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CHARACTERIZATION OF A GENERAL FORM OF DISTRIBUTIONS VIA LINEAR REGRESSION OF ORDER STATISTICS FROM OVERLAPPING SAMPLES AND PRODUCT MOMENTS

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ABSTRACT

In this paper, we have obtained a characterization result via linear regression of order statistics from overlapping samples for a general form of distribution which includes the triplet of Power function, Pareto and Exponential distributions. Product moments of the two non-adjacent order statistics from overlapping samples for a general form of distribution are also obtained.

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1. INTRODUCTION

Let $X_1, X_2, ...,$ are independently and identically distributed random variables. Further suppose that $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$, denote the order statistics for the sample of size $n \ge 1$. In this paper we have characterize a general form of continuous distribution functions

$$F(x) = 1 - [ax + b]^c \quad (1.1)$$

through the linear regression of order statistics from overlapping samples i.e. using the linear regression

$$E[X_{s:m} | X_{r:n} = x] = a_1 x + b_1, \quad 1 \le s \le m, \quad 1 \le r \le n, \quad s \ge r, \quad m < r. \quad (1.2)$$

The distribution (1.1) contains the triplet of Power, Pareto and Exponential distributions. We have also obtained the product moments $E[X_{r:n}^p X_{s:m}^q]$ for the distribution (1.1).

For the case $m = n$, many characterization results have been obtained by different researchers for both adjacent and non-adjacent order statistics. References may be made to Fisz [10], Rogers [21], Ferguson [8], Ferguson [9] Ahsanullah and Wesolowski [1], Dembinska and Wesolowski [5] López-Blázquez and Moreno-Rebollo [16] and many more. Dembinska and Wesolowski [6], characterized the triplet of exponential, Pareto and Power distributions for $m = n$. For $m = n$ Khan and Abu-Saleh [15] have characterized a general form of distribution by the adjacent order statistics which was further obtained for non-adjacent order statistics by Khan and Abouammoh [13].

For the case $m \ne n$, a less work have been done yet. Ahsanullah and Nevzorov [2] characterized the triplet of Power, Pareto and Exponential distributions for $r = s = 1$ and $m < n$. Wesołowski and Gupta [22] considered only a special case $s = m = 1$. Dołęgowski and Wesołowski [7] have shown that for $s \ge r$, $m < n$, the linearity of regression (1.2) holds for the triplet of Power, Pareto and Exponential distributions. They have shown that for Power distribution $a < 1$, for Pareto distribution $a > 1$ and for Exponential distribution $a = 1$.

In Section 2 of this paper, we have given a unified approach for the characterization of all the triplet of Power, Pareto and Exponential distributions through the linear regression (1.2) in one result by just characterizing the distribution function (1.1). For this we have exploited the very useful result of Dołęgowski and Wesołowski [7] which is given as:

Let $X_1, X_2, ... X_n$ be n independently identically distributed continuous random variables, thus for any $m < n \in N$, $1 \le s \le m$, $1 \le r \le n$, $s \ge r$,

$$E[X_{s:m} | X_{r:n}] = \sum_{l=i}^{n-m+i} \frac{\binom{l-1}{i-1}\binom{n-l}{m-i}}{\binom{n}{m}} E[X_{l:n} | X_{r:n}] \quad . \tag{1.3}$$

Further , the product moments of two order statistics from overlapping samples are also obtained in Section 3.

2. CHARACTERIZATION THEOREM

Theorem 2.1. Let $X_1, X_2, ...,$ be a sequence of independently and identically distributed continuous random variables with *df* $F(x)$ and *pdf* $f(x)$ with support (α, β), where α, β may be finite or infinite. Then for $1 \le r \le n$, $1 \le s \le m$, $r \le s$ and $m < n$,

$$E[X_{s:m} | X_{r:n} = x] = a_1 x + b_1 \tag{2.1}$$

if and only if

$$\bar{F}(x) = [ax + b]^c \text{ where } a \ne 0,\ c \ne 0,\ x \in (\alpha, \beta),\ \bar{F}(x) = 1 - F(x) \tag{2.2}$$

