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ON THE MASS PROPERTIES OF GEOMETRICALLY EXACT BEAMS WITH ARBITRARY SECTIONS

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ABSTRACT
Geometrically Exact Beam Theory is a general and powerful framework for writing the equations of motion of beams that can be subject to large motions, including large rotations in three dimensional space. Within this formalism, beams are described as 1D Cosserat continua, typically using a curvilinear abscissa s to parametrize the displacement of the centerline r=r(s)∈R^3 and the rotation of the section R=R(s)∈SO(R,3). We implement these functions via B-spline interpolation between a discrete number of control points. This formulation, among other things, requires the definition of the mass properties of the beam section(s). In a section at abscissa s we assume the most general case of a center-of-mass offset c=c(s)∈R^2, a mass per unit length, a tensor of inertia-per-unit-length, possibly rotated respect to section reference. The discretization of the equations of motion, with mass lumping for efficiency, leads to a second order ODE (or DAE) where the mass matrix is block diagonal, with 6x6 mass blocks per each node. For generic angular velocity of the section, gyroscopic torques may arise. We recover also the linearization of the inertial forces, obtaining 6x6 block-diagonal gyroscopic-damping matrices and 6x6 gyroscopic-stiffness matrices, whose formulation is quite intricate yet useful for implicit timesteppers or modal analysis.

Keywords: beams, inertia, gyroscopic damping, lumped masses.

INTRODUCTION
The Geometrically Exact Beam Theory (GEBT) originates from early works of Cosserat, that theorized continua that feature rotation information at each point [1]. Beams with arbitrary large deformation can be considered 1-D Cosserat continua, where at an abscissa s one has a position $\boldsymbol{r} = \boldsymbol{r}(s) \in \mathrm{R}^3$ and a rotation of the section $\boldsymbol{R} = \boldsymbol{R}(s) \in \mathrm{SO}(\mathrm{R}, 3)$.

Seminal works on the practical use of this theory in the field of computational mechanics were published decades later by Simo and coworkers [2]. Since then, many researchers have worked on the topic. Among the many sources in literature, we just mention the references [3-9] for additional details.

One of the goals of GEBT is the simulation of non-linear dynamics of beams subject to large rotations and displacements, such as in the case of blades of helicopters and of wind turbines. Among the many applications we also mention in passing the cases of the simulation of cables and cable looms when studying the electrical layouts of vehicles, where beams are subject to arbitrary large deformations, or the case of wires in mooring lines or in winches of nautical devices. The cases of the straight Eulero-Bernoulli beam or the case of the Timoshenko-Ehrenfest beam can be considered simpler bidimensional sub-cases of the GEBT.

When dealing with dynamics, inertial loads must be modeled: these amount to distributed loads that depend on how the mass is distributed along the beam. In this paper we do not make the assumption of uniform density, neither we assume a constant section shape.

Once the strong form of the dynamic equilibrium is discretized, one has two possible outcomes, as always in finite element theory: either one obtains a full per-element mass matrix, often called *consistent mass matrix*, or one obtains per-node *lumped mass matrices*. The former requires more computations, leads to more densely populated system matrices, but in terms of accuracy it offers only marginal improvements over the lumped mass model; for this reason in this work we will focus on the lumped mass approach.

In the following sections we will present a full treatment of inertial forces for generic beam sections. Moreover, we will present a systematic linearization of such forces, showing that the linearization produces a gyroscopic tangent damping matrix and a gyroscopic tangent stiffness matrix, the latter almost always neglected in literature. These tangent matrices can be useful for two reasons: either for implementing Newton iterations in implicit solvers, or to compute the complex eigenvalues of the system in view of a modal reduction or for stability analysis, for example.

RIGID FRAME KINEMATICS

We recall basic facts on the kinematics of rigid frames in three dimensional space.

A. Rotations

We use $\boldsymbol{R} \in \mathrm{SO}(\mathbb{R},3)$ to denote a rotation tensor from the special orthogonal Lie group. The infinitesimal rotation, also virtual variation of the rotation tensor, is denoted $\boldsymbol{\theta}^{\delta}$, with subscript *a* for absolute basis or *l* for local (intrinsic, material) basis, as in

$$\tilde{\boldsymbol{\theta}}_a^{\delta} = \delta \boldsymbol{R} \boldsymbol{R}^T$$

where the tilde operator builds a skew symmetric matrix as in

$$\tilde{\boldsymbol{\theta}} = \begin{bmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{bmatrix}$$

