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A NEW LIFETIME DISTRIBUTION WITH PROPERTIES AND APPLICATIONS

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ABSTRACT
In the field of distribution theory, various lifetime distributions having different failure rates like increasing, decreasing, bathtub and inverted bathtub etc. have been proposed. Among of them, one of the famous failure and applicable failure rate situation is bathtub type failure rate. In this paper, we are proposing a new two parameter distribution having increasing and bathtub failure rates. We have studied its basic statistical properties and used method of maximum likelihood for the estimation of its parameters. At last, a real dataset has been used to show the application of the proposed model over four other competitive models.

Keywords: Lifetime distribution; Bathtub failure rate; Estimation of parameter; Real data application.

1. INTRODUCTION
In statistics, to analyze various random phenomenon, a lots of lifetime models have been proposed which are applicable in various fields of real life like in economics, finance, social-sciences, science, marketing, engineering and the others fields. The lifetime models are proposed to used according to their failure rates and corresponding nature of their density functions respectively. In distribution theory, the famous failure rates are constant, decreasing, increasing, bathtub or inverted-bathtub.

The situation of bathtub failure rate (BFR) is applicable when the situation of wear-in, constant and wear-out phases occur together. A common and perfectly used example is whole human life. If we analyze the whole human life span, we found that firstly decreasing failure rate situation (wear-in) phase occurs in early ages and after that, life become stable in second phase and which is called the occurrence of constant failure rate and lately, the increasing failure rate situation occurs in late years which is a wear-out phase of life. The bathtub failure rate situation is not only application in medical or ageing phenomenon but also applicable in other filed like engineering.

In literature, for analyzing the bathtub failure rate situation, various authors attempted to develop some useful bathtub models. In this direction, one can see Barlow and Proschan (1975), Hjorth (1980), Rajarshi and Rajarshi (1988) and Mudholkar and Srivastava (1993), Maurya et al. (2017) etc. for further detailed about BFR.

In this paper, we are going to propose a new lifetime model having two parameter and increasing and bathtub failure rate. Here, we used New Logarithmic Transformation (NLT) proposed by Maurya et al. (2018). The NLT transformations is very flexible and does not incorporate any model parameter to the new model. For more details about the NLT based distributions, one can see Maurya and Nadarajah (2021), Maurya (2021) and Rai et al. (2022), etc.. The CDF of NLT method is given as:

$$F(x) = \frac{\log\left(1+G(x)\right)}{\log 2}; x > 0.$$

(1)

Here G(x) is the base cumulative distribution function (CDF) of the distribution. Here we have considered two parameter Power Function distribution as the baseline distribution. The CDF of the considered model is $G(x)=(x/k)^{\alpha}$; $0 \le x \le k$ & $\alpha \geq 0$. Then the CDF of the proposed model is

$$F(x) = \frac{\log(1+(x/k)^{\alpha})}{\log 2}; \qquad 0 \le x \le k; k; \ \alpha \ge 0.$$

(2)

And the corresponding proposed probability density function (PDF) is

$$f(x) = \frac{\alpha}{k \log 2} \frac{(x/k)^{(\alpha-1)}}{(1+(x/k)^{\alpha})}; \qquad 0 \le x \le k; \ k; \alpha \ge 0.$$

(3)

The associated hazard rate function is

$$h(x) = \frac{\alpha}{k\,(1+(x/k)^{\alpha})} \frac{(x/k)^{(\alpha-1)}}{[\log 2 - \log(1+(x/k)^{\alpha})]}; \qquad 0 \le x \le k; \ k; \alpha \ge 0.$$

(4)

The rest of the paper is organized as follows: In Section 2, we have discussed the shapes of PDF and failure rate of the model. In Section 3, some basic statistical properties of the proposed model like its mean, mode, variance, quantile function, median, distribution of its r^{th} order statistics, and Shannon entropy have been discussed. Section 4, describes the basic inferential procedure for the estimation of the parameter. Here we have considered method of moments and maximum likelihood estimation method. In Section 5, a real dataset with five other two parameter decreasing hazard rate lifetime distributions have been considered to show the suitability of the proposed model. Finally, the conclusions are summarized in Section 6.