Where $a_1 = \sum_{k=s}^{n-m+s} \frac{\binom{k-1}{s-1}\binom{n-k}{m-s}}{\binom{m}{n}} \prod_{j=0}^{k-r-1} \frac{c(n-r-j)}{c(n-r-j)+1}$

$$b_1 = -\frac{b}{a} \sum_{l=s}^{n-m+s} \sum_{j=0}^{l-r-1} \frac{\binom{l-1}{s-1}\binom{n-l}{m-s}}{\binom{n}{m}} \frac{1}{c(n-l+j+1)} \prod_{i=0}^{j} \frac{c(n-l+s+1)}{c(n-l+s+1)+1} .$$

Proof. First we will prove that (2.2) implies (2.1). From khan and Abouammoh [13], we have for the *df* given in (1.1)

$$E[X_{l:m} | X_{r:n} = x] = \left(\prod_{j=0}^{l-r-1} \frac{c(n-r-j)}{c(n-r-j)+1} \right) x - \frac{b}{a} \left[\sum_{j=0}^{l-r-1} \frac{1}{c(n-l+j+1)} \prod_{i=0}^{j} \frac{c(n-l+i+1)}{c(n-l+i+1)+1} \right] \tag{2.3}$$

Substituting (2.3) in (1.3), the relationship (2.1) is established.

We have used Rao and Shanbhag [20] result and López-Blázquez *et al.* [17] result to show that (2.1) implies (2.2). For the sake of completeness, we present here these results.

Rao and Shanbhag [20] proved that:

Let $\int_{R_+} G(u+v)\mu(du) = G(v) + c^*$ $a\ e$ [L] for $u \in R_+ = [0, \infty)$, where $G: R_+ \to R = (-\infty, \infty)$ is locally integrable Borel measurable function and μ is a $\sigma-$finite measure on R_+ with $\mu(\{0\}) < 1$, then

$$G(x)=\begin{cases}\gamma+\alpha'[1-e^{\eta x}] & a.e.[L]\ if\ \eta\neq 0 \quad (2.4)\\ \gamma+\beta' x & a.e.[L]\ if\ \eta=0 \quad (2.5)\end{cases}$$

where α', β', γ' are constant and η is such that

$$\int_{R_+} e^{\eta x}\mu(dx)=1 \qquad (2.6)$$

And López-Blázquez *et al.* [17] have given the conditional *pdf* of order statistics from overlapping samples as

$$f_{X_{s:m}|X_{r:m}=x}(y)=\sum_{k=s}^{s+n-m}\frac{\binom{k-1}{s-1}\binom{n-k}{m-s}}{\binom{n}{m}}f_{X_{k:n}|X_{r:n}=x}(y) \qquad . \qquad (2.7)$$

To prove (2.1) implies (2.2), we have

$$E[X_{s:m}|X_{r:n}=x]=a_1x+b_1$$

$$\Rightarrow \quad \int_x^\beta y f_{X_{s:m}|X_{s:m}}(y)dy=a_1x+b_1 \qquad (2.8)$$

Using (2.7) in (2.8), we have

$$\int_x^\beta y\sum_{k=s}^{s+n-m}\frac{\binom{k-1}{s-1}\binom{n-k}{m-s}}{\binom{n}{m}}f_{X_{k:n}|X_{r:n}=x}(y)dy=a_1x+b_1$$

$$\Rightarrow \quad \int_x^\beta y\sum_{k=s}^{s+n-m}\frac{\binom{k-1}{s-1}\binom{n-k}{m-s}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\frac{[F(y)-F(x)]^{k-r-1}[1-F(y)]^{n-k}}{[1-F(x)]^{n-r}}f(y)dy=a_1x+b_1$$

$$\sum_{k=s}^{s+n-m}\frac{\binom{k-1}{s-1}\binom{n-k}{m-s}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\int_x^\beta y\left(\frac{\bar F(y)}{\bar F(x)}\right)^{n-k}\left(1-\frac{\bar F(y)}{\bar F(x)}\right)^{k-r-1}\frac{f(y)}{\bar F(x)}dy=a_1x+b_1$$

Setting $\bar F(x)=e^{-v}$ and $\bar F(y)=e^{-(u+v)}$, we have

$$\sum_{k=s}^{s+n-m}\frac{\binom{k-1}{s-1}\binom{n-k}{m-s}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\int_0^\infty \bar F^{-1}[e^{-(u+v)}][1-e^{-u}]^{k-r-1}e^{-(n-k+1)u}du=a_1\bar F^{-1}(e^{-v})+b_1$$