Thus $\tilde{\boldsymbol{\theta}}^{\delta} \in \mathrm{so}(\mathbb{R},3)$, the Lie algebra of the $\mathrm{SO}(\mathbb{R},3)$ Lie group. Some properties that will be useful later are:

$$\begin{aligned} \tilde{\boldsymbol{a}}^T &= -\tilde{\boldsymbol{a}} \\ \tilde{\boldsymbol{a}}\boldsymbol{b} &= \boldsymbol{a} \times \boldsymbol{b} \\ \tilde{\boldsymbol{a}}\boldsymbol{b} &= -\tilde{\boldsymbol{b}}\boldsymbol{a} = -\boldsymbol{b} \times \boldsymbol{a} \\ \widetilde{\tilde{\boldsymbol{a}}\boldsymbol{b}} &= \tilde{\boldsymbol{a}}\tilde{\boldsymbol{b}} - \tilde{\boldsymbol{b}}\tilde{\boldsymbol{a}} \\ \tilde{\boldsymbol{a}}\boldsymbol{B} &= (B^T \tilde{\boldsymbol{a}}^T)^T = -(\boldsymbol{B}^T \tilde{\boldsymbol{a}})^T \\ \widetilde{\boldsymbol{R}\boldsymbol{a}} &= \boldsymbol{R}\tilde{\boldsymbol{a}}\boldsymbol{R}^T \end{aligned}$$

Similarly, for angular velocity:

$$\tilde{\boldsymbol{\omega}}_l = \boldsymbol{R}^T \dot{\boldsymbol{R}}$$

At this point it is not difficult to recover the following results:

$$\begin{aligned} \tilde{\boldsymbol{\omega}}_l &= \boldsymbol{R}^T \tilde{\boldsymbol{\omega}}_a \boldsymbol{R}, \\ \boldsymbol{\omega}_l &= \boldsymbol{R}^T \boldsymbol{\omega}_a \\ \tilde{\boldsymbol{\omega}}_a &= \boldsymbol{R} \tilde{\boldsymbol{\omega}}_l \boldsymbol{R}^T \\ \boldsymbol{\omega}_a &= \boldsymbol{R} \boldsymbol{\omega}_l \end{aligned}$$

and

$$\begin{aligned} \tilde{\boldsymbol{\theta}}_l^{\delta} &= \boldsymbol{R}^T \tilde{\boldsymbol{\theta}}_a^{\delta} \boldsymbol{R}, \quad & \boldsymbol{\theta}_l^{\delta} &= \boldsymbol{R}^T \boldsymbol{\theta}_a^{\delta} \\ \tilde{\boldsymbol{\theta}}_a^{\delta} &= \boldsymbol{R} \tilde{\boldsymbol{\theta}}_l^{\delta} \boldsymbol{R}^T, \quad & \boldsymbol{\theta}_a^{\delta} &= \boldsymbol{R} \boldsymbol{\theta}_l^{\delta} \end{aligned}$$

The variations of the angular velocity vectors are not intuitive at all, being:

$$\delta \boldsymbol{\omega}_a = \dot{\boldsymbol{\theta}}_a^{\delta} - \tilde{\boldsymbol{\omega}}_a \boldsymbol{\theta}_a^{\delta},$$

$$\delta \boldsymbol{\omega}_l = \dot{\boldsymbol{\theta}}_l^{\delta} + \tilde{\boldsymbol{\omega}}_l \boldsymbol{\theta}_l^{\delta},$$

B. Point kinematics in relative motion

The absolute position of a point attached to a rigid frame B that has a rotation $\boldsymbol{R}_B$ respect to the absolute frame, and a translation $\boldsymbol{r}_B$ respect to the origin of the absolute frame, is:

$$\boldsymbol{p} = \boldsymbol{r}_B + \boldsymbol{R}_B \boldsymbol{p}_{(B)}$$

where we use the subscript (B) to denote a vector expressed in a basis B, or we do not use that subscript if the basis is absolute, or obvious.

The speed of the point can be obtained via differentiation of the above expression and remembering the already listed properties of the Lie algebra $so(\boldsymbol{R},3)$, obtaining:

$$\dot{\boldsymbol{p}} = \dot{\boldsymbol{r}}_B + \tilde{\omega}_a \boldsymbol{R}_B \boldsymbol{p}_{(B)} + \boldsymbol{R}_B \dot{\boldsymbol{p}}_{(B)}$$

Taking another differentiation, one gets also the expression of the absolute acceleration of the point:

$$\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{r}}_B + \dot{\tilde{\omega}}_a \boldsymbol{R}_B \boldsymbol{p}_{(B)} + \tilde{\omega}_a \tilde{\omega}_a \boldsymbol{R}_B \boldsymbol{p}_{(B)} + 2\tilde{\omega}_a \boldsymbol{R}_B \dot{\boldsymbol{p}}_{(B)} + \boldsymbol{R}_B \ddot{\boldsymbol{p}}_{(B)}$$

