DIFFERENTIAL SUBORDINATION AND STRONG DIFFERENTIAL SUBORDINATION OF SUBCLASSES OF MULTIVALENT FUNCTIONS

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ABSTRACT

In this paper, we deduce some differential subordination and strong differential subordination of subclasses of multivalent functions subordination outcomes involving the generalized differential operator we have introduced new classes by using subordination and we have obtained coefficient estimates and properties which contains Distortion and Growth theorems, radius of starlikeness and radius of convexity, and other related results for $\mathcal{KM}(A, B, \alpha, \delta, p)$ *and* $\mathcal{M}(A, B, \alpha, \delta, p)$ *.for certain multivalent analytic functions in the open unit disk. These outcomes are applied to obtain differential sandwich theorems.*

Keywords: Analytic function, multivalent function, differential subordination, differential superordination, sandwich theorem, generalized differential operator.

INTRODUCTION

Let $\mathcal{D}_\rho$ denoted the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n , (p, n \in N = \{1,2, \dots\})\ z \in \mathcal{U} \quad (1.1)$$

For the function $f \in \mathcal{D}_\rho$ given by (1.1) and $g \in \mathcal{D}_\rho$ defined by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n ,$$

The Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n = (g * f)(z) \quad (1.2)$$

Motivated essentially by M. K. Aouf [6], Shams et al. [3] introduced the operator $I_p^\alpha : \mathcal{D}_\rho \rightarrow \mathcal{D}_\rho$ as follows:

$$I_p^\alpha(f(z)) = z^p + \sum_{k=1}^{\infty}\left(\frac{p+1}{k+p+1}\right)^\alpha a_{k+p} z^{k+p}, \alpha \in \mathbb{R} \quad (1.3)$$

Using the above definition relation, it is easy verify that the operator becomes an integral operator

$$I_p^\alpha(f(z)) = \frac{(p+1)^\alpha}{z\Gamma(\alpha)} \int_0^z (log\frac{z}{t})^\alpha a_{k+p} z^k + p\, f(t)dt, \text{ for } \alpha > 0 \quad (1.4)$$

$$I_p^\alpha(f(z)) = f(z), \text{ for } \alpha = 0,$$

and, moreover

$$(p + 1)\, I_p^{\alpha-2}(f(z)) = z\,(I_p^{\alpha-2} f(z))' + I_p^{\alpha-1}(f(z)) \text{ for } \alpha \in \mathbb{R}$$

We mention that the one- parameter family of integral operator $I^\alpha \equiv I_1^\alpha$ was defined by Salagean [4].

DEFINITION (1.1.1):

Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap \mu[0,p]$ and $\lambda > -p$. The class of admissible functions $\Phi\ [\Omega, q]$ consists of those functions $\emptyset : \mathbb{C}^2 \times \mathcal{U} \times \bar{\mathcal{U}} \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$\emptyset(u, v; z, \xi) \notin \Omega,$ (1.5)

Whenever $u = q(\xi), v = p + 1,$

and

$\mathcal{R}e\left\{\frac{(p+1)\{(p+1)-2v+u\}-zv-3u}{(p+1)2v-5u}\right\} \geq k\,\mathcal{R}e\left\{\frac{\xi q''(\xi)}{q'(\xi)}+1\right\},$ (1.6)

$z \in \mathcal{U}, \xi \in \partial\mathcal{U}\backslash E(q), \xi \in \bar{\mathcal{U}}$ and $k \geq p$.

THEOREM (1.1.1):

Let $\emptyset \in \Phi_k[\Omega, q]$. If $f \in \mathcal{D}_p$ satisfies

$\{\emptyset\, I_p^{\alpha-2} f(z), I_p^{\alpha-2} f(z);\ z \in \mathcal{U}, \xi \in \bar{\mathcal{U}}\} \subset \Omega,$ (1.7)

Then

$I_p^{\alpha-2} f(z) \prec q(z).$

PROOF: By using (1.5) we get the equivalent relation

$$I_p^{\alpha-2}(f(z)) = \frac{z\left(I_p^{\alpha-2} f(z)\right)' + I_p^{\alpha-1} f(z)}{p+1} \quad (1.8)$$

$$I_p^{\alpha-1}(f(z)) = \frac{z\left(I_p^{\alpha} f(z)\right)' + I_p^{\alpha} f(z)}{p+1} \quad (1.9)$$

$$\left(I_p^{\alpha-1} f(z)\right)' = \frac{z\left(I_p^{\alpha} f(z)\right)'' + 2\left(I_p^{\alpha} f(z)\right)'}{p+1} \quad (1.10)$$

Assume that F(z) = $I_p^{\alpha} f(z)$.