2. SHAPE OF DISTRIBUTION AND HAZARD RATE FUNCTION OF THE PROPOSED MODEL

In this section, we will discuss the nature and shape of the CDF, PDF and its failure rate function by using Eq. (2) - Eq. (4). We will discuss these by using the plots of corresponding functions.

2.1 SHAPE OF THE PDF AND CDF

The nature of the distribution is same as its baseline distribution. The proposed model is also having exponentially decreasing nature of the PDF as its baseline model. The plot the PDF for varying value of the parameter is shown in Figure 1. In this figure, the term “par” denotes the considered parameter α. The limiting behavior of the PDF (Eq. (3)) of the proposed model is

$$\lim_{\substack{x\to 0\\ \alpha>1}} f(x) = \lim_{\substack{x\to 0\\ \alpha>1}} \frac{\alpha}{\log 2} \frac{(x/k)^{(\alpha-1)}}{(1+(x/k)^{\alpha})} = 0.$$

$$\lim_{\substack{x\to 0\\ \alpha\to 1}} f(x) = \lim_{\substack{x\to 0\\ \alpha\to 1}} \frac{\alpha}{\log 2} \frac{(x/k)^{(\alpha-1)}}{(1+(x/k)^{\alpha})} = \frac{1}{k \log 2}.$$

And

$$\lim_{\substack{x\to 0\\ \alpha<1}} f(x) = \lim_{\substack{x\to 0\\ \alpha<1}} \frac{\alpha}{\log 2} \frac{(x/k)^{(\alpha-1)}}{(1+(x/k)^{\alpha})} = \text{Inf}.$$

(5)

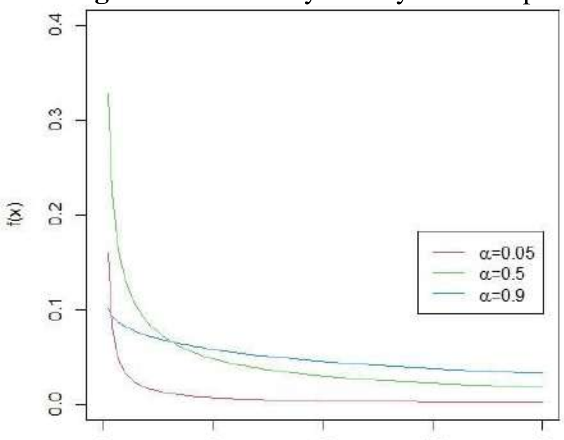
And

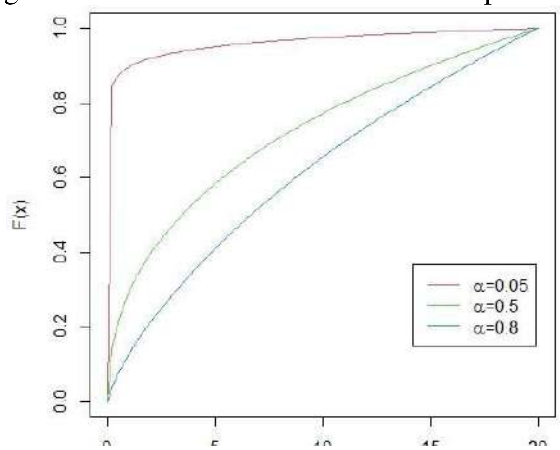
$$\lim_{x\to k} f(x) = \lim_{x\to k} \frac{\alpha}{k \log 2} \frac{(x/k)^{(\alpha-1)}}{(1+(x/k)^{\alpha})} = \frac{\alpha}{2k \log 2} = \frac{\alpha}{k \log 4}.$$

(6)

This is also reflecting in Figure 1.

