X,Y)\phi Z + 2g(\phi X,Y)Q\phi Z - S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX] - \frac{p+n-1}{n+3}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] - \frac{p-4}{n+3}[g(X,Z)Y - g(Y,Z)X] + \frac{p}{n+3}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X]$$

Where S is the Ricci tensor of type (0, 2), Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and $p = \frac{n+r-1}{n+1}$, r being the scalar curvature of the manifold.

2. SOME BASIC AND USEFUL PROPERTIES OF $(LCS)_n$ -MANIFOLDS

Here some necessary formulas about $(LCS)_n$-manifolds are given and these are used throughout in this paper.

An n-dimensional Lorentzian manifold M is a smooth, connected, paracompact, and Hausdorff manifold equipped with a Lorentzian metric g. This means that M admits a smooth symmetric tensor field g of type (0, 2) such that for every point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ defines a non-degenerate inner product of signature, (−, +, · · · , +) where T_pM denotes the tangent space of M at *p* and $\mathbb{R}$ represents the real number space. A non-zero vector $p \in T_pM$ is categorized as time-like (respectively, non-space-like, null, space-like) if it satisfies $g_p(v,v) < 0$ (respectively < 0, = 0, > 0) [12].

Definition 2.1. In a Lorentzian manifold (M,g), a vector field P is considered concircular if it satisfies the condition:

(2.1) $g(X,P) = A(X)$

for any $X \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the set of all differentiable vector fields on M. Here, A is a 1-form.

A concircular vector field P further obeys the equation:

(2.2) $(\tilde{\nabla}_X A)Y = \alpha[g(X,Y) + \omega(X)A(Y)]$

Where $\tilde{\nabla}$ represents the covariant differentiation operator of M with respect to the Lorentzian metric g, ω is a closed 1-form and α is a non-zero scalar function.

Given a Lorentzian manifold M with a unit time-like concircular vector field ξ, known as the characteristic vector field of the manifold, it follows that:

(2.3) $g(\xi,\xi) = -1.$

Let ξ be the unit concircular vector field. Then there exists a non-zero 1-form η such that for all $X, Y \in \Gamma(TM)$, the inner product of X with ξ is equal to the evaluation of η on X, expressed as

(2.4) $g(X,\xi) = \eta(X).$

Moreover, the covariant derivative $\tilde{\nabla}_X \eta$ of η with respect to the Lorentzian metric tensor g is given by

(2.5) $(\tilde{\nabla}_X \eta) = \alpha[g(X,Y) + \eta(X)\eta(Y)],$

Where α is a non-zero scalar function and $X, Y \in \Gamma(TM)$.

They satisfy the following relation

(2.6) $\tilde{\nabla}_X \alpha = (X\alpha) = \rho\eta(X)$

where ρ is a scalar function and it is given by $\rho = -(\xi\alpha)$. Let us take

(2.7) $\tilde{\nabla}_X \xi = \alpha$

Given equations (2.5) and (2.7), we obtain the expression:

(2.8) $\phi X = X + \eta(X)\xi,$

where ϕ represents a (1, 1) symmetric tensor. Consequently, the Lorentzian manifold M equipped with the unit time-like concircular vector field ξ, its corresponding 1-form η, and the (1, 1) tensor field ϕ is termed a Lorentzian concircular structure manifold, denoted briefly as an $(LCS)_n$-manifold. In an $(LCS)_n$-manifold, the following mathematical relations are established [8]:

(2.9) $\phi^2 = I + \eta \otimes \xi, \eta(\xi) = -1, \phi\xi = 0, \eta o \phi = 0,$

(2.10) $g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),$

(2.11) $\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)],$

(2.12) $R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y],$

(2.13) $R(\xi,Y)X = (\alpha^2 - \rho)[g(Y,X)\xi - \eta(X)Y],$

(2.14) $(\tilde{\nabla}_X \emptyset)Y = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X],$

(2.15) $S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X),$

(2.16) $S(\phi X, \phi Y) = S(X,Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y)$

for any $X, Y, Z \in \Gamma(\mathrm{TM})$, where R, S denote Riemannian curvature and Ricci tensor respectively.

