MATHEMATICAL MODELING FOR AN ECOLOGICAL SYSTEM INDUCED BY COMPETITION AND GROUP DEFENSE

S. N. Raw[1] and B. P. Sarangi[2]
[1]Department of Mathematics, National Institute of Technology Raipur, Chhattisgarh, India
[2]Centre for Data Science, Siksha O Anusandhan deemed to be University, Bhubaneswar, India
[1] sharaw.maths@nitrr.ac.in, [2] bpsarangi04@gmail.com

ABSTRACT
This article deals with modeling and analysis of an ecological system i.e., a prey-predator interactions including two competing prey and a predator species. We have incorporated linear and Monod–Haldane functional response to show interaction with the predator. The principal assumption is to consider two preys with independent behaviours where the first prey doesn't resist during predation and the second one uses the group defense mechanism for survival. Also, we have assumed that prey undergoes inter-specific competition. Stability and Hopf bifurcation analysis has been done for the coexisting equilibrium point. The significance of group defense and inter-specific competition have elaborately shown in the article. We have done extensive numerical simulations for the proposed model system which resemble many environmental applications. Predator acts as stabilizing agent when competition and resource management occur within a trophic level. This scenarios replicated by our model with real world situation.

Keywords: Bifurcation, Ecological Model, Group defense, Inter-specific competition, Stability analysis.

INTRODUCTION
Mathematical models and biological models are employed in population ecology to describe the relationships between populations in the environment. Mathematical formulas, validated ecologically, are utilized for the development of biological models, with researchers eager to evaluate factors influencing population dynamics. One crucial factor is the selection of the trophic function or functional response in an ecological model. The trophic function typically represents the feeding rate of predators, influenced by various factors such as prey detection, interference, capture, consumption, handling, and availability. All species in nature aspire to protect themselves from threats and predation in order to persist longer. Group defense mechanisms have been adopted by species to evade predation, wherein animals defend themselves concurrently from being killed and guard individuals within the group. Additionally, within these groups, species often undergo competition for habitat, food, mates, and resources. In short, they cooperate as well as compete with each other. Competition within a trophic level for resources is known as interspecific competition. This scenario is prevalent in the environment and represents a significant ecological issue. Prey predator model with the interspecific competition is considered in [1-6]. Grouping of species, benefits of prey defense and anti-predator behaviour in the prey-predator model is examined in [7,8].

The article has regulated in the following ways. Sec. II deals with the formulation of the model with specifications. Mathematical properties are examined briefly in Sec. III. All the results are simulated through MATLAB programming in Sec. IV. We describe the findings of the model with application in ecology in Conclusion.

MODEL FORMULATION
Biologically compatible and reliable models are influenced by various environmental factors, which are often complex and interconnected. In this context, we focus on a simple model comprising only two prominent factors: the grouping of species and competition.

Now, the proposed model is as follows:

$$\frac{dx_1}{dt} = r_1 x_1\left(1 - \frac{x_1}{k_1}\right) - \alpha_1 x_1 y - \beta_1 x_1 x_2, \tag{1}$$

$$\frac{dx_2}{dt} = r_2 x_2\left(1 - \frac{x_2}{k_2}\right) - \frac{\gamma_1 x_2 y}{b x_2^2 + c} - \beta_2 x_1 x_2, \qquad (2)$$

$$\frac{dy}{dt} = -dy + \alpha_2 x_1 y + \frac{\gamma_2 x_2 y}{b x_2^2 + c}, \qquad (3)$$

with $x_1, x_2, y \geq 0$, and parameters $r_1, r_2, k_1, k_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, b, c, d \in (0, \infty)$.

The primary assumptions behind constructing the model are

i. The proposed model consists of three population, specifically two preys x_1, x_2 , and a predator y.