Figure 1. Probability density function plot Figure 2. Cumulative distribution function plot





The shape of the proposed PDF/CDF for varying parameter value α ("par") is shown in Figure 2. Here the other parameter "k" is fixed at 20. The Figure 2 shows the CDF plot at various value of the parameter α and for the fixed value of k at 20. This figure also indicates that the. the CDF is greater for the lesser value of the parameter α for fixed value of the parameter "k". This leads to another property of distribution which is called stochastic ordering of the distribution (see Gupta et al. (1998)). In mathematical notation, if $\alpha_1 \leq \alpha_2$ then $S_1(x) \leq S_2(x)$, where $S_1(x)$ and $S_2(x)$ both are the proposed survival function for different parameter values α_1 & α_2 respectively.

2.2 SHAE OF THE HAZARD RATE OF THE MODEL

The shape of the hazard rate can be visualized by plotting the function given in Eq. (4) for various value of the parameter α for the fixed value of the parameter "k" at 20. The plot is given in Figure 3. This figure shows that the proposed distribution has increasing and bathtub failure rates.

Figure 3. Failure Rate Function Plot

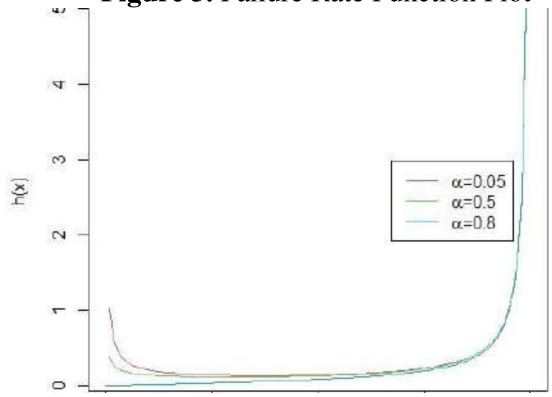
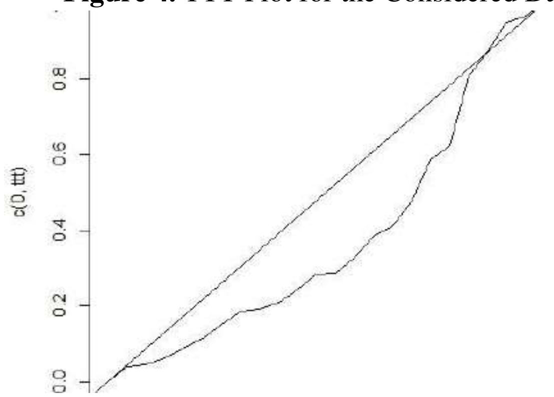


Figure 4. TTT Plot for the Considered Dataset



3. SOME BASIC STATISTICAL PROPERTIES OF THE DISTRIBUTION

In this section we will derive statistical properties of the proposed model. Here we have derived rth raw moments, quantile function, median, Moor's coefficient for measure of skewness and kurtosis, PDF of rth order statistics, Shannon entropy and stress-strength reliability.

3.1 MOMENTS OF THE PROPOSED DISTRIBUTION

The rth raw moments of the proposed distribution can be calculated easily by using the equation Eq. (3) as,

$$\mathbf{E}[\mathbf{X}^{\mathbf{r}}] = \frac{\alpha}{\mathbf{k}\log 2}\int_0^{\mathbf{k}} \frac{x^{\mathbf{r}}(x/k)^{(\alpha-1)}}{(1+(x/k)^{\alpha})}\mathbf{dx}.$$

(7)

After some basic mathematics and by using substitution method (t= $(x/k)^{\alpha}$) of solving integration, we get

$$\mathbf{E}[\mathbf{X}^{\mathbf{r}}] = \frac{\mathbf{k}^{\mathbf{r}}}{\log 2}\int_0^1 \frac{t^{\frac{r}{\alpha}}}{(1+\mathrm{t})}\mathrm{dt} = \frac{\mathbf{k}^{\mathbf{r}}\left[\mathrm{H}_{\frac{r}{2\alpha}} - \mathrm{H}_{\frac{1}{2}\left(\frac{r}{\alpha}-1\right)}\right]}{2\log 2}.$$