From (1.1), (1.2) and (1.3), we get

(2.17) $C(\xi, Y)Z = \frac{r-(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}[g(Y,Z)\xi - \eta(Z)Y] - \frac{1}{(n-2)}[S(Y,Z)\xi - \eta(Z)QY].$

(2.18) $C(\xi, Y)\xi = \frac{r}{(n-1)(n-2)}[\eta(Y)\xi + Y] - \frac{(\alpha^2-\rho)}{(n-2)}[n\eta(Y)\xi + Y] - \frac{1}{(n-2)}QY.$

(2.19) $\bar{C}(\xi, Y)Z = -\frac{(\alpha^2-\rho)}{(n-2)}[g(Y,Z)\xi - \eta(Z)Y] - \frac{1}{(n-2)}[S(Y,Z)\xi - \eta(Z)QY].$

(2.20) $\bar{C}(\xi, Y)\xi = -\frac{(\alpha^2-\rho)}{(n-2)}[n\eta(Y)\xi + Y] - \frac{1}{(n-2)}QY.$

(2.21) $B(\xi, Y)Z = \frac{(2p-4)+4(\alpha^2-\rho)}{(n+3)}[g(Y,Z)\xi - \eta(Z)Y] - \frac{2}{(n+3)}[S(Y,Z)\xi - \eta(Z)QY].$

(2.22) $B(\xi, Y)\xi = \frac{(2p-4)+4(\alpha^2-\rho)}{(n+3)}Y + \frac{(2p-4)-2(n-3)(\alpha^2-\rho)}{(n+3)}\eta(Y)\xi - \frac{2}{n+3}QY.$

3. SOME BASIC PROPERTIES OF INVARIANT SUBMANIFOLD OF $(LCS)_n$ -MANIFOLDS

Let g be the induced metric of the manifold N of a $(LCS)_n$-manifold M. Denote TN and $T^{\perp}N$ as the sets of all vector fields tangent to N and normal to N respectively. Let ∇ and $\tilde{\nabla}$ represent the Levi-Civita connections of N and M respectively. Then, the Gauss and Weingarten formulas are given by [13]:

(3.1) $\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$ for all $X, Y \in \Gamma(TN)$

(3.2) $\tilde{\nabla}_X Y = -A_V X + \nabla_X^{\perp} V$ for all $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^{\perp}N)$

Where h, A_V and $\nabla^{\perp}$ denote the second fundamental form, the shape operator and the normal connection respectively.

Definition 3.1. [5] An invariant submanifold N of a $(LCS)_n$-manifold M is defined as follows:

the structure vector field ξ is tangent to N at every point and for any vector field X tangent to N at each point, the Lie derivative of the symplectic form ϕ along X is also tangent to N, that is, $\phi(TN) \subset TN$ at every point on N.

Definition 3.2. [5] A submanifold N of a $(LCS)_n$-manifold M is deemed totally geodesic if its second fundamental form h satisfies $h \equiv 0$.

Relation Between Second Fundamental Form and Shape Operator:[28] The interplay between the second fundamental form h and the shape operator A_V is encapsulated by the equation:

(3.3) $g(h(X,Y), V) = g(A_V X, Y),$

where g is the metric tensor on M and $X, Y \in \mathrm{TM}$ with V being a vector orthogonal to N.

Covariant Derivative of the Second Fundamental Form: The covariant derivative of

h with respect to a vector field X is given by:

(3.4) $(\bar{\nabla}_X h)(Y,Z) = \nabla_X^{\perp}(h(X,Y)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$

Where $\overline{\nabla}$ signifies the Vander-Waerden-Bortolotti connection on M and $\nabla_X^{\perp}$ denotes the component of the covariant derivative normal to N. Here, X, Y, Z are vector fields belonging to the tangent space TN of N.

Normal Bundle Valued Tensor and Third Fundamental Form: The tensor $\overline{\nabla}h$, arising from the covariant derivative of the second fundamental form, is a tensor of type (0,3) that values in the normal bundle. It is referred to as the third fundamental form of

N. These reformulated conditions encapsulate the geometric properties of an invariant submanifold within a $(LCS)_n$-manifold, highlighting the interrelations between the second and third fundamental forms, the shape operator and the geometrical construct of being totally geodesic.