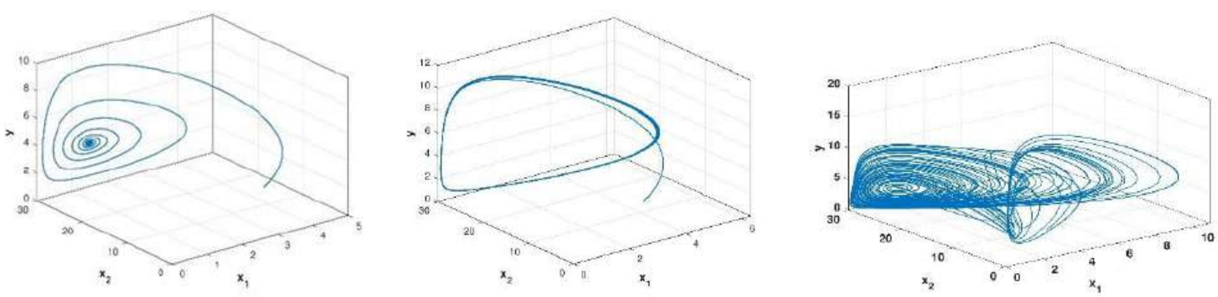
ii. Prey populations follows logistic growth rate with growth functions $r_1 x_1\left(1 - \frac{x_1}{k_1}\right)$ and $r_2 x_2\left(1 - \frac{x_2}{k_2}\right)$ respectively. r_1, k_1 and r_2, k_2 are intrinsic growth rates and carrying capacity of prey population x_1 and x_2 respectively.

iii. The preys are very competitive for resources. β_1 and β_2 are the competition coefficients.

iv. Prey x_1 is solitary whereas prey x_2 uses grouping tactics to avoid predations. So, we incorporate Holling type I and Monod–Haldane functional response to attest prey-predator interactions.

v. α_1 and γ_1 measures the reduction rate of prey populations x_1 and x_2, respectively. Parameters b and c denote the defense efficiency and half saturation constant of prey x_2.

vi. Terms $\alpha_2 x_1$ and $\frac{\gamma_2 x_2}{b x_2^2 + c}$ are the predator y growth due to prey consumptions where α_2 and γ_2 are conversion efficiencies of two preys. Parameter d is the mortality rate of the predator.



(a) (b) (c)

Figure 1: Dynamics of the model (1)-(3), (a) Stable focus at β_2 = 0.01, (b) Limit cycle at β_2 = 0.05, (c) Chaos at β_2 = 0.08, with parameter set (21).

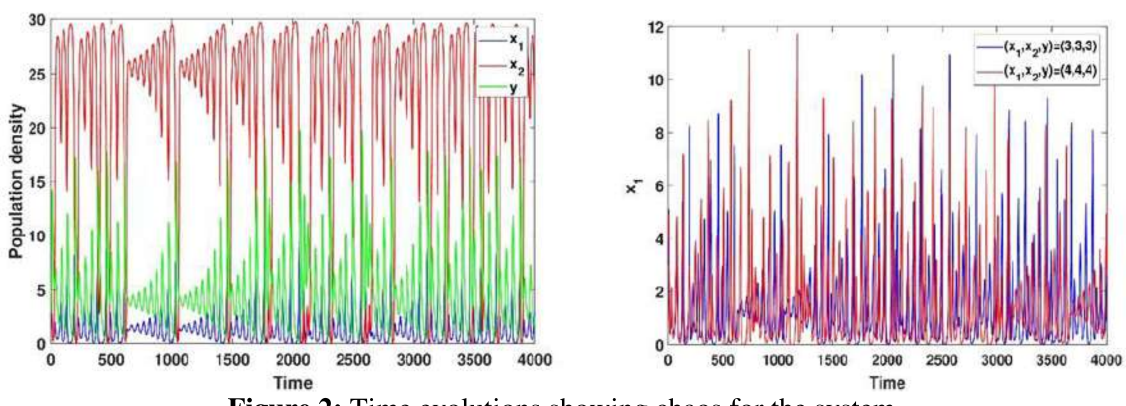


Figure 2: Time evolutions showing chaos for the system.

ANALYSIS OF THE MODEL

a. Boundedness

Using (1), we have $\frac{dx_1}{dt} \le r_1 x_1\left(1-\frac{x_1}{k_1}\right)$which implies $\limsup_{t\to\infty} x_1(t) \le k_1$. Similarly, for (2), $\limsup_{t\to\infty} x_2(t) \le k_2$.

Define a function $P(t) = \frac{\alpha_2}{\alpha_1}x_1(t) + \frac{\gamma_2}{\gamma_1}x_2(t) + y(t)$.