Setting $G(v)=\bar F^{-1}(e^{-v})$ and consequently $G(u+v)=\bar F^{-1}(e^{-(u+v)})$, we have

$$\int_0^\infty G(u+v)\mu(du)=G(v)+\frac{b_1}{a_1}$$

where $\mu(du)=\frac{1}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}[1-e^{-u}]^{k-r-1}e^{-(n-k+1)u}du$. (2.9)

For $\eta\neq 0$, we have

$G(x)=\gamma+\alpha'[1-e^{\eta x}]$

or $G(v)=\gamma+\alpha'[1-e^{\eta v}]$

or $e^{-v}=\bar{F}[\gamma+\alpha'(1-e^{\eta v})]$

Let $\gamma+\alpha'[1-e^{\eta v}]=z \quad \Rightarrow \frac{z-\gamma}{\alpha'}=1-e^{\eta v}=1-(e^{-v})^{-\eta} \;\Rightarrow e^{-v}=\left[1-\frac{(z-\gamma)}{\alpha'}\right]^{-1/\eta}=\bar{F}(z)$

$\Rightarrow \quad e^{-v}=[az+b]^{c}$, where $a=-\frac{1}{\alpha'}$, $b=1+\frac{\gamma}{\alpha'}$, $c=-\frac{1}{\eta}$

Now, to see the relationship amongst a_1, b_1 and a, b, c, we have from (2.6) and (2.9)

$\int_0^\infty e^{\eta u}\mu(du)=1$

or $\frac{1}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\int_0^\infty e^{-(n-k+1-\eta)u}(1-e^{-u})^{k-r-1}du=1$

Let $e^{-u}=t \Rightarrow du=-\frac{dt}{t}$.

Thus we have,

$\frac{1}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\int_0^1 t^{n-k-\eta}(1-t)^{k-r-1}dt=1$

$$a_1=\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}B(n-k-\eta+1,k-r) \quad =\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\prod_{j=0}^{k-r-1}\frac{c(n-r-j)}{c(n-r-j)+1}$$

For b_1, we have from Rao and Shanbhag [20] at $v=0$, $G(0)=\gamma$, at $\eta\neq 0$.

Also $\int_{R_+}G(u)\mu(du)=G(0)+\frac{b_1}{a_1}$

$$\frac{1}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\int_0^\infty F^{-1}(e^{-u})(1-e^{-u})^{k-r-1}e^{-(n+k+1)u}du=\gamma+\frac{b_1}{a_1}$$

or

$$\frac{1}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\left[\int_0^\infty[\gamma+\alpha'(1-e^{\eta u})](1-e^{-u})^{k-r-1}e^{-(n+k+1)u}du\right]=\gamma+\frac{b_1}{a_1}$$

Setting $e^{-u}=t$, we get

$$\frac{1}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\left[\int_0^1(\gamma+\alpha')t^{n-k}(1-t)^{k-r-1}dt\right.$$

$$\left.-\int_0^1\alpha' t^{n-k-\eta}(1-t)^{k-r-1}dt\right]=\gamma+\frac{b_1}{a_1}$$

$$\Rightarrow\quad \frac{1}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\left[(\gamma+\alpha')B(n-k+1,k-r)-\alpha' B(n-k-\eta+1,k-r)\right]$$

$$\Rightarrow\quad \gamma\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}B(n-k+1,k-r)$$

$$+\alpha'\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\left[B(n-k+1,k-r)-B(n-k-\eta+1,k-r)\right]=\gamma a_1+b_1$$

Thus, we get

$$b_1=(\gamma+\alpha')\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\left[B(n-k+1,k-r)-B(n-k-\eta+1,k-r)\right]$$

$$\Rightarrow\quad a=-\frac{1}{\alpha'},\ b=1+\frac{\gamma}{\alpha'},\ c=-\frac{1}{\eta}\quad\Rightarrow\gamma+\alpha'=-\frac{b}{a},\ a=-\frac{1}{\alpha'},\ c=-\frac{1}{\eta}$$

Hence

$$b_1 = -\frac{b}{a}\left[\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}} - a_1\right].$$

Remark 2.1. For $\eta = 0$ *i.e.* $c \to \infty$, we have by Rao and Shanbhag [20] result

$$G(v) = \bar{F}^{-1}(e^{-v}) = \gamma + \beta' v$$

or $\quad \bar{F}(\gamma + \beta' v) = e^{-v}$

Substituting $\gamma + \beta' v = z$, we get $\bar{F}(z) = e^{-\frac{z-\gamma}{\beta'}}$

.