INERTIAL FORCES

If a mass density ρ is associated to all points that move attached to the frame B, one can compute the total effect of the inertia force simply using Newtonian mechanics. In fact, considering a rigid volume V that follows the frame motion in the absolute space, one has:

$$F_I = -\int_V \rho \ddot{\boldsymbol{p}} dV$$

$$= -\int_V \rho \left(\ddot{\boldsymbol{r}}_B + \dot{\tilde{\omega}}_a \boldsymbol{R}_B \boldsymbol{p}_{(B)} + \tilde{\omega}_a \tilde{\omega}_a \boldsymbol{p}_{(B)} \right) dV$$

that expands into

$$-\left(\int_V \rho dV\right) \ddot{\boldsymbol{r}}_B - \dot{\tilde{\omega}}_a \boldsymbol{R}_B \left(\int_V \rho \boldsymbol{p}_{(B)} dV\right) - \tilde{\omega}_a \tilde{\omega}_a \boldsymbol{R}_B \left(\int_V \rho \boldsymbol{p}_{(B)} dV\right)$$

and finally,

$$F_I = -m\ddot{\boldsymbol{r}}_B + \boldsymbol{R}_B \tilde{\boldsymbol{s}}_l \boldsymbol{R}_B^T \dot{\omega}_a - \tilde{\omega}_a \tilde{\omega}_a \boldsymbol{R}_B \boldsymbol{s}_l$$

where we define the total mass m and $\boldsymbol{s}_l$, the static moment of masses, as

$$m = \int_V \rho dV$$

$$\boldsymbol{s}_l = \int_V \rho \boldsymbol{p}_{(B)} dV$$

Note that the position of the barycenter (center of mass) is a vector $\boldsymbol{c}_l$ for whom it holds:

$$\boldsymbol{s}_l = m\boldsymbol{c}_l$$

Along the same lines, it is possible to express the moment of the forces of inertia as

$$M_I = -\int_V \rho(\boldsymbol{R}_B\boldsymbol{p}_{(B)}) \times \ddot{\boldsymbol{p}}dV$$

$$= -\int_V \rho(\boldsymbol{R}_B\boldsymbol{p}_{(B)}) \times (\ddot{\boldsymbol{r}}_B + \dot{\tilde{\omega}}_a\boldsymbol{R}_B\boldsymbol{p}_{(B)} + \tilde{\omega}_a\tilde{\omega}_a\boldsymbol{R}_B\boldsymbol{p}_{(B)})dV$$

$$= -\int_V \rho(\boldsymbol{R}_B\boldsymbol{p}_{(B)}) \times \ddot{\boldsymbol{r}}_B dV - \int_V \rho(\boldsymbol{R}_B\boldsymbol{p}_{(B)}) \times (\dot{\tilde{\omega}}_a\boldsymbol{R}_B\boldsymbol{p}_{(B)})dV$$

$$- \int_V \rho(\boldsymbol{R}_B\boldsymbol{p}_{(B)}) \times (\tilde{\omega}_a\tilde{\omega}_a\boldsymbol{R}_B\boldsymbol{p}_{(B)})dV$$

$$= -\int_V \rho\boldsymbol{R}_B\tilde{\boldsymbol{p}}_{(B)}\ddot{\boldsymbol{r}}_B dV - \int_V \rho\boldsymbol{R}_B\tilde{\boldsymbol{p}}_{(B)}\left(\boldsymbol{R}_B\tilde{\boldsymbol{p}}_{(B)}\right)^T \dot{\boldsymbol{\omega}}_a dV$$

$$- \int_V \rho\tilde{\boldsymbol{\omega}}_a\boldsymbol{R}_B\tilde{\boldsymbol{p}}_{(B)}\left(\boldsymbol{R}_B\tilde{\boldsymbol{p}}_{(B)}\right)^T \boldsymbol{\omega}_a dV$$

$$= -\int_V \rho\boldsymbol{R}_B\tilde{\boldsymbol{p}}_{(B)}\boldsymbol{R}_B^T\ddot{\boldsymbol{r}}_B dV - \int_V \rho\boldsymbol{R}_B\tilde{\boldsymbol{p}}_{(B)}\boldsymbol{R}_B^T\boldsymbol{R}_B\tilde{\boldsymbol{p}}_{(B)}{}^T\boldsymbol{R}_B^T\dot{\boldsymbol{\omega}}_a dV$$

$$- \int_V \rho\tilde{\boldsymbol{\omega}}_a\boldsymbol{R}_B\tilde{\boldsymbol{p}}_{(B)}\boldsymbol{R}_B^T\boldsymbol{R}_B\tilde{\boldsymbol{p}}_{(B)}{}^T\boldsymbol{R}_B^T\ddot{\boldsymbol{r}}_B\boldsymbol{\omega}_a dV$$

$$= -\boldsymbol{R}_B\tilde{\boldsymbol{s}}_l\boldsymbol{R}_B^T\ddot{\boldsymbol{r}}_B - \boldsymbol{R}_B\boldsymbol{J}_l\boldsymbol{R}_B^T\dot{\omega}_a - \tilde{\omega}_a\boldsymbol{R}_B\boldsymbol{J}_l\boldsymbol{R}_B^T\boldsymbol{\omega}_a$$