Hence, $\frac{dP}{dt}$ is bounded by

$$\frac{dP}{dt} \le \frac{\alpha_2 r_1}{\alpha_1}x_1(t) + \frac{\gamma_2 r_2}{\gamma_1}x_2(t) - dy(t)$$

$$\Rightarrow \frac{dP}{dt} \le 2\left(\frac{\alpha_2 r_1 k_1}{\alpha_1} + \frac{\gamma_2 r_2 k_2}{\gamma_1}\right) - \rho P$$

$$\Rightarrow \frac{dP}{dt} + \rho P \le \xi,$$

where $\rho = min\,(r_1, r_2, d)$ and $\xi = 2\left(\frac{\alpha_2 r_1 k_1}{\alpha_1} + \frac{\gamma_2 r_2 k_2}{\gamma_1}\right)$.

The above analysis concludes that model (1)-(3) is uniformly bounded in $\mathbb{R}^+$.

Theorem 1. *The solutions of model system (1)-(3) is uniformly bounded in in* $\mathbb{R}^+$.

b. Equilibrium E^* with Stability

Here we calculate the positive equilibrium point $E^* = (x_1^*,\ x_2^*,\ y^*)$ by following [9] and $(x_1^*,\ x_2^*,\ y^*)$ is the solution of following system

$$r_1\left(1-\frac{x_1}{k_1}\right) - \alpha_1 y - \beta_1 x_2 = 0, \tag{4}$$

$$r_2\left(1-\frac{x_2}{k_2}\right) - \frac{\gamma_1 y}{bx_2^2+c} - \beta_2 x_1 = 0, \tag{5}$$

$$-d + \alpha_2 x_1 + \frac{\gamma_2 x_2}{bx_2^2+c} = 0. \tag{6}$$

Using (6), we have calculated

$$x_1 = \frac{1}{\alpha_2}\left(d - \frac{\gamma_2 x_2}{bx_2^2+c}\right). \tag{7}$$

Equations (4) and (5) give

$$G_1(x_2, y) = r_1 - \frac{r_1}{\alpha_2 k_1}\left(d - \frac{\gamma_2 x_2}{bx_2^2+c}\right) - \alpha_1 y - \beta_1 x_2, \tag{8}$$

$$G_2(x_2, y) = r_2\left(1-\frac{x_2}{k_2}\right) - \frac{\gamma_1 y}{bx_2^2+c} - \frac{\beta_2}{\alpha_2}\left(d - \frac{\gamma_2 x_2}{bx_2^2+c}\right). \tag{9}$$

In (8), when $y \to 0$, then $x_2 \to x_{2a}$. $G_1(x_2, 0) = 0$ implies $r_1 - \frac{r_1}{\alpha_2 k_1}\left(d - \frac{\gamma_2 x_2}{bx_2^2+c}\right) - \beta_1 x_2 = 0$, which is a cubic polynomial of variable x_2, i.e.,

$$x_2^3 - \frac{r_1}{\beta_1}\left(1 - \frac{d}{\alpha_2 k_1}\right)x_2^2 + \left(\frac{c}{b} - \frac{r_1\gamma_2}{\beta_1\alpha_2 k_1 b}\right)x_2 - \frac{r_1 c}{\beta_1}\left(1 - \frac{d}{\alpha_2 k_1}\right) = 0. \tag{10}$$