Then $I_p^{\alpha-1} f(z) = \frac{z(F'(z) + F(z))}{(p+1)}$

Therefore

$$I_p^{\alpha-2} f(z) = \frac{z}{(p+1)}\left\{\frac{z\left(I_p^{\alpha} f(z)\right)'' + 2\left(I_p^{\alpha} f(z)\right)'}{(p+1)}\right\} + \frac{z\left(I_p^{\alpha} f(z)\right)' + I_p^{\alpha} f(z)}{(p+1)}$$

Then we have by (1.8)

$$I_p^{\alpha-2} f(z) = \frac{z}{(p+1)}\left\{\frac{z(F''(z) + (z+1)zF'(z) + F(z))}{(p+1)}\right\} + \frac{z(F'(z) + F(z))}{(p+1)}$$

$$= \frac{1}{(p+1)}\left\{\frac{z^2(F''(z) + (z+1)zF'(z) + F(z))}{(p+1)}\right\} \quad (1.11)$$

Let

$$u = r, v = \frac{s+r}{(p+1)}, w = \frac{t+(z+1)s+r}{r(p+1)^2}$$

Assume that

$$\psi(r, s; z, \xi) = \emptyset(u, v; z, \xi) = \emptyset\left(r, \frac{s+r}{(p+1)}, \frac{t+(z+1)s+r}{r(p+1)^2}; z, \xi\right). \quad (1.12)$$

By using (1.8) and (1.9), we obtain

$$\psi(F(z), zF'(z), z^2F''(z);\ z, \xi) = \emptyset\ (I_p^{\alpha} f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z);\ z, \xi\} \quad (1.13)$$

Therefore, by making use (1.7), we get

$$\psi(F(z), zF'(z), z^2F''(z);\ z, \xi) \in \Omega. \quad (1.14)$$

Also, by using

$$w = \frac{(t + (z + 1)s + r)}{r(p + 1)^2}$$

and by simple calculations, we get

$$\frac{((p+1)\{(p+1) - 2v + u\} - zv - 3u(z-1)))}{((p+1)2v - 5u)} \quad (1.15)$$

and the admissibility condition for $\emptyset \in \Phi[\Omega, q]$ is equivalent to the admissibility condition for ψ then,

$F(z) \prec q(z)$.

Hence, we get

$I_p^{\alpha-2} f(z) \prec q(z)$.

If we assume that $\Omega \neq \mathbb{C}$ is a simply connected domain. So, $\Omega = h(\mathcal{U})$, for some conformal mapping h of $\mathcal{U}$ onto Ω. Assume the class is written as $\Phi[h, q]$.Therefore, we conclude immediately the following theorem.

THEOREM (1.1.2):

Let $\emptyset \in \Phi[h, q]$. If $f \in \mathfrak{D}_\rho$ satisfies

$$\{\emptyset\ I_p^{\alpha-2} f(z), I_p^{\alpha-1} f(z);\ I_p^{\alpha} f(z);\ z \in U, \xi\} \prec\prec h(z), \quad (1.16)$$

then $I_p^{\alpha-2} f(z) \prec q(z)$.

The next result is an extension of Theorem (1.1.1) to the case where the behavior of q on $\partial\mathcal{U}$ is not known.

COROLLARY (1.1.1):

Let $\Omega \subset \mathbb{C}$, q be univalent in $\mathcal{U}$ and $q(0) = 0$. Let $\emptyset \in \Phi[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathfrak{D}_\rho$ satisfies

$$\emptyset\ \{I_p^{\alpha-2} f(z), I_p^{\alpha-1} f(z);\ I_p^{\alpha} f(z);\ z, \xi\} \prec\prec \Omega, \quad (1.17)$$

then

$I_p^{\alpha-2} f(z) \prec q(z)$.