Defining the mass moment of inertia

$$J_l = \int_V \rho\tilde{\boldsymbol{p}}_{(B)}\tilde{\boldsymbol{p}}_{(B)}{}^T dV$$

we have:

$$M_I = -\boldsymbol{R}_B\tilde{\boldsymbol{s}}_l\boldsymbol{R}_B^T\boldsymbol{r}_B - \boldsymbol{R}_B\boldsymbol{J}_l\boldsymbol{R}_B^T\dot{\omega}_a - \tilde{\omega}_a\boldsymbol{R}_B\boldsymbol{J}_l\boldsymbol{R}_B^T\boldsymbol{\omega}_a$$

This leads to the Newton-Euler equations for a rigid body subject to a force ***F*** and a torque ***M,*** where the force is assumed applied to the center of mass and both force and torque are assumed expressed in the absolute reference:

$$m\ddot{\boldsymbol{r}}_B - \boldsymbol{R}_B\tilde{\boldsymbol{s}}_l\boldsymbol{R}_B^T\dot{\boldsymbol{\omega}}_a + \tilde{\omega}_a\tilde{\omega}_a\boldsymbol{R}_B\boldsymbol{s}_l = \boldsymbol{F}$$

$$\boldsymbol{R}_B\tilde{\boldsymbol{s}}_l\boldsymbol{R}_B^T\ddot{\boldsymbol{r}}_B + \boldsymbol{J}_a\dot{\omega}_a + \tilde{\omega}_a\boldsymbol{J}_a\boldsymbol{\omega}_a = \boldsymbol{M}$$

where we simplified the notation by using the transformation of the inertia tensor into the absolute frame $\boldsymbol{J}_a = \boldsymbol{R}_B\boldsymbol{J}_l\boldsymbol{R}_B^T$.

Note that if the center of mass corresponds to the origin of the *B* frame, one has the form that is often available in literature, for simple shapes, and where one has the contribution of the gyroscopic torque $\tilde{\omega}_a\boldsymbol{J}_a\boldsymbol{\omega}_a$ anyway:

$$m\ddot{\boldsymbol{r}}_B = \boldsymbol{F}$$

$$\boldsymbol{J}_a\dot{\omega}_a + \tilde{\omega}_a\boldsymbol{J}_a\boldsymbol{\omega}_a = \boldsymbol{M}$$

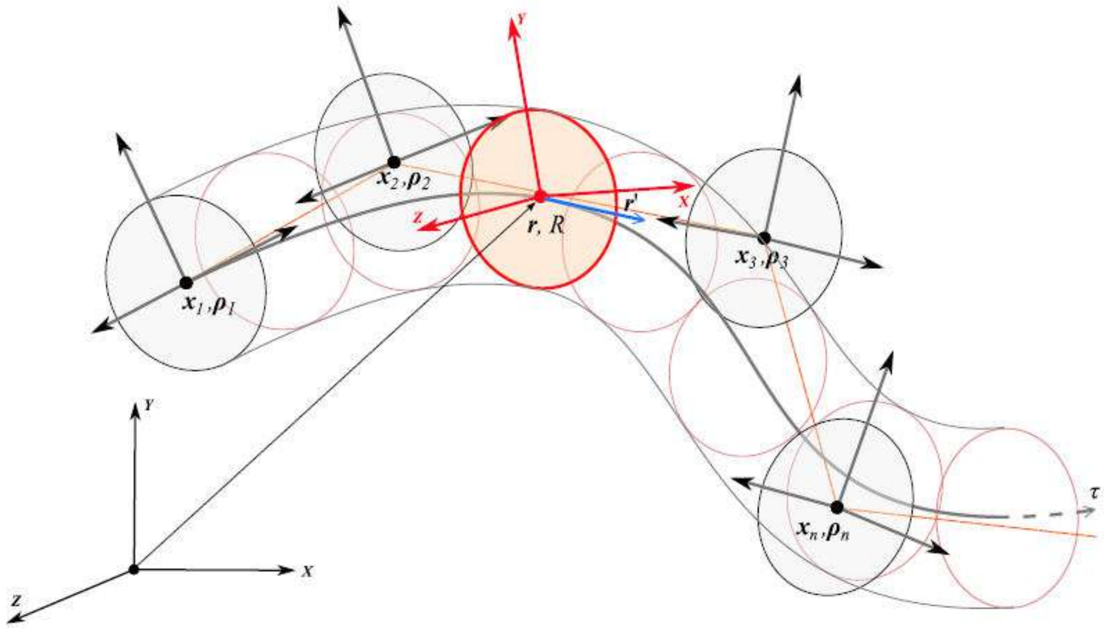

Figure 1: geometrically exact beam as a B-spline interpolation of centerline position and section rotation.