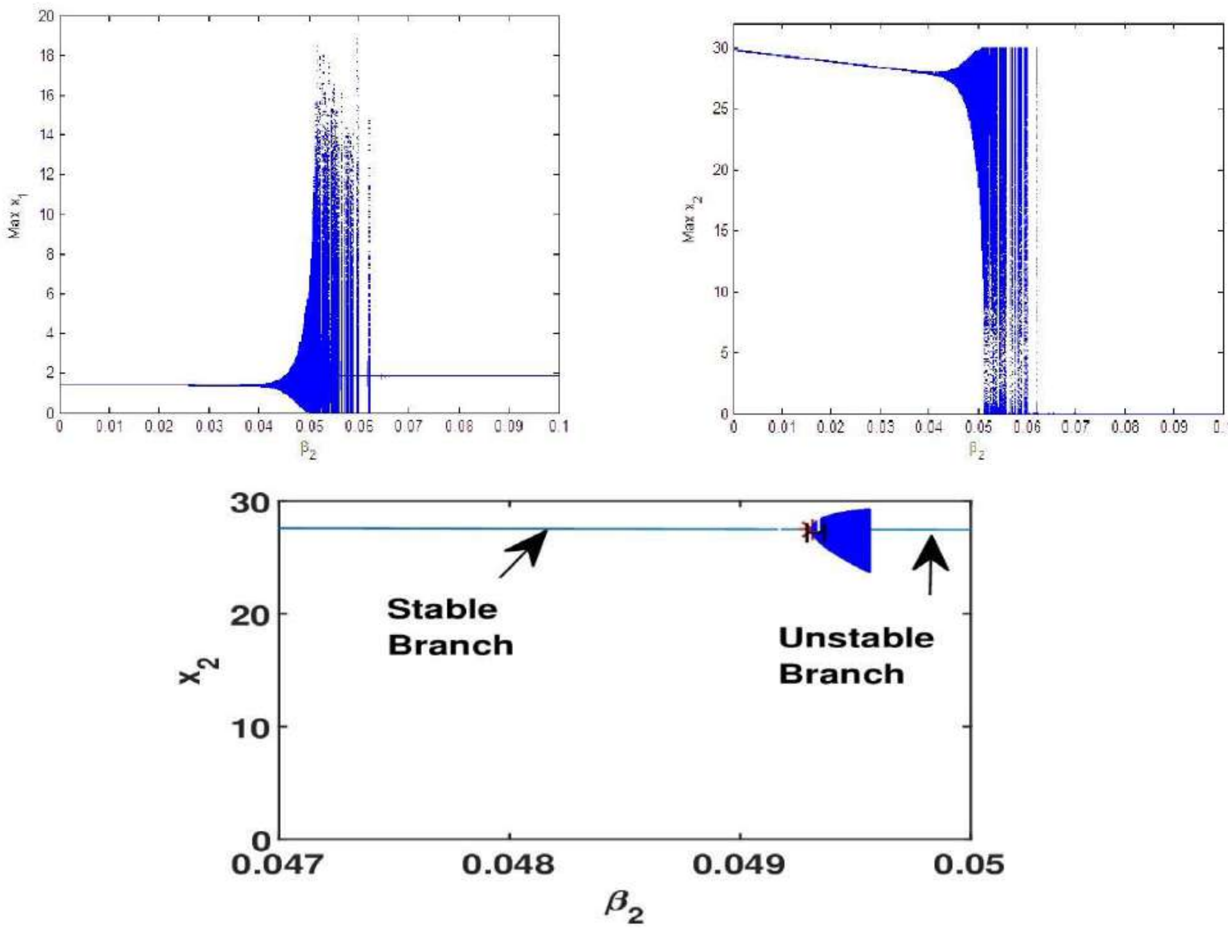


Figure 3: Bifurcation diagrams concerning parameter β_2.

Equation (10) has at least one positive root (say x_{2a}) of x_2, following Descartes rule of sign, if we choose $\alpha_2 k_1 > d$ and $c\beta_1\alpha_2 k_1 > r_1\gamma_2$.

Again $G_1(0, y) = 0$ implies a positive value of y i.e., $y = \frac{r_1}{\alpha_1}\left(1 - \frac{d}{\alpha_2 k_1}\right)$ provided $\alpha_2 k_1 > d$.

Also $\frac{dx_2}{dy} = -\frac{\partial G_1}{\partial y}/\frac{\partial G_1}{\partial x_2} = \frac{\alpha_1}{W_0}$, where $W_0 = -\beta_1 + \frac{r_1\gamma_2}{\alpha_2 k_1}\frac{c-bx_2^2}{(bx_2^2+c)^2}$ and $\frac{dx_2}{dy} < 0$ if we choose $c \le bx_2^2$.

Similarly, for (9) as $y \to 0$, $x_2 \to x_{2b}$. $G_2(x_2, 0) = 0$ implies $\left(1 - \frac{x_2}{k_2}\right) - \frac{\beta_2}{\alpha_2}\left(d - \frac{\gamma_2 x_2}{bx_2^2+c}\right) = 0$ which is a polynomial of degree 3 and given by

$$x_2^3 - k_2\left(1 - \frac{\beta_2 d}{r_2\alpha_2}\right)x_2^2 + \left(\frac{c}{b} - \frac{k_2\beta_2\gamma_2}{r_2\alpha_2 b}\right)x_2 - \frac{ck_2}{b}\left(1 - \frac{\beta_2 d}{r_2}\right) = 0.$$

Hence $G_2(x_2, 0)$ has a real root x_{2b} if $r_2 > \beta_2 d$ and $cr_2\alpha_2 > k_2\beta_2\gamma_2$. Further $G_2(0, y) = 0 \Rightarrow y = \frac{c}{\gamma_2}\left(r_2 - \frac{\beta_2 d}{\alpha_2}\right)$ and $y > 0$ if $r_2\alpha_2 > \beta_2 d$. We also have $\frac{dx_2}{dy} = -\frac{\partial G_2}{\partial y}/\frac{\partial G_2}{\partial x_2} = \frac{\gamma_1}{bx_2^2+c}\frac{1}{W_1}$, and $\frac{dx_2}{dy} > 0$ if conditions $W_1 = \frac{\partial G_2}{\partial x_2} > 0$ holds.