In an invariant submanifold *N* of a $(LCS)_n$-manifold *M* the following conditions hold [11]:

(3.5) $h(X,\xi) = 0,$

(3.6) $R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y],$

(3.7) $S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), i.e., Q\xi = (n-1)(\alpha^2 - \rho)\xi,$

(3.8) $(\nabla_X \phi)Y = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\},$

(3.9) $h(X,\phi Y) = \phi h(X,Y),$

(3.10) $\nabla_X \xi = \alpha\phi X.$

An invariant submanifold N of a $(LCS)_n$-manifold M is also a $(LCS)_n$-manifold [11].

A $(LCS)_n$-manifold satisfying $P(\xi,X)S = 0$ is Einstein manifold [30].

So, an invariant submanifold of a $(LCS)_n$-manifold satisfying $P(\xi,X)S = 0$ is also an Einstein manifold. The following conditions also hold in this class of submanifold:

(3.11) $QX = (n-1)(\alpha^2 - \rho)X,$

(3.12) $(\overline{\nabla}h)(Z,\xi,U) = -\alpha h(\phi Z,U).$

4. INVARIANT SUBMANIFOLDS OF $(LCS)_n$-MANIFOLDS SATISFYING $C(X,Y).h = 0$ AND $C(X,Y).\overline{\nabla}h = 0$ WITH THE CONDITION THAT THE AMBIENT MANIFOLD SATISFYING $P(\xi,X)S = 0$

This section deals with invariant submanifolds of $(LCS)_n$-manifold satisfying $C(X,Y).h = 0$ and $C(X,Y).\overline{\nabla}h = 0$ with the condition that the ambient manifold satisfying $P(\xi,X)S = 0$.

Theorem 4.1. Let N be an invariant submanifold of a $(LCS)_n$-manifold M with the condition that the ambient manifold satisfying $P(\xi,X)S = 0$. Then $C(X,Y).h = 0$ holds on N if and only if N is totally geodesic provided $r \neq n(n-1)(\alpha^2 - \rho)$.

Proof. Let N be an invariant submanifold of a $(LCS)_n$-manifold satisfying $C(X,Y).h = 0$ such that $r \neq n(n-1)(\alpha^2 - \rho)$ with the condition that the ambient manifold satisfying $P(\xi,X)S = 0$. Then we get

(4.1) $R^{\perp}(X,Y)h(Z,U) - h(C(X,Y)Z,U) - h(Z,C(X,Y)U) = 0.$

Now setting $X = U = \xi$ in (4.1) and using (2.17) and (3.5), we get

(4.2) $h(Z,C(\xi,Y)\xi) = 0.$

By virtue of (2.18), it follows from (4.2) that

(4.3) $$\frac{r}{(n-1)(n-2)}h(Z,\eta(Y)\xi+Y)-\frac{(\alpha^2-\rho)}{(n-2)}h(Z,n\eta(Y)\xi+Y)$$
$$-\frac{1}{(n-2)}h(Z,QY)=0.$$

Using (3.5) and (3.11) in (4.3), it follows that

(4.4) $$\frac{r-n(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}h(Z,Y)=0,$$

which gives $h(Z,Y)=0$, provided $r\neq n(n-1)(\alpha^2-\rho)$. Hence the manifold N is totally geodesic provided $r\neq n(n-1)(\alpha^2-\rho)$. The converse statement is trivial and therefore the theorem is proved.

Theorem 4.2. Let N be an invariant submanifold of a $(LCS)_n$-manifold M with the condition that the ambient manifold satisfying $P(\xi,X)S=0$. Then $C(X,Y).\overline{\nabla}h=0$ holds on N if and only if N is totally geodesic provided $r\neq n(n-1)(\alpha^2-\rho)$.

Proof. Let N be an invariant submanifold of a $(LCS)_n$-manifold satisfying $C(X,Y).\overline{\nabla}h=0$ such that $r\neq n(n-1)(\alpha^2-\rho)$ with the condition that the ambient manifold satisfying $P(\xi,X)S=0$. Then we get

(4.5) $$R^{\perp}(X,Y)(\overline{\nabla}h)(Z,U,W)-(\overline{\nabla}h)(C(X,Y)Z,U,W)$$
$$-(\overline{\nabla}h)(Z,C(X,Y)U,W)-(\overline{\nabla}h)(Z,U,C(X,Y)W)=0.$$