Therefore $\quad \bar{F}(z) = \left[1 - \frac{1}{\beta' c}(z-\gamma)\right]^c$ as $c \to \infty$

$= [az + b]^c$,

where $\quad a = -\frac{1}{\beta' c}$, $b = 1 + \frac{\gamma}{\beta' c}$.

Thus $\quad a_1 = \sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}$ as $c \to \infty$.

To find b_1, we have by putting $v = 0$

$$\int_0^\infty G(u)\mu\, du = G(0) + \frac{b_1}{a_1} = \gamma + \frac{b_1}{a_1}$$

$$\int_0^\infty (\gamma + \beta' u)\mu\, du = G(0) + \frac{b_1}{a_1} = \gamma + \frac{b_1}{a_1}$$

or $\quad \gamma + \frac{b_1}{a_1} = \frac{\gamma}{a_1} + \frac{\beta'}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\int_0^\infty u\, e^{-(n-k+1)u}(1-e^{-u})^{k-r-1} du$.

Let $e^{-u} = t$, we get

$$\gamma+\frac{b_1}{a_1}=\frac{\gamma}{a_1}-\frac{\beta'}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}\int_0^1(\ln t)t^{n-k}(1-t)^{k-r-1}du$$

$$\gamma+\frac{b_1}{a_1}=\frac{\gamma}{a_1}-\frac{\beta'}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\frac{(n-r)!}{(k-r-1)!(n-k)!}$$

$$\times B(n-k+1,k-r)\left[\psi(n-k+1)-\psi(n-r+1)\right],$$

where $\psi(x)=\frac{d}{dx}\ln\Gamma(x)$ [Gradsthteyn and Ryzhik [11], P. 538].

Also from Gradsteyn and Ryzhik [11], P. 945, we have

$$\psi(x-p)-\psi(x)=-\sum_{j=1}^{p}\frac{1}{x-k_j}.$$

Therefore, we have

$$\gamma+\frac{b_1}{a_1}=\frac{\gamma}{a_1}-\frac{\beta'}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\left[\psi(n-k+1)-\psi(n-r+1)\right]$$

$$\gamma+\frac{b_1}{a_1}=\frac{\gamma}{a_1}+\frac{\beta'}{a_1}\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\sum_{j=1}^{k-r}\frac{1}{n-r+1-j}.$$

Since $a_1\to 1$ as $c\to\infty$, we have

$$b_1=\beta'\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\sum_{j=1}^{k-r}\frac{1}{n-r+1-j}.$$

Remark2.2: For single sample *i.e.* for $m=n$, Theorem 2.1 reduces to the result obtained by Khan and Abouammoh [13].

Examples

1. Power function distribution

$$\bar{F}(x)=\left(\frac{\nu-x}{\nu-\mu}\right)^{\theta}=\left[-\frac{1}{\nu-\mu}x+\frac{\nu}{\nu-\mu}\right]^{\theta},\quad \mu\le x\le\nu,$$

where $a=-\frac{1}{\nu-\mu}$, $b=\frac{\nu}{\nu-\mu}$, $c=\theta$

$$a_1=\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\prod_{j=0}^{k-r-1}\frac{\theta(n-r-j)}{\theta(n-r-j)+1}<1$$

$$b_1=\nu\left[\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}-a_1\right].$$

2. Pareto distribution

$$\bar{F}(x)=\left(\frac{x+\delta}{\mu+\delta}\right)^{\theta}=\left[\frac{1}{\mu+\delta}x+\frac{\delta}{\mu+\delta}\right]^{\theta}, \quad \mu\le x<\infty,$$

where $a=\frac{1}{\mu+\delta}$, $b=\frac{\delta}{\mu+\delta}$, $c=-\theta$.