We can extend the same reasoning to beam sections, where we will express the inertia properties on a per-unit-length basis, namely:

- $\underline{m}$ is the mass per unit length
- $\underline{\boldsymbol{s}}_l$ is the static moment of masses per unit length
- $\underline{\boldsymbol{J}}_a$ is the section moment of inertia per unit length

In particular, given a section at abscissa *s* with a density distribution $\underline{\rho} = \underline{\rho}(s, y, z)$ where *y* an z are orthogonal unit vectors on the section surface area *S*, one has to compute:

$$\underline{m} = \int_S \underline{\rho} dS$$

$$\underline{\boldsymbol{s}}_l = \int_S \underline{\rho} \boldsymbol{p}_{(B)} dS$$

or $\underline{\boldsymbol{s}}_l = \underline{m\boldsymbol{c}}_l$

$$\underline{\boldsymbol{J}}_l = \int_S \underline{\rho} \tilde{\boldsymbol{p}}_{(B)} \tilde{\boldsymbol{p}}_{(B)}{}^T dS$$

$$\underline{\boldsymbol{J}}_a = \boldsymbol{R}_B \underline{\boldsymbol{J}}_l \boldsymbol{R}_B^T$$

These values, in the material basis, are invariants that can be precomputed and stored at the beginning of the simulation.

Exceptions are sections that change during large deformations, like in isochoric models where the section is squeezed because of extreme curvatures, or in functionally-graded sections whose shape change continuously along the abscissa (although, in the latter case, one can still precompute the inertia invariants at the Gauss points, or in general at the countable points that are needed later when discretizing the problem at a finite element level).

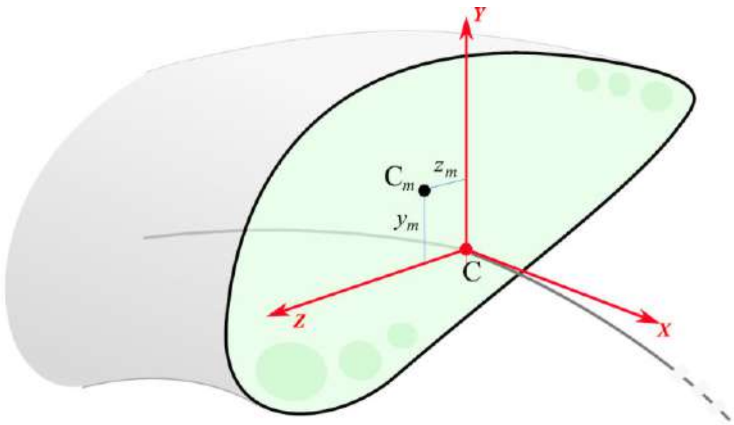

Figure 2: example of a generic section, with center of mass C being offset from the centerline reference of the section.

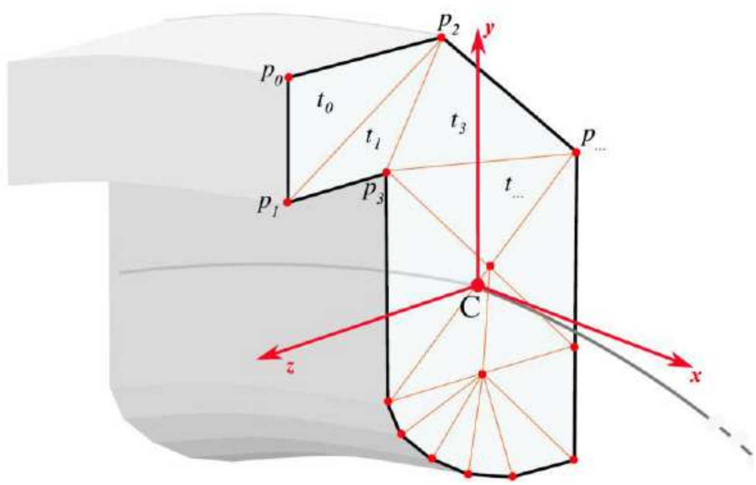

Figure 3: meshing of the section with triangles, to allow an automatic computation of integrals for the inertia invariants.

In case the section has a complex shape, with cavities, slots and inserts of different densities, the surface integrals may be less immediate than in the case for whom one can use analytical results. Anyway, for complex shapes we simply operate the integral on a triangulated mesh of the cross section, or sub-components of it, because the integrals of the formulas above have a closed form expression for single triangles, and these triangle integrals can be just summed (see Fig.3).