Putting $X=U=\xi$ in (4.5), we get

(4.6)
$$R^{\perp}(\xi,Y)(\overline{\nabla}h)(Z,\xi,W)-(\overline{\nabla}h)(C(\xi,Y)Z,\xi,W)-(\overline{\nabla}h)(Z,C(\xi,Y)\xi,W)-(\overline{\nabla}h)(Z,\xi,C(\xi,Y)W)=0.$$

By virtue of (3.4), (2.17), (2.18) and (3.5), we get

(4.7)
$$(\overline{\nabla}h)(C(\xi,Y)Z,\xi,W)=\left(\overline{\nabla}_{C(\xi,Y)Z}h\right)(\xi,W)=\nabla^{\perp}_{C(\xi,Y)Z}h(\xi,W)-h\left(\nabla_{C(\xi,Y)Z}\xi,W\right)-h\left(\xi,\nabla_{C(\xi,Y)Z}W\right)=$$
$$-\alpha h(\phi C(\xi,Y)Z,W)=\alpha\frac{r-(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}\eta(Z)h(\phi Y,W)-\frac{\alpha}{(n-2)}\eta(Z)h(\phi QY,W),$$

(4.8)
$$(\overline{\nabla}h)(Z,C(\xi,Y)\xi,W)=(\overline{\nabla}_Z h)(C(\xi,Y)\xi,W)=\nabla^{\perp}_Z h(C(\xi,Y)\xi,W)-h(\nabla_Z C(\xi,Y)\xi,W)-$$
$$h(C(\xi,Y)\xi,\nabla_Z W)=$$
$$\frac{r}{(n-1)(n-2)}[\nabla^{\perp}_Z h(Y,W)-h(\nabla_Z(\eta(Y)\xi+Y),W)-h(Y,\nabla_Z W)]-\frac{(\alpha^2-\rho)}{(n-2)}[\nabla^{\perp}_Z h(Y,W)-h(\nabla_Z(n\eta(Y)\xi+$$
$$Y),W)-h(Y,\nabla_Z W)]-\frac{1}{(n-2)}[\nabla^{\perp}_Z h(QY,W)-h(\nabla_Z QY,W)-h(QY,\nabla_Z W)]$$

and

(4.9)

$$(\overline{\nabla}h)(Z,\xi,C(\xi,Y)W) = (\overline{\nabla}_Z h)(\xi,C(\xi,Y)W) = \nabla_Z^{\perp} h(\xi,C(\xi,Y)W) - h(\nabla_Z\xi,C(\xi,Y)W) - h(\xi,\nabla_Z C(\xi,Y)W) = -h(\nabla_Z\xi,C(\xi,Y)W) = -\alpha h(\phi Z,C(\xi,Y)W) = \alpha\frac{r-(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}\eta(W)h(\phi Z,Y) - \frac{\alpha}{(n-2)}\eta(W)h(\phi Z,QY).$$

Taking account of (3.12), (3.5) and (4.7) – (4.9), we get from (4.6) that

(4.10)

$$-\alpha R^{\perp}(\xi,Y)h(\phi Z,W) - \alpha\frac{r-(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}\eta(Z)h(\phi Y,W) + \frac{\alpha}{(n-2)}\eta(Z)h(\phi QY,W) - \frac{r}{(n-1)(n-2)}[\nabla_Z^{\perp}h(Z,W) - h(\nabla_Z(n\eta(Y)\xi+Y),W) - h(Y,\nabla_Z W)] + \frac{(\alpha^2-\rho)}{(n-2)}[\nabla_Z^{\perp}h(Y,W) - h(\nabla_Z(n\eta(Y)\xi+Y),W) - h(Y,\nabla_Z W)] + \frac{1}{(n-2)}[\nabla_Z^{\perp}h(QY,W) - h(\nabla_Z QY,W) - h(QY,\nabla_Z W)] - \alpha\frac{r-(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}\eta(W)h(\phi Z,Y) + \frac{\alpha}{(n-2)}\eta(W)h(\phi Z,QY) = 0.$$

Putting $W = \xi$ in (4.10) and using (3.5) and (3.10), we get

(4.11) $-2\alpha\frac{r-(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}h(Y,\phi Z) + \frac{2\alpha}{(n-2)}h(QY,\phi Z) = 0.$