Hence, we see that the two isoclines (8) and (9) interest at a unique point (x_2^*, y^*) if the inequality

$$x_{2b} < x_{2a}, \qquad (11)$$

holds with some additional conditions

$\alpha_2 k_1 > d$, $c\beta_1\alpha_2 k_1 > r_1\gamma_2$, $bx_2^{*2} \geq c$

$r_2 > \beta_2 d$, $cr_2\alpha_2 > k_2\beta_2\gamma_2$. (12)

The value of x_1^* can be calculated using (7) and, for a positive value of x_1^*, we must set

$d > \frac{\gamma_2 x_2^*}{bx_2^{*2}+c}$. (13)

Hence, the equilibrium point $E^* = (x_1^*, x_2^*, y^*)$ exists if all the above conditions and restrictions are met.

Now, we study the local stability of our proposed model (1)-(3) around the coexisting equilibrium point. As our model is formulated as a system of nonlinear ODEs we proceed by computing the Jacobian matrix.

The Jacobian matrix around the equilibrium $E^* = (x_1^*, x_2^*, y^*)$ is $J(E^*) = (j_{ij})_{3\times3}$ with

$j_{11} = -\frac{r_1 x_1^*}{k_1}$, $j_{12} = -\beta_1 x_1^*$, $j_{13} = -\alpha_1 x_1^*$, $j_{21} = -\beta_2 x_2^*$,

$j_{22} = \frac{2\gamma_1 b x_2^{*2} y^*}{\left(bx_2^{*2}+c\right)^2} - \frac{r_2 x_2^*}{k_2}$, $j_{23} = \frac{\gamma_2 x_2^*}{bx_2^{*2}+c}$, $j_{31} = \alpha_2 y^*$,

$j_{32} = \frac{(c-bx_2^{*2})\gamma_2 y^*}{(bx_2^{*2}+c)^2}$, $j_{33} = 0$.

The characteristic polynomial of matrix $J(E^*)$ is

$$\lambda^3 + \xi_1\lambda^2 + \xi_2\lambda + \xi_3 = 0, \quad (14)$$

where

$\xi_1 = -(j_{11} + j_{22})$,

$\xi_2 = j_{11}j_{22} - j_{23}j_{32} - j_{13}j_{31} - j_{12}j_{21}$,

$\xi_2 = j_{11}j_{23}j_{32} + j_{13}j_{22}j_{31} - j_{12}j_{23}j_{31} - j_{13}j_{21}j_{33}$.

If we consider the restriction $\xi_1 > 0$, $\xi_2 > 0$, $\xi_3 > 0$ and

$\xi_1\,\xi_2 > \xi_3$ (Routh-Hurwitz stability condition) then E^* is stable.

Theorem 2. *If* $E^* = (x_1^*, x_2^*, y^*)$ *is locally stable, then it is globally stable if*

$2c > bk_2x_2^*$, $\frac{r_2 c}{k_2} > \frac{\gamma_1 b\, y^*(k_2+x_2^*)}{bx_2^{*2}+c}$, $d > \alpha_2 k_1 + \frac{\gamma_2 x_2^*}{bx_2^{*2}+c}$, (15)

$\beta_1\eta_1 + \beta_2\eta_2 = 0$. (16)

Proof. Define a positive Lyapunov function as

$$F = \eta_1\left[x_1 - x_1^* - x_1^*\ln\left(\frac{x_1}{x_1^*}\right)\right] + \eta_2\left[x_2 - x_2^* x_2^*\ln\left(\frac{x_2}{x_2^*}\right)\right]$$