THEOREM (1.1.3):

Let h and q be univalent in $\mathcal{U}$, with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\emptyset: \mathbb{C}^2 \times \mathcal{U} \times \bar{\mathcal{U}} \to \mathbb{C}$ satisfy one of the following conditions:

(1) $\emptyset \in \Phi[\Omega, q_\rho]$ for some $\rho \in (0,1)$ or

(2) there exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathfrak{D}_\rho$ satisfies (1.16), then

$I_p^{\alpha-2}f(z) \prec q(z)$.

PROOF:

Case (1): By using Theorem (1.1.1), we get

$I_p^{\alpha-2}f(z) \prec q_\rho$.

Since

$q_\rho \prec q(z)$,

then we get the result.

Case (2):

Assume that $F(z) = I_p^{\alpha}f(z)\ and\ F\rho(z) = F(\rho z)$. So,

$\emptyset(F\rho(z), zF_\rho'(z), z^2F_\rho''(z), \rho z) = \emptyset(F(\rho z), zF'(\rho z), z^2 F''(\rho z), \rho z) \in h\rho(\mathcal{U})$.

By using Theorem (1.1.1) with associated

$\emptyset(F(z), zF'(z), z^2 F''), w(z)) \in \Omega$, where w is any function mapping from $\mathcal{U}$ onto $\mathcal{U}$, with $w(z) = \rho z$, we obtain $F_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$.

By letting $\rho \to 1^-$, we get

$I_p^{\alpha-2}f(z) \prec\prec q(z)$.

The next theorem gives the best dominant of the differential subordination (1.13).

THEOREM (1.1.4):

Let h be univalent in $\mathcal{U}$ and $\emptyset: \mathbb{C}^2 \times \mathcal{U} \times \bar{\mathcal{U}} \to \mathbb{C}$. Suppose that the differential equation

$$\emptyset\left(q(z), \frac{z\left(q'(z)\right)+q(z)}{(p+1)}, \frac{1}{(p+1)}\left\{\frac{z^2\left(Q''(z)+(z+1)z\,Q'(z)+Q(z)\right)}{(p+1)}\right\}; z, \xi\right) = h(z), \qquad (1.18)$$

has a solution q with $q(0) = 0$ and satisfy one of the following conditions:

(1) $q \in Q_0$ and $\emptyset \in \Phi_k[h, q]$.

(2) q is univalent in $\mathcal{U}$ and $\emptyset \in \Phi_k[h, q_\rho]$ for some $\rho \in (0,1)$.

(3) q is univalent in $\mathcal{U}$ and there exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi_k[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$. If $f \in \mathfrak{D}_\rho$ satisfies (1.16), then $I_p^{\alpha-2}f(z) \prec q(z)$ and q is the best dominant.

PROOF: By using theorem (1.12) and theorem (1.13), we get that q is a dominant of (1.16). since q satisfies (1.18), it is also a solution of (1.16) and therefore by the help of (1.16) q will be dominant by all dominants of (1.16). Hence, q is the best dominant.

DEFINITION (1.1.2):

Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_k[\Omega, M]$ consists of those functions $\emptyset: \mathbb{C}^2 \times \mathcal{U} \times \bar{\mathcal{U}} \to \mathbb{C}$ such that

$\emptyset\left(Me^{i\theta},\frac{(k+1)Me^{i\theta}}{(p+1)}z,\xi\right)\notin\Omega,$ (1.19)

whenever $z\in\mathcal{U},\xi\in\bar{\mathcal{U}},k\geq 1.$

COROLLARY (1.1.2):
Let $\emptyset\in\Phi[\Omega,M]$. If $f\in A$ and $f\in\mathfrak{D}_{\rho}$ satisfies that

I. $\emptyset\left(I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);\,z,\xi\right)\in\Omega$, then $I_p^{\alpha-1}f(z)\prec Mz.$

II. $\emptyset\left(I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);\,z,\xi\right)$, then $I_p^{\alpha-1}f(z)<M$

COROLLARY (1.1.3):
Let $M>0$, and Let $C(\xi)$ be an analytic function in $\bar{\mathcal{U}}$ with $Re\{\xi C(\xi)\}\geq 0$ for $\xi\in\partial\mathcal{U}$. If $f\in\mathfrak{D}_{\rho}$ satisfies