Putting (3.11) in (4.11), we get

(4.12) $2\alpha\frac{r-(n-1)(\alpha^2-\rho)}{(n-1)(n-2)}h(Y,\phi Z) = 0,$

which implies that

(4.13) $h(Y,\phi Z) = 0,$

provided $r \neq n(n-1)(\alpha^2-\rho)$, since $\alpha \neq 0$. Replacing Z by ϕZ in (4.12) and using (2.8) and (3.5) get $h(Y,Z) = 0$, provided $r \neq n(n-1)(\alpha^2-\rho)$, since $\alpha \neq 0$. This gives that N is totally geodesic. The converse part of this theorem is obvious. Therefore, the proof is complete.

5. INVARIANT SUBMANIFOLDS OF $(LCS)_n$-MANIFOLDS SATISFYING $\bar{C}(X,Y).h = 0$ AND $\bar{C}(X,Y).\overline{\nabla}h = 0$ WITH THE CONDITION THAT THE AMBIENT MANIFOLD SATISFYING $P(\xi,X)S = 0$

This section deals with invariant submanifolds of $(LCS)_n$-manifold satisfying $\bar{C}(X,Y).h = 0$ and $\bar{C}(X,Y).\overline{\nabla}h = 0$ with the condition that the ambient manifold satisfying $P(\xi,X)S = 0$.

Theorem 5.1. Let N be an invariant submanifold of a $(LCS)_n$-manifold M with the condition that the ambient manifold satisfying $P(\xi,X)S = 0$. Then $\bar{C}(X,Y).h = 0$ holds on N if and only if N is totally geodesic provided $\alpha^2 \neq \rho$.

Proof. Let N be an invariant submanifold of a $(LCS)_n$-manifold satisfying $\bar{C}(X,Y).h = 0$ such that $\alpha^2 \neq \rho$ with the condition that the ambient manifold satisfying $P(\xi,X)S = 0$. Then we get

(5.1) $R^{\perp}(X,Y)h(Z,U) - h(\bar{C}(X,Y)Z,U) - h(Z,\bar{C}(X,Y)U) = 0.$

Setting $X = U = \xi$ in (5.1) and using (2.19) and (3.5), we get

(5.2) $h(Z,\bar{C}(\xi,Y)\xi)=0.$

By virtue of (2.20), it follows from (5.2) that

(5.3) $\frac{(\alpha^2-\rho)}{(n-2)}h(Z,Y)+n\frac{(\alpha^2-\rho)}{(n-2)}h(Z,\eta(Y)\xi)+\frac{1}{(n-2)}h(Z,QY)=0.$

By virtue of (3.5) and (3.11) in (5.3), it follows that

(5.4) $n\frac{(\alpha^2-\rho)}{(n-2)}h(Z,Y)=0,$

which gives $h(Z,Y)=0$, provided $\alpha^2\neq\rho$, since $n\neq 0$. Hence the submanifold N is totally geodesic provided $\alpha^2\neq\rho$. The converse statement is trivial and therefore the theorem is proved.

Theorem 5.2. Let N be an invariant submanifold of a $(LCS)_n$-manifold M with the condition that the ambient manifold satisfying $P(\xi,X)S=0$. Then $\bar{C}(X,Y).\bar{\nabla}h=0$ holds on N if and only if N is totally geodesic provided $\alpha^2\neq\rho$.

Proof. Let N be an invariant submanifold of a $(LCS)_n$-manifold satisfying $\bar{C}(X,Y).\bar{\nabla}h=0$ such that $\alpha^2\neq\rho$ with the condition that the ambient manifold satisfying $P(\xi,X)S=0$. Then we get

(5.5)
$R^{\perp}(X,Y)(\bar{\nabla}h)(Z,U,W)-(\bar{\nabla}h)(\bar{C}(X,Y)Z,U,W)-(\bar{\nabla}h)(Z,\bar{C}(X,Y)U,W)-(\bar{\nabla}h)(Z,U,\bar{C}(X,Y)W)=0.$

Putting $X=U=\xi$ in (5.5), we get

(5.6)
$R^{\perp}(\xi,Y)(\bar{\nabla}h)(Z,\xi,W)-(\bar{\nabla}h)(\bar{C}(\xi,Y)Z,\xi,W)-(\bar{\nabla}h)(Z,\bar{C}(X,Y)\xi,W)-(\bar{\nabla}h)(Z,\xi,\bar{C}(\xi,Y)W)=0$