(8)

Here, H_n is nth harmonic number. By using the above equation, one can calculate the mean of the proposed distribution say M, by just putting r =1, as,

$$\mathbf{E}[X] = \mathbf{M} = \frac{\mathbf{k}\left[\mathrm{H}_{\frac{1}{2\alpha}} - \mathrm{H}_{\frac{1}{2}\left(\frac{1}{\alpha}-1\right)}\right]}{2\log 2}.$$

(9)

Now, by using the equation Eq. (7) one can easily get the value of variance of the distribution as $\mathrm{E}[\mathrm{X}^2] - \mathrm{E}[\mathrm{X}]^2$, where

$$\mathbf{E[X^2] = M = \frac{k^2\left[H_{\frac{2}{2\alpha}} - H_{\frac{1}{2}\left(\frac{2}{\alpha}-1\right)}\right]}{2\log 2}}.$$

(10)

Similarly, other values of the population moments can be easily obtained.

3.2 QUANTILE FUNCTION OF THE DISTRIBUTION

THE PTH QUANTILE FUNCTION Q(P), OF THE DISTRIBUTION CAN BE OBTAINED BY SOLVING THE EQUATION $\mathbf{F(Q(p)) = p}$. BY USING EQ. (3) WE GET,

$$\mathbf{F(Q(p)) = \frac{\log(1+(Q(p)/k)^{\alpha})}{\log 2} = p;\ 0 \le p \le 1.}$$

(11)

AND BY SOLVING IT WE, GET

$$\mathbf{Q(p) = k\,(2^p - 1)^{1/\alpha};\ 0 \le p \le 1.}$$

(12)

From Eq. (12), we get the median (Md) of the distribution by just putting p=1/2. Thus, the median is

$$\mathbf{M_d = k\,(\sqrt{2} - 1)^{1/\alpha}}.$$

(13)

The quantile function can also be used for finding coefficient of skewness and kurtosis. The Bowley's coefficient of skewness (B) (see Bowley, A. L. (1920)), which is defined as

$$\mathbf{B = \frac{Q(3/4) + Q(1/4) - Q(2/4)}{Q(3/4) - Q(1/4)}}.$$

(14)

Similarly, Moors' coefficient of kurtosis (M) (see Moors, J. J. A. (1988)), based on quantile function is defined as

$$\mathbf{M = \frac{Q(3/8) + Q(7/8) - Q(1/8) - Q(5/8)}{Q(6/8) - Q(2/8)}}.$$

(15)

3.3 ORDER STATISTICS AND ITS PDF OF THE DISTRIBUTION

LET $X_1, X_2, \ldots, X_N$ BE N RANDOM SAMPLES FROM THE PROPOSED DISTRIBUTION. LET $X_{(R)}$ BE THE RTH ORDER STATISTICS. THEN THE PDF $\mathbf{f_{r:n}(x)}$ IS DEFINED AS

$$\mathbf{f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!} F^{r-1}(x)[1-F(x)]^{n-r} f(x)}$$

By putting the value of CDF and PDF, we get

$$\mathbf{f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!} \frac{\alpha}{k\log 2} \frac{(x/k)^{(\alpha-1)}}{(1+(x/k)^{\alpha})} \left[\frac{\log(1+(x/k)^{\alpha})}{\log 2}\right]^{r-1} \left[1-\frac{\log(1+(x/k)^{\alpha})}{\log 2}\right]^{n-r}}$$

(16)