$$+ \frac{1}{2}[y - y^*]^2. \quad (17)$$

Hence

$$\frac{dF}{dt} = \frac{r_1}{k_1}\eta_1(x_1 - x_1^*)^2 - (\alpha_1\eta_1 + \beta_2\eta_2)(x_1 - x_1^*)(y - y^*) - \beta_1\eta_1(x_1 - x_1^*)(x_2 - x_2^*) - \frac{r_2}{k_2}\eta_2(x_2 - x_2^*)^2 + \frac{\gamma_1 b\, y^*(x_2 + x_2^*)\eta_2}{(bx_2{}^2 + c)(bx_2^{*2} + c)}(x_2 - x_2^*)^2 - \frac{(\gamma_1 bx_2^{*2} + \gamma_1 c)\eta_2}{(bx_2{}^2 + c)(bx_2^{*2} + c)}(x_2 - x_2^*)(y - y^*) - (d - \alpha_2 x_1^*)(y - y^*)^2 + \alpha_2 y(x_1 - x_1^*)(y - y^*) + \frac{\gamma_2}{(bx_2{}^2 + c)(bx_2^{*2} + c)} + \frac{\gamma_2(c - bx_2^*(x_2 + x_2^*))}{(bx_2{}^2 + c)(bx_2^{*2} + c)}(x_2 - x_2^*)(y - y^*) + \frac{\gamma_2 x_2^*}{(bx_2^{*2} + c)}(y - y^*)^2 \qquad (18)$$

$$= -f_{11}(x_1 - x_1^*)^2 + f_{12}(x_1 - x_1^*)(x_2 - x_2^*) - f_{22}(x_2 - x_2^*)^2 + f_{13}(x_1 - x_1^*)(y - y^*) - f_{33}(y - y^*)^2 + f_{23}(x_2 - x_2^*)(y - y^*), \qquad (19)$$

where

$$f_{11} = \frac{r_1}{k_1}\eta_1,\ f_{12} = -\beta_1\eta_1 - \beta_2\eta_2,\ f_{13} = \alpha_2 y - \alpha_1\eta_1$$

$$f_{22} = \frac{r_2}{k_2}\eta_2 - \frac{\gamma_1 b\, y^*(x_2 + x_2^*)\eta_2}{(bx_2{}^2 + c)(bx_2^{*2} + c)},\ f_{33} = d - \alpha_2 x_1 - \frac{\gamma_2 x_2^*}{(bx_2^{*2} + c)},$$

$$f_{23} = \frac{\gamma_2}{(bx_2{}^2 + c)} + \frac{\gamma_2(c - bx_2^*(x_2 + x_2^*))}{(bx_2{}^2 + c)(bx_2^{*2} + c)} - \frac{(\gamma_1 bx_2^{*2} + \gamma_1 c)\eta_2}{(bx_2{}^2 + c)(bx_2^{*2} + c)}.$$

$\frac{dF}{dt}$ is negative definite if

$$\begin{cases} f_{11} > 0,\ f_{22} > 0,\ f_{33} > 0, \\ f_{12}{}^2 < f_{11}f_{22},\ f_{13}{}^2 < f_{11}f_{33},\ f_{23}{}^2 < f_{22}f_{33}\ . \end{cases} \qquad (20)$$

We get $\eta_1 = \frac{\alpha_2 y}{\alpha_1}$ and $\eta_2 = \frac{\gamma_2}{\gamma_1}\frac{2c - bx_2x_2^*}{bx_2^{*2} + c}$ with $2c > bx_2x_2^*$.

If conditions (15)-(16) hold, then E^* is also globally asymptotically stable.

C. Hopf bifurcation

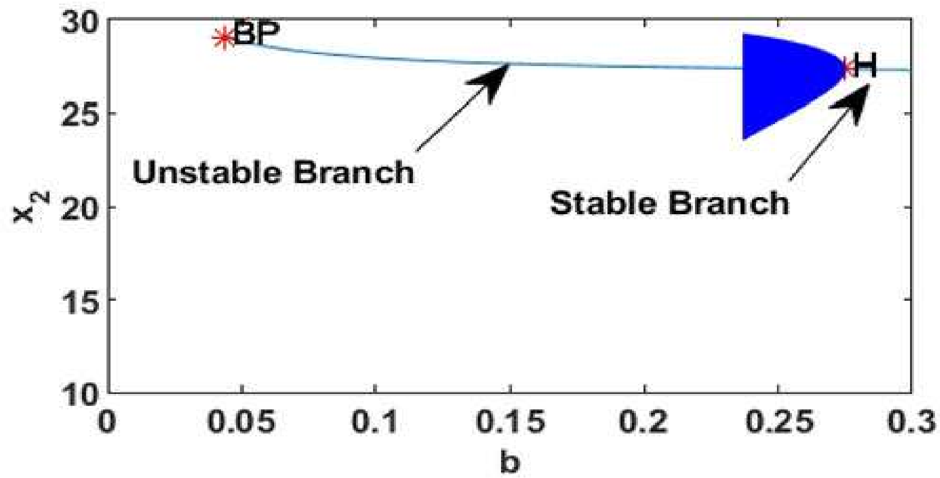
The interior equilibrium point E^*experiences a Hopf bifurcation at a critical value of defense efficiency of prey x_2 i.e., $b = b_h \in (0, \infty)$, following Liu's criterion if