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By virtue of (2.19), (2.20), (3.4) and (3.5), gives

(5.7)
$(\bar{\nabla}h)(\bar{C}(\xi,Y)Z,\xi,W)=\left(\bar{\nabla}_{\bar{C}(\xi,Y)Z}h\right)(\xi,W)=\nabla^{\perp}_{\bar{C}(\xi,Y)Z}h(\xi,W)-h\left(\nabla_{\bar{C}(\xi,Y)Z}\xi,W\right)-h\left(\xi,\nabla_{\bar{C}(\xi,Y)Z}W\right)=-\alpha h(\phi\bar{C}(\xi,Y)Z,W)=-\alpha\frac{(\alpha^2-\rho)}{(n-2)}\eta(Z)h(\phi Y,W)-\frac{\alpha}{(n-2)}\eta(Z)h(\phi QY,W),$

(5.8)
$(\bar{\nabla}h)(Z,\bar{C}(\xi,Y)\xi,W)=(\bar{\nabla}_Z h)(\bar{C}(\xi,Y)\xi,W)=\nabla^{\perp}_Z h(\bar{C}(\xi,Y)\xi,W)-h(\nabla_Z\bar{C}(\xi,Y)\xi,W)-h(\bar{C}(\xi,Y)\xi,\nabla_Z W)=-\frac{(\alpha^2-\rho)}{(n-2)}[\nabla^{\perp}_Z h(Y,W)-h(\nabla_Z Y,W)-h(Y,\nabla_Z W)]+n\alpha\frac{(\alpha^2-\rho)}{(n-2)}\eta(Y)h(\phi Z,W)-\frac{1}{(n-2)}[\nabla^{\perp}_Z h(QY,W)-h(\nabla_Z QY,W)-h(QY,\nabla_Z W)]$

and

(5.9)
$(\bar{\nabla}h)(Z,\xi,\bar{C}(\xi,Y)W) = (\bar{\nabla}_Z h)(\xi,\bar{C}(\xi,Y)W) = \nabla_Z^{\perp} h(\xi,\bar{C}(\xi,Y)W) - h(\nabla_Z \xi,\bar{C}(\xi,Y)W) - h(\xi,\nabla_Z \bar{C}(\xi,Y)W) = -h(\nabla_Z \xi,\bar{C}(\xi,Y)W) = -\alpha h(\phi Z,\bar{C}(\xi,Y)W) = -\alpha \frac{(\alpha^2-\rho)}{(n-2)}\eta(W)h(\phi Z,Y) - \frac{\alpha}{(n-2)}\eta(W)h(\phi Z,QY).$

Taking account of (3.12), (3.10) and (5.7) – (5.9), we get from (5.6)

(5.10)
$-\alpha R^{\perp}(\xi,Y)h(\phi Z,W) + \alpha \frac{(\alpha^2-\rho)}{(n-2)}\eta(Z)h(\phi Y,W) + \frac{\alpha}{(n-2)}\eta(Z)h(\phi QY,W) + \frac{(\alpha^2-\rho)}{(n-2)}[\nabla_Z^{\perp} h(Y,W) - h(\nabla_Z Y,W) - h(Y,\nabla_Z W)] - n\alpha \frac{(\alpha^2-\rho)}{(n-2)}\eta(Y)h(\phi Z,W) + \frac{1}{(n-2)}[\nabla_Z^{\perp} h(QY,W) - h(\nabla_Z QY,W) - h(QY,\nabla_Z W)] + \alpha \frac{(\alpha^2-\rho)}{(n-2)}\eta(W)h(\phi Z,Y) + \frac{\alpha}{(n-2)}\eta(W)h(\phi Z,QY) = 0.$

Putting $W = \xi$ in (5.10) and using (3.5) and (3.10), it follows that

(5.11) $\frac{2n\alpha(\alpha^2-\rho)}{(n-2)}h(Y,\phi Z) = 0,$

which gives $h(Y,\phi Z) = 0$, provided $\alpha^2 \neq \rho$, since $n \neq 0$ and $\alpha \neq 0$. Replacing Z by ϕZ in (5.11) and using (2.8) and (3.5) get $h(Y,Z) = 0$, provided $\alpha^2 \neq \rho$, since $n \neq 0$ and $\alpha \neq 0$. Hence the submanifold N is totally geodesic. The converse statement is trivial and therefore the theorem is proved.