3.4 SHANNON ENTROPY OF THE DISTRIBUTION

ENTROPY MEASURES THE RANDOMNESS OF THE MODEL. ONE OF THE FAMOUS AND OLDEST MEASURE OF ENTROPY IS SHANNON ENTROPY PROPOSED BY SHANNON, C.E. (1951). THE SHANNON ENTROPY IS DEFINED AS $\mathbf{E[-\log f(x)]}$. NOW, FROM EQ. (3) THE SHANNON ENTROPY FOR OUR PROPOSED DISTRIBUTION IS

$$\mathbf{-\log f(x) = -\log\frac{\alpha}{k\log 2} - (\alpha-1)\log\left(\frac{x}{k}\right) + \log\left(1+\left(\frac{x}{k}\right)^{\alpha}\right)}.$$

(17)

THUS,

$$\mathbf{E[-\log f(x)] = -\log\frac{\alpha}{k\log 2} - (\alpha-1)E\left[\log\left(\frac{x}{k}\right)\right] + E\left[\log\left(1+\left(\frac{x}{k}\right)^{\alpha}\right)\right]}.$$

(18)

NOW,

$$\mathbf{E\left[\log\left(\frac{x}{k}\right)\right] = \frac{\alpha}{k\log 2}\int_0^k \frac{\log(x/k)(x/k)^{(\alpha-1)}}{(1+(x/k)^{\alpha})}dx}.$$

(19)

AFTER SOME BASIC MATHEMATICS AND BY USING SUBSTITUTION METHOD (T= $(x/k)^{\alpha}$) OF SOLVING INTEGRATION, WE GET

$$\mathbf{E\left[\log\left(\frac{x}{k}\right)\right] = \frac{1}{\log 2}\int_0^1 \frac{\log t^{(1/\alpha)}}{1+t}dx = \frac{1}{\alpha\log 2}\int_0^1 \frac{\log t}{t}dt = \frac{-\pi^2}{12\,\alpha\log 2} = \frac{-1.19}{\alpha}}.$$

(20)

AND

$$E[\log(1+\left(\frac{x}{k}\right)^{\alpha})] = \frac{\alpha}{k\log 2}\int_0^k \frac{\log\left(1+\left(\frac{x}{k}\right)^{\alpha}\right)(x/k)^{(\alpha-1)}}{(1+(x/k)^{\alpha})}dx.$$

(21)

AGAIN, BY USING THE SAME SUBSTITUTION, WE GET

$$E[\log(1+\left(\frac{x}{k}\right)^{\alpha})] = \frac{1}{\log 2}\int_0^1 \frac{\log t}{t}dt = \frac{\log 2}{2}.$$

(22)

THUS, BY USING EQ. (20) AND EQ. (22) IN EQ. (18) WE GET THE EXPRESSION OF SHANNON ENTROPY IS

$$E[-\log f(x)] = -\log\frac{\alpha}{k\log 2} + \frac{1.19\ (\alpha-1)}{\alpha} + \frac{\log 2}{2}.$$

(23)

4. Estimation of the unknown parameter

In this section, we will discuss the methods for the estimation of the unknown parameter of the distribution. Here, we used two different methods of estimation of parameter. One is the method of moments and other is the method of maximum likelihood estimation.

4.1 Method of moments

Method of moments is the oldest and simplest method of estimation if population moments of the distribution are possible to obtain. In this method we compare the sample moments to correspond to its population moments. In our proposed distribution, both the parameters can be estimated by solving two normal equations simultaneously, which are obtained just by comparing the first two raw moments. The population mean is given in equation Eq. (9) and let $\bar{x}$ be the sample mean, then

$$\bar{x} = \frac{k\left[H_{\frac{1}{2\alpha}} - H_{\frac{1}{2}\left(\frac{1}{\alpha}-1\right)}\right]}{2\log 2}$$

(24)

And let the second sample raw moments is x^2, then form the equation Eq. (10), we get,

$$x^2 = \frac{k\left[H_{\frac{1}{2\alpha}} - H_{\frac{1}{2}\left(\frac{1}{\alpha}-1\right)}\right]}{2\log 2}$$

(25)

Which is not in closed form. So, it needs some iteration technique to solve it.