The strong form of the dynamic equilibrium of a section of the beam is then

$$\underline{m}\ddot{\boldsymbol{r}}_B - \boldsymbol{R}_B\tilde{\underline{\boldsymbol{s}}}_l\boldsymbol{R}_B^T\dot{\boldsymbol{\omega}}_a + \tilde{\omega}_a\tilde{\omega}_a\boldsymbol{R}_B\underline{\boldsymbol{s}}_l = \boldsymbol{n}_a{}' + \underline{\boldsymbol{n}}_a \tag{1.1}$$

$$\boldsymbol{R}_B\tilde{\underline{\boldsymbol{s}}}_l\boldsymbol{R}_B^T\ddot{\boldsymbol{r}}_B + \underline{\boldsymbol{J}}_a\dot{\omega}_a + \tilde{\omega}_a\underline{\boldsymbol{J}}_a\omega_a = \boldsymbol{m}_a{}' + \boldsymbol{r}_B{}' \times \boldsymbol{n}_a + \underline{\boldsymbol{m}}_a \tag{1.2}$$

Here, $\boldsymbol{n}_a$ and $\boldsymbol{m}_a{}'$ are generalized shear/tension stresses and curvature stresses, that depend on the generalized strains $\boldsymbol{\epsilon}_a$ and curvature $\boldsymbol{\kappa}_a$ of the geometrically exact beam at the section, and on the chosen constitutive law. The constitutive law is often expressed in the local basis as

$$\{\boldsymbol{\epsilon}_l, \boldsymbol{\kappa}_l\} \in R^6 \rightarrow \{\boldsymbol{n}_l, \boldsymbol{m}_l\} \in R^6$$

And, for linear elastic cases (like in the special cases of Timoshenko-Ehrenfest beam model or in the Euler-Bernoulli beam model) this becomes a linear mapping with a 6x6 symmetric matrix with *E,F,G* blocks:

$$\begin{pmatrix} \boldsymbol{n}_l \\ \boldsymbol{m}_l \end{pmatrix} = \begin{bmatrix} E & F^T \\ F & G \end{bmatrix} \begin{pmatrix} \epsilon_l \\ \kappa_l \end{pmatrix}$$

Also, $\underline{\boldsymbol{n}}_a$ and $\underline{\boldsymbol{m}}_a$ are per-unit-length distributed forces and distributed torques, if any (the gravitational load can be an example)

In case of simple beams, with uniform density and simple symmetric cross section (such as T beams, I beams, rectangular sections etc.) one has $\underline{\boldsymbol{s}}_l = 0$ and therefore the equilibrium becomes simpler:

$$\underline{m}\boldsymbol{r}_B = \boldsymbol{n}_a{}' + \underline{\boldsymbol{n}}_a$$

$$\underline{\boldsymbol{J}}_a\dot{\omega}_a + \tilde{\omega}_a\underline{\boldsymbol{J}}_a\omega_a = \boldsymbol{m}_a{}' + \boldsymbol{r}_B{}' \times \boldsymbol{n}_a + \underline{\boldsymbol{m}}_a$$

Either way, one will use the equilibrium equations to obtain a discretized form of the problem (for instance, using a Galërkin method for finite elements, or a collocation method, but this is not part of the discussion of these pages).

SYSTEM-LEVEL MASS MATRICES

The complete expression of the equations of motion of a multibody system involving beams and other components (other finite elements, rigid bodies, constraints, motors, etc.) is a Differential Algebraic Equation (DAE) of the form:

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}_q(\boldsymbol{q},t)^T\boldsymbol{\gamma} &= \boldsymbol{f}(\boldsymbol{q},\dot{\boldsymbol{q}},t) \\ \boldsymbol{C}(\boldsymbol{q},t) &= 0 \end{aligned} \qquad (2)$$

where we denote constraints as a vector of nonlinear algebraic equations $\boldsymbol{C}(\boldsymbol{q},t)$, their jacobians are represented by the $\boldsymbol{C}_q$ matrix, and constraint multipliers with $\boldsymbol{\gamma}$ [10]. The term $\boldsymbol{f}(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ contains all the applied forces, including gyroscopic terms. We assume that the configuration of the system is expressed by coordinates $\boldsymbol{q}$.

More in general, using quasi-velocities $\boldsymbol{v}$ and accelerations $\boldsymbol{a}$, with the linear mapping $\dot{\boldsymbol{q}} = \Gamma(\boldsymbol{q})\boldsymbol{v}$ one has a more intuitive form of DAE:

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} - \boldsymbol{C}_v(\boldsymbol{q},t)^T\boldsymbol{\lambda} &= \boldsymbol{f}(\boldsymbol{q},\boldsymbol{v},t) \\ \boldsymbol{C}(\boldsymbol{q},t) &= 0 \\ \dot{\boldsymbol{q}} &= \Gamma(\boldsymbol{q})\boldsymbol{v} \end{aligned} \qquad (3)$$

where, for instance, each beam node inserts $\dot{\boldsymbol{\omega}}_a$ in $\boldsymbol{a}$, inserts $\boldsymbol{\omega}_a$ in $\boldsymbol{v}$, and where the rotation of the node is inserted in $\boldsymbol{q}$ through some convenient parametrization, for example quaternions or three Tait angles.