6. INVARIANT SUBMANIFOLDS OF $(LCS)_n$-MANIFOLDS SATISFYING $B(X,Y).h = 0$ AND $B(X,Y).\bar{\nabla}h = 0$ WITH THE CONDITION THAT THE AMBIENT MANIFOLD SATISFYING $P(\xi,X)S = 0$

This section deals with invariant submanifolds of $(LCS)_n$-manifold satisfying $B(X,Y).h = 0$ and $B(X,Y).\bar{\nabla}h = 0$ with the condition that the ambient manifold satisfying $P(\xi,X)S = 0$.

Theorem 6.1. Let N be an invariant submanifold of a $(LCS)_n$-manifold M with the condition that the ambient manifold satisfying $P(\xi,X)S = 0$. Then $B(X,Y).h = 0$ holds on N if and only if N is totally geodesic provided $2(n-3)(\alpha^2-\rho) \neq (2p-4)$.

Proof. Let N be an invariant submanifold of a $(LCS)_n$-manifold satisfying $B(X,Y).h = 0$ such that $2(n-3)(\alpha^2-\rho) \neq (2p-4)$ with the condition that the ambient manifold satisfying $P(\xi,X)S = 0$. Then we get

(6.1) $R^{\perp}(X,Y)h(Z,U) - h(B(X,Y)Z,U) - h(Z,B(X,Y)U) = 0.$

Setting $X = U = \xi$ in (6.1) and using (3.5), we get

(6.2) $h(Z,B(\xi,Y)\xi) = 0.$

By virtue of (2.22), it follows from (6.2) that

(6.3) $-\frac{(2p-4)+4(\alpha^2-\rho)}{(n+3)}h(Y,Z) + \frac{2}{(n+3)}h(QY,Z) = 0.$

Using (3.11) in (6.3), we get

$$\frac{2(n-3)(\alpha^2-\rho)-(2p-4)}{(n+3)}h(Y,Z)=0,$$

which gives $h(Y,Z)=0$, provided $2(n-3)(\alpha^2-\rho)\neq(2p-4)$. Hence the submanifold N is totally geodesic provided $2(n-3)(\alpha^2-\rho)\neq(2p-4)$. The converse statement is trivial and therefore the theorem is proved.

Theorem 6.2. Let N be an invariant submanifold of a $(LCS)_n$-manifold M with the condition that the ambient manifold satisfying $P(\xi,X)S=0$. Then $B(X,Y).\overline{\nabla}h=0$ holds on N if and only if N is totally geodesic provided $2(n-3)(\alpha^2-\rho)\neq(2p-4)$.

Proof. Let N be an invariant submanifold of a $(LCS)_n$-manifold satisfying $B(X,Y).\overline{\nabla}h=0$ such that $2(n-3)(\alpha^2-\rho)\neq(2p-4)$with the condition that the ambient manifold satisfying $P(\xi,X)S=0$. Then we get

(6.4)
$$R^{\perp}(X,Y)(\overline{\nabla}h)(Z,U,W)-(\overline{\nabla}h)(B(X,Y)Z,U,W)-(\overline{\nabla}h)(Z,B(X,Y)U,W)-(\overline{\nabla}h)(Z,U,B(X,Y)W)=0.$$

Putting $X=U=\xi$ in (6.4), we get

(6.5)
$$R^{\perp}(\xi,Y)(\overline{\nabla}h)(Z,\xi,W)-(\overline{\nabla}h)(B(\xi,Y)Z,\xi,W)-(\overline{\nabla}h)(Z,B(\xi,Y)\xi,W)-(\overline{\nabla}h)(Z,\xi,B(\xi,Y)W)=0.$$