4.2 Method of maximum likelihood

The method of maximum likelihood is the very famous method of estimation due to its properties. Let X_1, X_2, …, X_n be n random samples from the proposed distribution. Then the maximum likelihood estimator is the value that maximizes likelihood function or logarithmic of likelihood function (say log L). The logarithmic of likelihood function is given as

$$\log L = n \log \frac{\alpha}{k \log 2} + (\alpha - 1) \sum_{i=1}^{n} \log (x_i / k) - \sum_{i=1}^{n} \log (1 + (x_i / k)^{\alpha}).$$

(26)

The first derivative of the likelihood function equating to zero, gives the likelihood equation as

$$\frac{\partial}{\partial \alpha} \log L = \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{1}{(1 + (x_i / k)^{\alpha})} = 0.$$

(27)

We get MLE (say $\hat{\alpha}$) of the parameters α after solving Eq. (27). The other parameter "k" can be estimated just by taking the maximum value of the observed sample say $\hat{k}$=max(X_1, $X_{2,}$ …, X_n). The likelihood equation given in Eq. (27) cannot be solved analytically and it requires some iteration technique. Here, we have proposed to use Newton-Raphson method through R software (R Core Team (2023)).

5. Real data application

In this section, we have considered a real dataset for checking suitability of the proposed model in the field of decreasing hazard model. Further, to check the superiority of our proposed model over other competitive models, we have taken four well known lifetime models which have bathtub type failure rate. The considered models are Generalized DUS exponential (GDUSE) distribution proposed by (Maurya et al. (2017)), generalized Lindley (Nadarajah et al. (2012), extension of exponential distribution (Nadarajah & Haghighi (2010)) and Chen model (Chen (2000)).

All the considered models are two parameter and at least bathtub type failure rate. We have also considered a bathtub failure rate data of 23 size of times between failure of secondary reactor pumps data and were taken by Suprawhardana and Prayoto (1999). The dataset is given in Table 1. We have also plotted scaled TTT plot for the considered dataset in Figure 4. This figure shows that the dataset has bathtub type failure rate. For more details about scaled TTT plot one can see Aarset (1987). So, the considered dataset is suitable for all the considered distributions.

Table 1. 23 real dataset of secondary reactor pumps

2.160 0.150 4.082 0.746 0.358 0.199 0.402 0.101 0.605 0.954 1.359 0.273 0.491 3.465 0.070 6.560 1.060 0.062 4.992 0.614 5.320 0.347 1.921

We have calculated ML estimate(s) along with K-S distance with p-value and AIC and BIC in the Table 2. Here, AIC and BIC are the model selection criterion and is least for the most suitable model. Here all the values of the parameters are rounded at three decimal places and rest values are at two decimal places.

Table 2. MLE, KS statistics with p-value and AIC, BIC for fitted dataset

Distribution	**ML Estimate**		**KS- Test**		**AIC**	**BIC**
	α	**k**	**Statistics**	**p-value**		
Proposed	0.452	15.08	0.15	0.19	**204.54**	**208.36**
GDUSE	0.424	0.215	0.16	0.16	210.68	214.51
GL	0.411	0.297	0.16	0.13	211.96	215.78
EED	0.537	0.194	0.14	0.24	208.75	212.57

Chen	0.361	0.286	0.14	0.26	210.84	214.67

From the Table 2, we see that, among all the considered bathtub type failure models, all fit well to the considered dataset (as p value is greater than 0.05 for all the considered distributions). The proposed distribution has minimum value of model selection criterion i.e. AIC and BIC for the considered dataset. However, we may recall that the KS test statistic provides the maximum difference between the empirical and fitted model. Therefore, even if the model fits well at all points except at one point the value of KS test statistic will be large with the low value of the p-value, although KS- values are closed for all the considered distributions. In this sense, it may not be proper to consider it as a better tool for establishing the superiority of a model over other than AIC and BIC. Thus, the proposed model is the better fit to the considered real data in comparison to other competitive bathtub type failure rate models.