I. $\xi_1(b_h) > 0, \xi_3(b_h) > 0,$

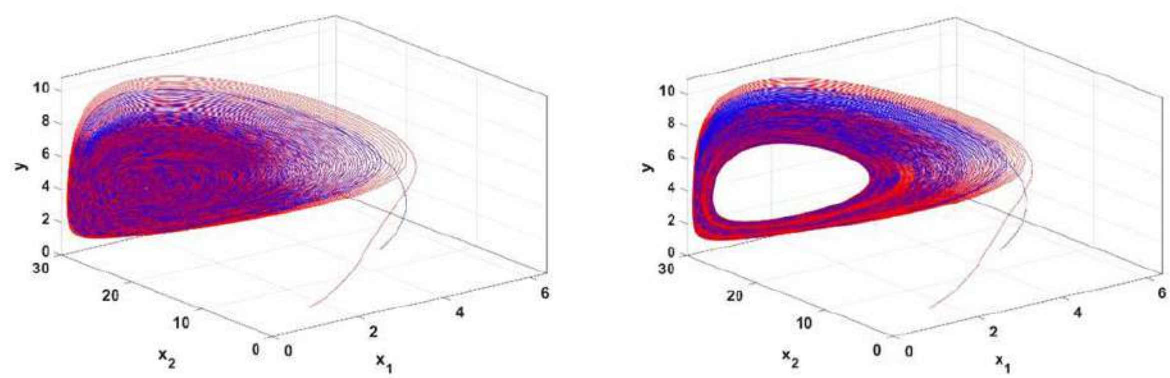
II. $\mathcal{b}(b_h) = \xi_1(b_h)\xi_2(b_h) - \xi_3(b_h) = 0,$

III. $\frac{d}{d\mathcal{b}}(\mathcal{b}(b))|_{b=b_h} \neq 0,$

hold. Here, ξ_i, $i = 1,2,3$ are coefficients of characteristic polynomial of E^* given in (14), also the smooth function of b is defined on the positive real axis. We discuss the appearance of Hopf bifurcation numerically in Sec. IV.



(a)



(b)

Figure 4: Hopf-bifurcation for parameter

COMPUTER SIMULATIONS

In this section, we demonstrate the dynamics of the model system (1)-(3) and signify the control parameters practising computer programming. We have used MATLAB and MATCONT software to affirm the numerically approximate solutions to the model system.

$$\begin{cases} r_1 = 0.5, r_2 = 0.9, k_1 = 30, k_2 = 30, \alpha_1 = 0.056, \\ \alpha_2 = 0.06, \beta_1 = 0.01, \beta_2 = 0.05, \gamma_1 = 0.3, \gamma_2 = 0.15, \\ b = 0.2, c = 3, d = 0.11 \end{cases} \tag{21}$$

We have simulated the parameter set (21) via phase portrait, time series and bifurcation to understand sparse dynamics. This model pursues a rich and multifaceted dynamics. This model system (1)-(3) has the interior equilibrium $E^* = (x_1^* = 1.4141,\ x_2^* = 29.3127,\ y^* = 3.2679)$. The eigenvalues of Jacobian matrix $J(E^*)$ are -0.8570, $-0.0092 + 0.0388i$ and $0.0092 - 0.0388i$. Hence E^* is locally stable. This result is simulated in Figure 1a. When the saturation constant c exceeds values 90, then the equilibrium point E^* becomes globally stable and satisfies the global stability conditions given in (15)-(16).

We have observed that the competition coefficient (β_2) of prey x_2 as the control parameter after a vivid numerical simulation. Parameter β_2 has a strong dynamical effect on the model system. It changes the system dynamics from stability to chaotic given in Figure 1.