$\left|\emptyset((p+1)I_p^{\alpha-1}f(z)-\lambda^2 I_p^{\alpha}f(z)+C(\xi)\right|<$ M,

Then

$|I_p^{\alpha-1}f(z)|<M.$

PROOF:
From Corollary (1.1.2) by taking $\emptyset(u,v,z,\xi)=(p+1)v-\lambda^2u+C(\xi)$ and $\Omega=h(\mathcal{U})$,

where $h(z)=Mz$. By using Corollary (1.1.2), we need to show that

$\emptyset\in\Phi[\Omega,M]$, that is, the admissible condition (1.19) is satisfied. We get

$\emptyset\left(Me^{i\theta},\frac{(k+1)Me^{i\theta}}{(p+1)};\,z,\xi\right)$

$=\left|(k+1)\,Me^{i\theta}-\lambda^2Me^{i\theta}+C(\xi)\right|$

$=\left|(k-\lambda^2)\,Me^{i\theta}+C(\xi)\right|\geq(k-\lambda^2)\,\mathrm{M}+Re\{ke-i\theta\}+Re\,\{C(\xi)e-i\theta\}\geq\lambda\mathrm{M}\geq.$

Hence by Corollary (1.1.3), we get the result.

DEFINITION (1.1.3):
Let Ω be a set in $\mathbb{C}$ and $q\in[0,p]$ with $q'(z)\neq 0$. The class of admissible functions $\Phi'\,[\Omega,q]$ consists of those functions $\emptyset\colon C^2\times\mathcal{U}\times\bar{\mathcal{U}}\to\mathbb{C}$ that satisfy the admissibility condition:

$\emptyset(u,v;\,\zeta,\xi)\notin\Omega,$ (1.20)

whenever $u=q(z),v=\frac{\frac{1}{m}zq'(z)+q(z)}{1+p}$,

and

$Re\left\{\frac{(p+1)\{(p+1)-2v+u\}-zv-3u\}}{(p+1)2v-5u}\right\}\geq\frac{1}{m}Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}+1\right\}$ (1.21)

$z\in\mathcal{U},\zeta\in\partial\mathcal{U}\backslash E(q),\xi\in\bar{\mathcal{U}}\ and\ m\geq p.$

THEOREM (1.1.4):
Let $\emptyset\in\Phi_k'[h,q]\,.If\ f\in\mathfrak{D}_{\rho},I_p^{\alpha-1}f(z)\in Q_0$ and

$\emptyset(I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);\ z,\xi)$

is univalent in $\mathcal{U}$,

Then

$\Omega \subset \emptyset(I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);\ z,\xi)$

implies that

$q(z) \prec I_p^{\alpha-1}f(z)$.

PROOF:
By (1.13) and $\Omega \subset \{\emptyset(I_p^{\alpha-1}f(z),I_p^{\alpha}f(z)\};\ (z \in \mathcal{U},\xi \in \mathcal{U})$,

we have $\Omega \subset \{\psi(F(z),zF'(z),z,\xi)(z \in \mathcal{U},\xi \in \bar{\mathcal{U}})\}$.

From $u = r$, $v = \frac{(s+r)}{(p+1)}$, we see that the admissibility for $\emptyset \in \Phi'_k[\Omega,q]$ is equivalent to admissibility condition for ψ. Hence, $\psi \in \Psi'[\Omega,q]$ and so we have

$q(z) \prec I_p^{\alpha-2}f(z)$.

The following Theorem is an immediate consequence of Theorem (1.1.5).

THEOREM (1.1.5):
Let $q \in \mu\,[0,p]$, h be analytic in $\mathcal{U}$ and $\emptyset \in \Phi'_k[h,q]$ .If $f \in \mathfrak{D}_\rho, I_p^{\alpha-1}f(z) \in Q_0$: and $\{\emptyset(I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);\ z \in \mathcal{U},\xi \in \bar{\mathcal{U}})\}$ is univalent in $\mathcal{U}$,

then

$$h(z) \prec\prec \{\emptyset(I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);\ z \in \mathcal{U},\xi \in \bar{\mathcal{U}})\} \qquad (1.22)$$

implies that

$q(z) \prec I_p^{\alpha-1}f(z)$.