By virtue of (2.21), (2.22), (3.5), (3.11) and (3.10), we get

(6.6)
$$(\overline{\nabla}h)(B(\xi,Y)Z,\xi,W)=\left(\overline{\nabla}_{B(\xi,Y)Z}h\right)(\xi,W)=\nabla^{\perp}_{B(\xi,Y)Z}h(\xi,W)-h\left(\nabla_{B(\xi,Y)Z}\xi,W\right)-h\left(\xi,\nabla_{B(\xi,Y)Z}W\right)=-\alpha h(\phi B(\xi,Y)Z,W)=\alpha\frac{(2p-4)+4(\alpha^2-\rho)}{(n+3)}\eta(Z)h(\phi Y,W)-\frac{2\alpha}{(n+3)}\eta(Z)h(\phi QY,W),$$

(6.7)
$$(\overline{\nabla}h)(Z,B(\xi,Y)\xi,W)=(\overline{\nabla}_Z h)(B(\xi,Y)\xi,W)=\nabla^{\perp}_Z h(B(\xi,Y)\xi,W)-h(\nabla_Z B(\xi,Y)\xi,W)-h(B(\xi,Y)\xi,\nabla_Z W)=\frac{(2p-4)+4(\alpha^2-\rho)}{(n+3)}[\nabla^{\perp}_Z h(Y,W)-h(\nabla_Z Y,W)-h(Y,\nabla_Z W)]-\alpha\frac{(2p-4)-2(\alpha^2-\rho)(n+3)}{(n+3)}\eta(Y)h(\phi Z,W)-\frac{2}{(n+3)}[\nabla^{\perp}_Z h(QY,W)-h(\nabla_Z QY,W)-h(QY,\nabla_Z W)]$$

and

(6.8)
$$(\overline{\nabla}h)(Z,\xi,B(\xi,Y)W)=(\overline{\nabla}_Z h)(\xi,B(\xi,Y)W)=\nabla^{\perp}_Z h(\xi,B(\xi,Y)W)-h(\nabla_Z\xi,B(\xi,Y)W)-h(\xi,\nabla_Z B(\xi,Y)W)=-h(\nabla_Z\xi,B(\xi,Y)W)=-\alpha h(\phi Z,B(\xi,Y)W)=\alpha\frac{(2p-4)+4(\alpha^2-\rho)}{(n+3)}\eta(W)h(\phi Z,Y)-\frac{2\alpha}{(n+3)}\eta(W)h(\phi Z,QY).$$

Taking account of (2.7), (3.12) and (6.6)-(6.8), we get from (6.5)

(6.9)

$$-\alpha R^{\perp}(\xi,Y)h(\phi Z,W)-\alpha\frac{(2p-4)+4(\alpha^2-\rho)}{(n+3)}\eta(Z)h(\phi Y,W)+\frac{2\alpha}{(n+3)}\eta(Z)h(\phi QY,W)-\frac{(2p-4)+4(\alpha^2-\rho)}{(n+3)}[\nabla_Z^{\perp}h(Y,W)-h(\nabla_Z Y,W)-h(Y,\nabla_Z W)]+\alpha\frac{(2p-4)-2(\alpha^2-\rho)(n-3)}{(n+3)}\eta(Y)h(\phi Z,W)-\alpha\frac{(2p-4)+4(\alpha^2-\rho)}{(n+3)}\eta(W)h(\phi Y,Z)+\frac{2}{(n+3)}[\nabla_Z^{\perp}h(QY,W)-h(\nabla_Z QY,W)-h(QY,\nabla_Z W)]+\frac{2\alpha}{(n+3)}\eta(W)h(\phi Z,QY)=0.$$

Putting $W=\xi$ in (6.9) and using (3.5) and (3.10), it follows that

(6.10) $$\alpha\frac{(2p-4)+4(\alpha^2-\rho)}{(n+3)}[h(Y,\phi Z)+h(\phi Y,Z)]-\frac{2\alpha}{(n+3)}[h(\phi Z,QY)+h(QY,\phi Z)]=$$

Replacing Z by ϕZ and using (2.8), (3.5) and (3.11) in (6.10), we get

(6.11) $$\alpha\frac{(2p-4)-2(\alpha^2-\rho)(n-3)}{(n+3)}h(Y,Z)=0,$$

which gives $h(Y,Z)=0$, provided $2(n-3)(\alpha^2-\rho)\neq(2p-4)$, since $\alpha\neq 0$. Hence the submanifold N is totally geodesic provided $2(n-3)(\alpha^2-\rho)\neq(2p-4)$. The converse statement is trivial and therefore the theorem is proved.

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