In detail, looking at (1.1) and (1.2) it is easy to see that each i-th beam node adds the following 6x6 block on the diagonal of the global $\boldsymbol{M}$ matrix:

$$\boldsymbol{M}_i = \begin{bmatrix} \underline{m}\boldsymbol{I} & \tilde{\underline{\boldsymbol{s}}}_a^T \\ \tilde{\underline{\boldsymbol{s}}}_a & \underline{\boldsymbol{J}}_a \end{bmatrix}_i l_i$$

where the l value has the dimension of a length and it works like a weight that tells how much mass is lumped to the node. For example, in a simple straight finite element of length L with two end nodes, one can set $l_i = \frac{1}{2}L$; the total mass would be preserved but the rotational inertia would be approximated.

A positive remark about the lumped mass approach is that, despite the approximation of the real mass properties, the error tends to zero for an increasingly dense discretization of the beam. Moreover various Authors have experienced that computing the exact mass stiffness of a finite element (the so called *consistent* mass matrix) does not provide relevant improvements on the accuracy respect to the lumped mass matrix, but at the same time it impacts a lot on the performance of the code, because the lumped mass assumption guarantees that the mass matrix is block diagonal and very sparse: this is a significant aspect when using explicit integrators that, in case of small mass blocks on the diagonal, could even operate with a precomputed inverse of the mass matrix.

Similarly, for lumped masses at beam nodes, each i-th node adds the following 1x6 vector of centrifugal/gyroscopic forces and torques to the $\boldsymbol{f}$ global forces in (3) :

$$\boldsymbol{f}_i = \begin{Bmatrix} -\tilde{\omega}_a\tilde{\omega}_a \boldsymbol{R}_B \underline{\boldsymbol{s}}_l \\ \tilde{\omega}_a \underline{\boldsymbol{J}}_a \boldsymbol{\omega}_a \end{Bmatrix}_i l_i$$

LINEARIZATION OF THE DAE

The linearization of the DAE (3), that is in the case of quasi-velocities, leads to:

$$\boldsymbol{M}_a(\boldsymbol{q})\delta\boldsymbol{a} + \boldsymbol{D}_v(\boldsymbol{q},\boldsymbol{v},t)\delta\boldsymbol{v} + \boldsymbol{K}_{w^\delta}(\boldsymbol{q},\boldsymbol{v},\boldsymbol{a},\boldsymbol{\lambda},t)\boldsymbol{w}^\delta$$

$$+\boldsymbol{C}_v(\boldsymbol{q},t)^T\delta\boldsymbol{\gamma} = 0$$

$$\boldsymbol{C}_v(\boldsymbol{q},t)\delta\boldsymbol{v} = 0$$

$$\dot{\boldsymbol{q}} = \boldsymbol{\Gamma}(\boldsymbol{q})\boldsymbol{v}$$

$$\boldsymbol{q} = \boldsymbol{\Gamma}(\boldsymbol{q})\boldsymbol{w}^\delta$$

where, for instance, each beam node inserts the virtual rotation $\boldsymbol{\theta}_a^\delta$ in $\boldsymbol{w}^\delta$, where $\boldsymbol{w}^\delta$ is the global vector of virtual increments of rotations and positions.

Here the mass matrix $\boldsymbol{M}_a$ is the same $\boldsymbol{M}$ used in (3). However the expressions of the tangent damping matrix $\boldsymbol{D}_v(\boldsymbol{q},\boldsymbol{v},t)$ and of the tangent stiffness matrix $\boldsymbol{K}_{w^\delta}(\boldsymbol{q},\boldsymbol{v},\boldsymbol{a},\boldsymbol{\lambda},t)$ are not as trivial as for $\boldsymbol{M}_a(\boldsymbol{q})$, because they include the quite complex linearization of inertial forces and torques, that is, the linearization of the left hand side of (1.1) and (1.2) that contain gyroscopic/centrifugal terms.

Assuming a linearization about $\boldsymbol{\theta}_a^\delta, \delta\boldsymbol{\omega}_a, \delta\dot{\boldsymbol{\omega}}_a$, and focusing only on the node inertial effects, each *i*-th node adds the following 6x6 block on the diagonal of the global $\boldsymbol{D}_v(\boldsymbol{q},\boldsymbol{v},t)$ matrix.