THEOREM (1.1.7):
Let h be analytic in $\mathcal{U}$ and $\emptyset$: $\mathbb{C}^2 \times \mathcal{U} \times \bar{\mathcal{U}} \to \mathbb{C}$. Suppose that the differential equation

$\emptyset(q(z),\frac{z\,(q'(z))+q(z)}{(p+1)},z,\xi) = h(z)$, has a solution $q \in Q_0$.$If\ \emptyset \in \Phi'_k[h,q], f \in \mathfrak{D}_\rho, I_p^{\alpha-1}f(z) \in Q_0$ and $\{\emptyset(I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);\ z,\xi)$ is univalent in U ,

then $h(z) \prec\prec \emptyset\left(I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);\ z,\xi\right)$ (1.23)

implies that $q(z) \prec I_p^{\alpha-1}f(z)$, and q is the best dominant.

PROOF:
The proof of this Theorem is the same of proof Theorem (1.1.4). Theorem (1.1.2) and Theorem (1.1.5), we obtained the following Theorem.

THEOREM (1.1.8):
Let h_1 and q_1 be analytic functions in $\mathcal{U}$, h_2 be a univalent function in $\mathcal{U}$, $q_2 \in Q_0$ with $q_1(0) = q_2(0) = 0$ and $\emptyset \in \Phi_k\,[h_2,q_2] \cap \Phi_k{}'\,[h_1,q_1]$. If $f \in \mathfrak{D}_\rho$, $I_p^{\alpha-1}f(z) \in \mu\,[0,p] \cap Q_0$ and $\emptyset(I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);\ z,\xi)$ is univalent in $\mathcal{U}$,

Then

$h_1(z) \prec\prec \emptyset(I_p^{\alpha-1}f(z), I_p^{\alpha}f(z); z, \xi) \prec\prec h_2(z),$ (1.24)

implies that

$q_1(z) \prec I_p^{\alpha-1}f(z) \prec q_2(z).$

Some Applications of Differential Subordination Involving Hadamard Product

Let $\mathfrak{D}_\rho$ denoted the class of functions of the form

$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, (a_n \geq 0\ p \in \mathbb{N} = \{1,2,\dots\})\ z \in \mathcal{U}$ (1.25)

For the function $f \in \mathfrak{D}_\rho$ given by (1.1) and $g \in \mathfrak{D}_\rho$ defined by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}$$

The Hadamard product (or convolution) of f *and* g is given by

$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p}$ (1.26)

Let A, B, σ and ε, δ, τ be fixed real numbers. $f(z) \in \mathfrak{D}(p,1)$ Contained in

$\mathcal{L}_{\varepsilon,\delta,\tau}$ $(p; A, B)$ gives $\mathcal{L}_{\varepsilon,\delta,\tau,p}(f) \prec \frac{1+2Az}{1+2Bz}, z \in \mathcal{U}$ (1.27)

$$\mathcal{L}_{\varepsilon,\delta,\tau,p}(f) = [1 - \sigma]\frac{\mathcal{H}_p^{\mu+p-1}}{z^p} + \sigma\frac{\mathcal{H}_p^{\mu+p-1}}{z^p}$$

Where

$$\mathcal{H}_p^{\mu+p-1} f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(\mu+p+n)}{\Gamma(\mu+p)n!} a_n + p\, z^{p+n}$$

$\alpha_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1,$

$$\left(\tau \geq 0, \delta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B \leq A \leq \frac{1}{2}\right)$$

Hence from above relation, we have been obtain

$z(\mathcal{H}_p^{\mu+p-1} f(z))' = (\mu + p)\mathcal{H}_p^{\mu+p} f(z) - \mu\mathcal{H}_p^{\mu+p-1} f(z)$ (1.28)

This work is due to the [8] and [5]. Where we have used the techniques of differential subordination to obtain several interesting properties. A holomorphic function f is said to be close-to-convex of order α $(0 \leq \alpha < 1)$ if there exists a convex function $h \in D(1,1)$ and a real β such that

$Re\left(\frac{f'(z)}{e^{i\beta} h'(z)}\right) > \alpha$ *for* $z \in \mathcal{U}.$

THEOREM 1.2.1:

Let the function $f(z) \in \mathfrak{D}(p,1)$. Then

$$z\left(z^{1-p}\mathcal{H}_p^{\mu+p-1} f(z)\right)'' = (\mu + p)\left(z^{1-p}\mathcal{H}_p^{\mu+p} f(z)\right)' - (\mu + p)\left(z^{1-p}\mathcal{H}_p^{\mu+p-1} f(z)\right)'$$

(1.29)

PROOF: we know that

$$z\left(\mathcal{H}_p^{\mu+p-2}\ f(z)\right)' = (\mu + p - 1)\,\mathcal{H}_p^{\mu+p-1}\ f(z) - \mu\mathcal{H}_p^{\mu+p-1} f(z)$$

since

$$z^{2-p}\mathcal{H}_p^{\mu+p-2}\ f(z) + (2 - \mathrm{p})\,\mathcal{H}_p^{\mu+p-2}\ f(z) = (\mu + p - 1)\,\mathcal{H}_{\mathrm{p}}^{\mu+p-1} f(z) + (2 - \mu + p)\, z^{1-p}\mathcal{H}_{\mathrm{p}}^{\mu+p-2}\ f(z)$$

But owing to

$$\left(\mathcal{H}_p^{\mu+p-2}\ f(z)\right)' + (2 - \mathrm{p})\mathcal{H}_{\mathrm{p}}^{\mu+p-2} f(z) = z^p\left(z^{2-p}\mathcal{H}_p^{\mu+p-2}\ f(z)\right)'$$

We obtain

$$\left(z^{2-p}\mathcal{H}_p^{\mu+p-2}\ f(z)\right)' = (\mu + p - 1)\left(z^{2-p}\mathcal{H}_p^{\mu+p-2}\ f(z)\right) + (2-\mu+p)\left(z^{2-p}\mathcal{H}_p^{\mu+p-2}\ f(z)\right)$$

Differentiating both sides of above equation we get

$$\left(z^{2-p}\mathcal{H}_p^{\mu+p-2}\ f(z)\right)'' = (\mu + p - 1)\left(z^{2-p}\mathcal{H}_p^{\mu+p-1}\ f(z)\right)' - \mu(z^{2-p}\mathcal{H}_p^{\mu+p-1}\ f(z))'$$

COROLLARY 1.2.1:

Let $f(z) \in \mathfrak{D}\,(p,1)$. and $z^{1-p}\mathcal{H}_p^{\mu+p-1} f(z)$ is convex univalent function. Then$z^{1-p}\mathcal{H}_p^{\mu+p-1} f(z)$ is close-to-convex of order $\frac{\mu+p-1}{|(\mu+p)|}$ with respect to $z^{1-p}\mathcal{H}_p^{\mu+p-1} f(z)$.

PROOF:

Since

$z(z^{2-p}\mathcal{H}_p^{\mu+p-2} f(z))'' = (\mu + p)\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}\ f(z)\right)' - \mu\,(z^{2-p}\mathcal{H}_p^{\mu+p-2} f(z))'$. We obtain

$$\frac{(z^{2-p}\mathcal{H}_p^{\mu+p-2} f(z))'}{(z^{2-p}\mathcal{H}_p^{\mu+p-1} f(z))'} = \frac{(z^{2-p}\mathcal{H}_p^{\mu+p-1} f(z))''}{(\mu+p)(z^{2-p}\mathcal{H}_p^{\mu+p-1} f(z))''} + 1$$

Since

$$z^{2-p}\mathcal{H}_{\mathrm{p}}^{\mu+p-1} f(z)$$

is a convex function,

$$Re\left\{\frac{(\mu+p)(z^{2-p}\mathcal{H}_p^{\mu+p-1} f(z))'}{|\mu+p|(z^{2-p}\mathcal{H}_p^{\mu+p-1} f(z))'}\right\} = Re\left\{\frac{z(z^{2-p}\mathcal{H}_p^{\mu+p-1} f(z))''}{|\mu+p|(z^{2-p}\mathcal{H}_p^{\mu+p-1} f(z))'}\right\}$$

$$Re\left\{\frac{\mu+p-2}{|(\mu+p)|}\right\}$$

Therefore, by definition of close-to-convex we get the required result.