6. CONCLUSIONS

In the present paper, we have proposed a new distribution. The proposed model is two parameters having Increasing and bathtub type failure rates. We have studied its nature of PDF, CDF and its failure rate functions. We also discussed the limiting case of the PDF and stochastic order relationship. After that, we have derived some basic statistical properties like r^{th} order raw moments, and quantile function. After that, by using the quantile function, we have derived median and find the expression of measure of skewness and measure of kurtosis. We have also derived the distribution of r^{th} order statistics and find the expression for the Shannon entropy.

In addition to these, method of moments and maximum likelihood estimation have been discussed for the estimation of the parameter purpose. Lastly, a real dataset and four other two parameter distributions having at least bathtub type failure rate namely GDUS exponential, generalized Lindley, extension of exponential and Chen distributions have been taken to check the fitting of the model over these other. In model comparison, we found that, the proposed distribution fitted well to the considered dataset. Hence, we recommend the proposed new lifetime distribution.

7. REFERENCES

1. M. V. Aarset; How to identify a bathtub hazard rate, IEEE Transactions on Reliability, 36(1), 106–108 (1987).
2. L. Bowley; Elements of statistics, volume 2, P S King, (1920).
3. R. E. Barlow; F. Proschan (1975); Statistical theory of reliability and life testing: probability models, Technical report, Florida State Univ Tallahassee.
4. Z. Chen; A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function, Statistics & Probability Letters. 49(2):155–161 (2000).
5. R. C. Gupta; P. L. Gupta; R. D. Gupta; Modeling failure time data by Lehmann alternatives, Communications in Statistics-Theory and Methods, 27, 887–904 (1998).
6. U. Hjorth, U; A reliability distribution with increasing, decreasing, constant and bathtub-shaped failure rates, Technometrics, 22(1):99-107 (1980).
7. S. K. Maurya; One parameter bathtub model and its real life application, Journal of Scientific Research, 65(3), 202-208 (2021).
8. S. K. Maurya; A. Kaushik; S. K. Singh; U. Singh; A new class of distribution having decreasing, increasing, and bathtub-shaped failure rate, Communications in Statistics- Theory and Methods, 46(20), 10359–10372 (2017).
9. S. K. Maurya; S. Nadarajah; Poisson generated family of distributions: A review, Sankhya B, 83(2), 484–540 (2021).

10. J. J. A. Moors; A quantile alternative for kurtosis, The statistician, 37(1), 25–32 (1988).

11. G. S. Mudholkar; D. K. Srivastava; Exponentiated Weibull family for analyzing bathtub failure-rate data, IEEE Transactions on Reliability, 42(2):299-302 (1993).

12. S. Nadarajah; H. S. Bakouch; R. Tahmasbi; A generalized Lindley distribution, Sankhya B, 73(2), 331–359 (2011).

13. S. Nadarajah; F. Haghighi; An extension of the exponential distribution, Statistics, 45(6), 543-558 (2011).

14. R Core Team; R: A Language and Environment for Statistical Computing, R Foundation for Statistical Computing, Vienna, Austria (2023) URL http://www.R-project.org/.

15. P. Rai; K.K. Dubey; S.K. Maurya; An Application of Decreasing Failure Rate Distribution, Journal of Survey in Fisheries Sciences, 10(1S), 5295-5299 (2023).

16. S. Rajarshi; M. Rajarshi; Bathtub distributions: A review, Communications in Statistics-Theory and Methods, 17(8):2597-2621 (1988).

17. E. Shannon; Prediction and entropy of printed English, The Bell System Technical Journal, 30, 50–64 (1951).

18. M. S. Suprawhardana; S. Prayoto; Total time on test plot analysis for mechanical components of the RSG-GAS reactor, Atom Indones, 25(2), 81-90 (1999).