$$a_1=\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\prod_{j=0}^{k-r-1}\frac{\theta(n-r-j)}{\theta(n-r-j)-1}>1,\ b_1=\delta\left(a_1-\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\right).$$

3. Exponential distribution

$$\bar{F}(x)=e^{-\lambda(x-\mu)}, \quad x\ge\mu$$

$$=\left(1-\frac{\lambda(x-\mu)}{c}\right)^{c}, \quad c\to\infty\ a=-\frac{\lambda}{c},\quad b=\frac{c+\lambda\mu}{c}$$

Obviously $a_1=1,\ b_1=\beta'\sum_{k=s}^{s+n-m}\frac{\binom{n-k}{m-s}\binom{k-1}{s-1}}{\binom{n}{m}}\sum_{j=1}^{k-r}\frac{1}{n-r+1-j}.$

3. Product moments

From López-Blázquez *et al.* [17], we have the joint density of order statistics from overlapping samples

$$f_{X_{s:m},X_{r:n}}(y,x)=\sum_{k=s}^{s+n-m}\frac{\binom{k-1}{s-1}\binom{m-k}{n-s}}{\binom{m}{n}}f_{k,j:n}(y,x) \qquad (3.1)$$

From Arnold *et. al* [3] and David and Nagaraja [2004], we have

$$f_{k,r:n}(y,x)=\frac{n!}{(r-1)!(k-r-1)!(n-k)!}[F(x)]^{r-1}[F(y)-F(x)]^{k-r-1}\times[1-F(y)]^{n-k}f(x)f(y) \tag{3.2}$$

Theorem: For $1\le s\le m$, $1\le r\le n$, $m\le n$ and for the distribution given in (2.2), $s\ge r$

$$E[X_{r:n}^{p}X_{s:m}^{q}]=(-1)^{p+q}\left(\frac{b}{a}\right)^{p+q}\sum_{k=s}^{s+m-n}\sum_{u=0}^{p}\sum_{v=0}^{q}(-1)^{u+v}\frac{\binom{k-1}{s-1}\binom{m-k}{n-s}}{\binom{m}{n}}\frac{n!}{(n-k)!}\frac{1}{b^{u+v}}\binom{p}{u}\binom{q}{v}$$

$$\times\frac{\Gamma(n+1+v/c)\Gamma(n+1+(u+v)/c-r)}{\Gamma(n+1+v/c-r)\Gamma(n+1+(u+v)/c)} \tag{3.3}$$

Proof: From khan *et al.* [14], we have

$$E[X_{r:n}^{p}X_{s:n}^{q}]=(-1)^{p+q}\left(\frac{b}{a}\right)^{p+q}\frac{n!}{(n-s)!}\sum_{u=0}^{p}\sum_{v=0}^{q}(-1)^{u+v}\frac{1}{b^{u+v}}\binom{p}{u}\binom{q}{v}$$

$$\times\frac{\Gamma(n+1+v/c-s)\Gamma(n+1+(u+v)/c-r)}{\Gamma(n+1+v/c-r)\Gamma(n+1+(u+v)/c)} \tag{3.4}$$

Now consider

$$E[X_{r:n}^{p}X_{s:m}^{q}]=\int_{\alpha}^{\beta}\int_{x}^{\beta}x^{p}y^{q}f_{X_{r:n},X_{s:m}}(x,y)\,dy\,dx \tag{3.5}$$

Using (3.1) in (33), we have

$$E[X_{r:n}^{p}X_{s:m}^{q}]=\sum_{k=s}^{s+n-m}\frac{\binom{k-1}{s-1}\binom{m-k}{n-s}}{\binom{m}{n}}\int_{\alpha}^{\beta}\int_{x}^{\beta}x^{p}y^{q}f_{r,k:n}(x,y)\,dy\,dx=\sum_{k=s}^{s+n-m}\frac{\binom{k-1}{s-1}\binom{m-k}{n-s}}{\binom{m}{n}}E[X_{r:n}^{p}X_{k:n}^{q}] \tag{3.6}$$

Now using (3.4) in (3.6), we get the required result.

Remark: For single sample *i.e.* for $m=n$, (3.3) reduces to the result obtained by Khan *et al.* [14].