THEOREM 1.3

Let $f_1(z), f_2(z) \in D\,(p,1)$, $\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_1(z)) \prec h_1(z)$and

$\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_2(z)) \prec h_2(z)$, where $h_1(z)$, $h_2(z)$ are convex univalent in $\mathcal{U}$ and if $\frac{\mu+p}{\lambda} \geq 0$, $\mu + p > q > \lambda > 0$,

then $\mathcal{L}_{\varepsilon,\delta,\tau,p}\,(\mathcal{H}_q^{\mu+p-2}(f_1 * f_2)) \prec \frac{\mu+p}{\lambda} \int_0^z t^{\frac{\mu+p}{\lambda}} h_1(t) * h_2(t)dt \prec h_1(t) * h_2(t)$.

PROOF:

Since

$\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_1(z)) \prec h_1(z)$ *and* $\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_2(z)) \prec h_2(z)$

then we have

$\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_1(z)) * \mathcal{L}_{\varepsilon,\delta,\tau,p}(f_2(z)) \prec h_1(z) * h_2(z)$

and, the convolution of convex univalent functions is also the convex univalent function. Now, let

$$p(z) = \mathcal{L}_{\varepsilon,\delta,\tau,p}\,(\mathcal{H}_q^{\mu+p-2}(f_1 * f_2)\,(z)) =$$
$$(1-\lambda)\frac{\left(\mathcal{H}_q^{\mu+p-2}\,(\mathcal{H}_q^{\mu+p-2}\,(f_1 * f_2)\,(z)\right)}{z^p} + \lambda\frac{\left(\mathcal{H}_q^{\mu+p-2}\,(\mathcal{H}_q^{\mu+p}\,(f_1 * f_2)\,(z)\right)}{z^p}$$

Then $p(z)$ is holomorphic function and $p(0) = 1$ $in\ \mathcal{U}$.

By using (1.2), we have

$$p(z) + \frac{\lambda p}{\mu+p}\,p'(z)$$

$$= \mathcal{L}_{\varepsilon,\delta,\tau,p}\left(\mathcal{H}_q^{\mu+p-2}(f_1 * f_2)(z)\right) + \frac{\lambda p}{\mu+p}\left(\mathcal{L}_{\varepsilon,\delta,\tau,p}\,(\mathcal{H}_q^{\mu+p-2}\,(f_1 * f_2)\,(z)\right)'$$

$$= \left(1-\frac{\lambda p}{\mu+p}\right)z^{-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right) + \frac{\lambda p}{\mu+p} z^{2-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right)' +$$
$$\frac{\lambda p}{\mu+p}\left[\left(1-\frac{\lambda p}{\mu+p}\right)z^{-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right) +\right.$$
$$\left.\left(\frac{\lambda p}{\mu+p} z^{2-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right)'\right)'\right]$$

$$=$$
$$\left(1-\frac{\lambda p}{\mu+p}\right)z^{-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right) + \frac{\lambda}{\mu+p} z^{1-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right)' +$$
$$\frac{\lambda p}{\mu+p}\left[\left(1-\frac{\lambda p}{\mu+p}\right)(-pz^{-p-1})\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right) +\right.$$
$$z^{-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right)' + \frac{\lambda}{\mu+p}\left((1-p)z^{-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right)' +\right.$$
$$\left.z^{1-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right)''\right]$$

$$= \left[1-\frac{\lambda p}{\mu+p}-\frac{\lambda p}{\mu+p}+\frac{\lambda^2 p^2}{(\mu+p)^2}\right]z^{-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right) + \left[-\frac{\lambda}{\mu+p}-\frac{\lambda}{\mu+p}\left(1-\frac{\lambda}{\mu+p}\right) +\right.$$
$$\left.\frac{\lambda^2}{(\mu+p)^2}(1-p)\right]z^{1-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right)' +$$
$$\frac{\lambda^2}{(\mu+p)^2} z^{2-p}\left(\mathcal{H}_q^{\mu+p-2} f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2(z)\right)''$$