We evaluate that for $\beta_2 = 0.01$, the model is stable (see Figure 1a) and for $\beta_2 = 0.05$, the model shows an oscillating behaviour (see Figure 1b). Increasing the parameter value, the system starts showing chaotic behaviour and finally for $\beta_2 = 0.08, \gamma_1 = 0.8, c = 9$, we can see a butterfly-like structure, also known as a strange attractor (see Figure 1c). Time evolutions for this strange attractor yields in Figure 2 and it is observed that chaos has a sensitive dependence on the initial conditions. To ensure these dynamical changes, we have also plotted bifurcation dynamics regarding β_2. Figure 3 shows the exact dynamical changes as we have seen in phase portrait. The system switches stability at $\beta_2 = 0.0495$ and Hopf-bifurcation occurs.

Figure 4 is generated to showcase the Hopf-bifurcation analysis for the given system. The system (1)-(3) goes through a Hopf bifurcation for the threshold value $b = b_h = 0.2747$ at Hopf point (H) at the critical point (1.505, 27.3184, 3.6023) with Lyapunov coefficient -1.672809e-05. (see Figure 4b). We have plotted the stable limit cycle for $b = 0.25$ surrendering an unstable equilibrium point (0.3485, 28.2943, 9.6211). Also, we get stable equilibrium point at $b = 0.3$ (see Figure 4b). Theoretically this indicates the appearance of supercritical Hopf-bifurcation for the given system. Additionally, from simulations, we observe a negative first Lyapunov coefficient, which further confirms the direction of the Hopf bifurcation. For $b = 0.0434,$ prey x_1 extinct and the critical point converted to Branch point (BP) at (0.0000, 29.0561, 3.7399). The system exposes unstable behaviour for $b < 0.27$ and becomes steady afterwards.

CONCLUSION

The article discusses the modeling and dynamics of a three-species prey-predator model with two prey and a predator. For example, rabbits are solitary prey, while rhinos and hippopotamuses employ defense tactics to escape predation. All these species are herbivorous and share the same food and habitat, so they experience interspecific competition. Lions are the common predator here.

We discuss the dynamics of the model at interior equilibrium point E^* with conditions and restrictions. We have observed that the competition coefficient (β_2) among prey species x_2 is the key parameter and it enacts the complexity and dynamical stability of the model system (1)-(3). It is observed that prey defense mechanisms have a stronger influence on the persistence of any species. For our model, we observe that due to the prey defense mechanism, prey x_2 persists in nature for a longer time, whereas prey x_1 fails to sustain against the predator. Inter-specific competition and defense efficiency b have opposite effects on prey x_2. Still, for both parameters, the population density of prey x_2 continuously remains higher. Figures 3 and 4 magnify this phenomenon wonderfully. Finally, we have concluded that the defense mechanism sustains the endurance of prey species, and interspecific competition can influence the stability of the model.

REFERENCES

[1] J. D. Parish, S. B. Saila, Interspecific competition, predation and species diversity, J. Theoret. Biol., 27, 207-220, 1970.

[2] D. Kesh, A.K. Sarkar, A.B. Roy, Persistence of two prey-one predator system with ratio-dependent predator influence, Math. Methods Appl. Sci., 23, 347-356, 2000.

[3] J.P. Tripathi, D. Jana, V. Tiwari, A Beddington–DeAngelis type one-predator two-prey competitive system with help. Nonlinear Dyn., 94, 553-573, 2018.

[4] D. Jana, A. Banerjee, G.P. Samanta, Degree of prey-refuges: control the competition among prey and foraging ability of predator, Chaos Solitons Fractals, 104, 350-362, 2017.

[5] T.K. Kar, A. Batabyal, Persistence and stability of a two prey one predator system, International Journal of Engineering, Science and Technology, vol. 2, no. 2, pp. 174-190, 2010.

[6] J. Sokol, J.A. Howell, Kinetics of phenol oxidation by washed cell, Biotecnhnol. Bioeng., 23, 203-249, 1980.

[7] P. Mishra, S.N. Raw, Dynamical complexities in a predator-prey system involving teams of two prey and one predator, J. Appl. Math. Comput. 61, 1-24, 2019.

[8] S.N. Raw, P. Mishra, B. Tiwari, Mathematical Study About a Predator-Prey Model with Anti-predator Behavior, Int. J. Appl. Comput. Math 6, 68, 2020.

[9] B. Dubey, R.K. Upadhyay, Persistence and extinction of one-prey and two-predators system. Nonlinear Anal Model Contr 9(4), 307-329, 2004.