Examples

1. Power function distribution

$$\bar{F}(x)=\left(\frac{\beta-x}{\beta-\alpha}\right)^{\theta}=\left(-\frac{1}{\beta-\alpha}x+\frac{\beta}{\beta-\alpha}\right)^{\theta},\quad \alpha\le x\le\beta$$

Here $a=-\dfrac{1}{\beta-\alpha}$, $b=-\dfrac{\beta}{\beta-\alpha}$, $c=\theta$ and

$$E[X_{r:n}^{p}X_{s:m}^{q}]=(-1)^{p+q}\beta^{p+q}\sum_{k=s}^{s+m-n}\sum_{u=0}^{p}\sum_{v=0}^{q}(-1)^{u+v}\frac{\binom{k-1}{s-1}\binom{m-k}{n-s}}{\binom{m}{n}}$$

$$\times\frac{n!}{(n-k)!}\left(\frac{\beta-\alpha}{\beta}\right)^{u+v}\binom{p}{u}\binom{q}{v}\frac{\Gamma(n+1+v/\theta-k)\Gamma(n+1+(v+u)/\theta-r)}{\Gamma(n+1+v/\theta-r)\Gamma(n+1+(v+u)/\theta)}.$$

Taking $\alpha=0$ and $\theta=1$, we get the result for $U(0,\beta)$ distribution as

$$E[X_{r:n}^{p}X_{s:m}^{q}]=(-)^{p+q}\beta^{p+q}\sum_{k=s}^{s+m-n}\sum_{u=0}^{p}\sum_{v=0}^{q}(-1)^{u+v}\frac{\binom{k-1}{s-1}\binom{m-k}{n-s}}{\binom{m}{n}}$$

$$\times\frac{n!}{(n-k)!}\binom{p}{u}\binom{q}{v}\frac{\Gamma(n+1+v-k)\Gamma(n+1+u+v-r)}{\Gamma(n+1+v-r)\Gamma(n+1+u+v)}.$$

For $p=1, q=1$ and $m=n$, we get the result obtained by Malik [19].

And for $U(0,1)$ *i.e.* for $\beta=1$, we have

$$E[X_{r:n}^{p}X_{s:m}^{q}]=\sum_{k=s}^{s+n-m}\frac{\binom{k-1}{s-1}\binom{m-k}{n-s}}{\binom{m}{n}}\frac{r(k+1)}{(n+1)(n+2)}$$

2. Pareto distribution

$$\bar{F}(x)=\left(\frac{\mu+\delta}{x+\delta}\right)^{\theta}=\left(\frac{1}{\mu+\delta}+\frac{\delta}{\mu+\delta}\right)^{-\theta},\quad \mu\le x\le\infty,$$

Here $a=\frac{1}{\mu+\delta}$, $b=\frac{\delta}{\mu+\delta}$, $c=-\theta$ and

$$E[X_{r:n}^{p}X_{s:m}^{q}]=\mu^{p+q}\sum_{k=s}^{s+m-n}\frac{n!}{(n-k)!}\frac{\Gamma(n+1-q/\theta-k)\Gamma(n+1-(p+q)/\theta-r)}{\Gamma(n+1-q/\theta-r)\Gamma(n+1-(p+q)/\theta)}.$$

For $m=n$, we get the result obtained by Huang [12] and For $p=1, q=1$ and $m=n$, we get the result obtained by Malik [18].

3. Exponential distribution

$$\bar{F}(x)=[ax+b]^{c}$$

Let $a=-\frac{\lambda}{c}$, $b=1$, then we have

$$\lim_{c\to\infty}\bar{F}(x)=e^{-\lambda x}.$$

And using the result of khan *et al.* [14]

$$E[X_{r:n}^{p} X_{s:m}^{q}] = \left(\frac{1}{\lambda}\right)^{p+q} p!q! \frac{\sum_{k=s}^{s+m-n} \sum_{u=0}^{s-r-1} \sum_{v=0}^{r-1} \binom{k-1}{s-1}\binom{m-k}{n-s}}{\binom{m}{n}} \binom{k-r-1}{u}\binom{r-1}{v} \frac{n!}{(n-k)!(k-r-1)!(r-1)!}$$

$$\times \frac{n!}{(n-k)!(k-r-1)!(r-1)!(n-r+v+1)^{\alpha+1}(n-k+u+1)^{\beta+1}}.$$

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