Now

$$\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_1(z)) * \mathcal{L}_{\varepsilon,\delta,\tau,p}(f_2(z))$$

$$= \left[\left(1-\frac{\lambda p}{\mu+p}\right)\left(z^{-p}\left(\mathcal{H}_q^{\mu+p-2}\ f_1(z) * \frac{\lambda p}{\mu+p} z^{1-p}\left(\mathcal{H}_q^{\mu+p-2}\ f_1(z)\right)'\right)\right)\right] * \left[\left(z^{-p}\left(\mathcal{H}_q^{\mu+p-2}\ f_2\ (z) * \frac{\lambda}{\mu+p} z^{1-p}(\mathcal{H}_q^{\mu+p-2}\ f_2\ (z)\right)\right)'\right] = \left[\left(1-\frac{\lambda p}{\mu+p}\right)^2 z^{-p}\left(\mathcal{H}_q^{\mu+p-2}\ f_1(z) * \mathcal{H}_q^{\mu+p-2} f_2\ (z)\right) + 2\left(1-\frac{\lambda p}{\mu+p}\right)\frac{\lambda}{\mu+p}\mathcal{H}_q^{\mu+p-2}\ f_1(z) * \left(\mathcal{H}_q^{\mu+p-2}\ f_2(z)\right)' + \left(\frac{\lambda p}{\mu+p}\right)^2 z^{1-p}\mathcal{H}_q^{\mu+p-2}\ f_1(z) * \left(\mathcal{H}_q^{\mu+p-2}\ f_2(z)\right)'\right]'$$

$$= \left(1-\frac{\lambda p}{\mu+p}\right)^2 z^{-p}\left(\mathcal{H}_q^{\mu+p-2}\ f_1(z) * \mathcal{H}_q^{\mu+p-2}\ f_2(z)\right) + \left[2\left(1-\frac{\lambda p}{\mu+p}\right) z^{1-p}\left(\frac{\lambda p}{\mu+p}\right)^2\right] z^{1-p}\mathcal{H}_q^{\mu+p-2}\ f_1(z) * \left(\mathcal{H}_q^{\mu+p-2}\ f_2(z)\right)' + \left(\frac{\lambda p}{\mu+p}\right)^2 z^{2-p}\left(\mathcal{H}_q^{\mu+p-2}\ f_1(z) * \mathcal{H}_q^{\mu+p-2}\ f_2(z)\right)''.$$

Then we get

$p(z) + \frac{\lambda p}{\mu+p} p'(z) = \mathcal{L}_{\varepsilon,\delta,\tau,p}(f_1(z)) * \mathcal{L}_{\varepsilon,\delta,\tau,p}(f_2(z)) \prec h_1(z) * h_2(z)\ p(z) \prec \frac{\mu+p}{\lambda} z^{\frac{-\mu+p}{\lambda}} \int_0^z t^{\frac{\mu+p}{\lambda}}\ h_1(t) * h_2(t)dt \prec h_1(t) * h_2(t).$

REFERENCES

1. G. I. Oros and Gh. Oros, Strong differential subordination, Turkish J. Math. 33(3) (2009), 249–257.
2. A. W. Goodman, “On the Schwarz-Christoffel transformation and p-valent functions,” Trans. Am. Math. Soc., vol. 68, no. 2, pp. 204–223, 1950.
3. Saeid Shams & S. R. Kulakarni, Jay M. Jahangiri, “Subordination properties of p-valent functions defined by integral operators”, International Journal of Mathematics and Mathematical Sciences, Hindawi, vol. 2006, pages 1-3.
4. G. S. Salagean, Subclasses of univalent functions, Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1, Lecture Notes in Math., vol.1013, Springer, Berlin,1983, 1981, pp: 362-372.
5. S. S. Miller and P. T. Mocanu, Differential subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.
6. M. K. Aouf, R. M. El-Ashwah and A. M. Abd-Eltawab, Differential subordination and superordination results of higher-order derivatives of p-valent functions involving a generalized differential operator, Southeast Asian Bull. Math. 36(4) (2012), 475–488.
7. N. E. Cho, O. S. Kwon and H. M. Srivastava, Strong differential subordination and superordination for multivalently meromorphic functions involving the Liu-Srivastava operator, Integral Transforms Spec. Funct. 21(8) (2010), 589–601.
8. K. Piejko and J. Sokół, “On the Dziok–Srivastava operator under multivalent analytic functions,” Appl. Math. Comput., vol. 177, no. 2, pp. 839–843, 2006.
9. S. S. Miller and P. T. Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl. 65 (1978), 289–305.