The expression of that term requires lengthy algebraic manipulations, here we report only the results in sake of compactness:

$$\boldsymbol{D}_i = \begin{bmatrix} 0 & \underline{m}\tilde{\boldsymbol{\omega}}_a\tilde{\boldsymbol{c}}_a^T + \underline{m}[\widetilde{\tilde{\boldsymbol{\omega}}_a\boldsymbol{c}_a}] \\ 0 & \tilde{\omega}_a\underline{\boldsymbol{J}}_a - \underline{\boldsymbol{J}}_a\tilde{\boldsymbol{\omega}}_a \end{bmatrix}_i l_i$$

Similarly, again for a linearization about $\boldsymbol{\theta}_a^\delta, \delta\boldsymbol{\omega}_a, \delta\dot{\boldsymbol{\omega}}_a$, each *i*-th node adds the following 6x6 block on the diagonal of the global $\boldsymbol{K}_{w^\delta}(\boldsymbol{q},\boldsymbol{v},\boldsymbol{a},\boldsymbol{\lambda},t)$ matrix:

$$\boldsymbol{K}_i = \begin{bmatrix} 0 & \underline{m}\dot{\tilde{\omega}}_a\tilde{\boldsymbol{c}}_a^T + \underline{m}\tilde{\omega}_a\tilde{\omega}_a\tilde{\boldsymbol{c}}_a^T \\ 0 & \underline{m}\tilde{\boldsymbol{c}}_a\ddot{\tilde{\boldsymbol{r}}}_a + \underline{\boldsymbol{J}}_a\dot{\tilde{\omega}}_a - \underline{\boldsymbol{J}}_a\dot{\boldsymbol{\omega}}_a + \tilde{\omega}_a(\underline{\boldsymbol{J}}_a\tilde{\omega}_a - \underline{\boldsymbol{J}}_a\tilde{\boldsymbol{\omega}}_a) \end{bmatrix}_i l_i$$

The effect of $\boldsymbol{D}_i$ and $\boldsymbol{K}_i$ matrices is not different from the effect of damping and stiffness matrices caused by structural damping, structural stiffness, dampers and springs. As such, they affect the natural frequencies of the system, they affect stability etc.

In rotordynamics literature, a version of the $\boldsymbol{D}_i$ matrix that includes only the lower-right 3x3 part, and assuming a simplification of some terms for a specific orientation of the X axis along the rotation shaft, is cited as *gyroscopic damping matrix*. However the $\boldsymbol{K}_i$ is most often neglected in literature. One can see that $\boldsymbol{K}_i$ can be null in some circumstances, but not always.

For the special case where the section centroid corresponds with the center of mass, one has the following gyroscopic damping and stiffness:

$$\boldsymbol{D}_i = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\boldsymbol{\omega}}_a \underline{\boldsymbol{J}}_a - \underline{\boldsymbol{J}}_a \tilde{\boldsymbol{\omega}}_a \end{bmatrix}_i l_i$$

$$\boldsymbol{K}_i = \begin{bmatrix} 0 & 0 \\ 0 & +\underline{\boldsymbol{J}}_a \dot{\tilde{\boldsymbol{\omega}}}_a - \underline{\boldsymbol{J}}_a \tilde{\dot{\boldsymbol{\omega}}}_a + \tilde{\boldsymbol{\omega}}_a (\underline{\boldsymbol{J}}_a \tilde{\boldsymbol{\omega}}_a - \underline{\boldsymbol{J}}_a \tilde{\boldsymbol{\omega}}_a) \end{bmatrix}_i l_i$$

Another thing to consider is that a linearization about $\boldsymbol{\theta}_a^{\delta}, \delta\boldsymbol{\omega}_a, \delta\dot{\boldsymbol{\omega}}_a$ does not produce the same mass, damping and stiffness matrices that we would obtain doing a linearization about $\delta\boldsymbol{q},\ \delta\dot{\boldsymbol{q}},\ \delta\ddot{\boldsymbol{q}}$ starting from the DAE in the form (2). Only the second type of linearization is helpful if we want to use those matrices for an eigenvalue analysis of the structure, as in modal problems. To this end, an option could be to start from matrices for the $\boldsymbol{\theta}_a^{\delta}, \delta\boldsymbol{\omega}_a, \delta\dot{\boldsymbol{\omega}}_a$ linearization, as presented in these pages, then assume that $\boldsymbol{\theta}_a^{\delta}, \dot{\boldsymbol{\theta}}_a^{\delta}, \ddot{\boldsymbol{\theta}}_a^{\delta}$ of each node are used in $\delta\boldsymbol{q},\ \delta\dot{\boldsymbol{q}},\ \delta\ddot{\boldsymbol{q}}$, and use the property $\delta\boldsymbol{\omega}_a = \dot{\boldsymbol{\theta}}_a^{\delta} - \tilde{\boldsymbol{\omega}}_a \boldsymbol{\theta}_a^{\delta}$ to obtain the final matrices.

CONCLUSION

We presented an analytical expression of gyroscopic and centrifugal terms for generic sections of geometrically-exact beams, under the assumption of mass lumping. The mass properties are defined via per-unit-length inertia tensors, per-unit-length mass, and center of mass offset respect to the centroid reference. Finally, we also present the tangent damping and tangent stiffness matrices originating from such gyroscopic/centrifugal effects, leading to non-trivial expressions. These tangent matrices can be used in Newton iterations required by implicit integrators, or in eigenproblems originating from modal analysis or from reduced order modeling.

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