ISSN: 2633-4828 Vol. 4 No.3, December, 2022

International Journal of Applied Engineering & Technology

ALGEBRAIC INSIGHTS INTO DESIGN GRAPHS: STRUCTURAL PROPERTIES AND APPLICATIONS

Syeda Farra Fatima Nahri
Department of Mathematics, Milliya Arts, Science, and Management Science College, Beed: 431122, Maharashtra, India.
E mail: syedafarra@gmail.com

ABSTRACT

This paper explores the algebraic approach to design graphs, focusing on their structural and combinatorial properties. Design graphs are regular bipartite graphs where any two distinct vertices of the same color share the same number of common neighbors. These graphs are characterized by parameters such as half-size (mmm), degree (ddd), and shared neighbors (ccc), which satisfy the relation c(m−1)=d(d−1)c(m-1) = d(d-1)c(m−1)=d(d−1). Extremal design graphs, particularly those with diameter 3 and girth 6, minimize vertex count while maximizing structural efficiency. Examples like the cycle C6C_6C6, Heawood graph, and Tutte-Coxeter graph illustrate these concepts. Additionally, finite fields (FFF) are employed to construct graph families and analyze their symmetries through field extensions, Frobenius automorphism, trace, and norm functions. The incidence graphs derived from finite-dimensional vector spaces over finite fields also exhibit design graph properties. These graphs demonstrate regularity, symmetry, and algebraic richness, making them valuable in combinatorics and network theory. The paper also investigates quadratic signatures and their algebraic properties, providing a robust framework for understanding graph-theoretical problems through algebraic methods. By leveraging algebraic tools, the study enhances the characterization of design graphs and contributes to applications in coding theory, cryptography, and combinatorial optimization.

INTRODUCTION

Design graphs, a fascinating class of regular bipartite graphs, play a crucial role in algebraic graph theory due to their symmetry, regularity, and combinatorial richness. A design graph is defined as a regular bipartite graph where any two distinct vertices of the same color share an equal number of common neighbors. While the complete bipartite graph Kn, nK_ {n,n}Kn,n satisfies this definition, it is generally excluded from consideration by convention. These graphs are characterized by three key parameters: the half-size (mmm), degree (ddd), and the number of common neighbors (ccc) shared by monochromatic vertex pairs, satisfying the algebraic relation c(m−1)=d(d−1)c(m-1) = d(d-1)c(m−1)=d(d−1).

Design graphs exhibit unique extremal properties, especially in cases where the graph has a diameter of 3 and a girth of either 4 or 6. When the parameter ccc equals 1, these graphs minimize the vertex count for regular graphs of girth 6, earning the designation of *extremal design graphs*. Examples of extremal design graphs include the cycle C6C_6C6 and the Heawood graph, both of which demonstrate exceptional structural balance and symmetry.

Partial design graphs extend the concept by allowing two distinct values (c1c_1c1 and c2c_2c2) for shared neighbor counts. Notable examples include the cube graph (QnQ_nQn) and the Tutte-Coxeter graph. The latter showcases intriguing algebraic properties with parameters c1=0c_1 = 0c1=0 and c2=1c_2 = 1c2=1, as well as structural characteristics such as a diameter of 4 and specific adjacency patterns between vertices.

Finite fields provide an essential algebraic tool for constructing and analyzing design graphs. Finite fields, defined by q=pdq = p^dq=pd (where ppp is a prime number and ddd is a positive integer), exhibit cyclic multiplicative groups and automorphisms, such as the Frobenius automorphism, which facilitate symmetry analysis in graph structures. These algebraic properties enable the construction of incidence graphs, where vertices represent subspaces of finite-dimensional vector spaces, and edges represent inclusion relationships.

The study of design graphs through algebraic tools not only advances our understanding of their structural intricacies but also contributes to practical applications in combinatorial optimization, network design, and cryptographic systems. This paper aims to bridge the gap between algebraic theory and combinatorial graph properties, offering a comprehensive framework for analyzing and constructing design graphs using algebraic methods.

METHODS

Design Graphs.
Next, we consider bipartite analogues of strongly regular graphs. A design graph is a regular bipartite graph with the property that any two distinct vertices of the same colour have the same number of common neighbours. The complete bipartite graph Kn,n fits the definition, but we exclude it by convention. For a design graph, we let m denote the half-size, d the degree, and c the number of neighbours shared by any monochromatic pair of vertices. Note that the parameter c is, in fact, determined by the following relation:

$c(m-1) = d(d-1)$

This is obtained by counting in two ways the paths of length 2 joining a fixed vertex with the remaining $m-1$ vertices of the same colour. As an immediate consequence of the definition, we have following fact.

Proposition 1: The next exercise focusses on another extremal property. A design graph with diameter 3 and girth 4 or 6 depends on whether c > 1 or c = 1. On the other hand, a regular bipartite graph with diameter 3 and girth 6 is a design graph with parameter c = 1. These design graphs are especially intriguing because the previous proposition identifies them as being extremal for the girth among regular bipartite graphs of diameter 3.

Exercise 1. Design graphs with parameter c = 1 and degree d are the ones that minimize the number of vertices among all d-regular graphs of girth 6. It seems appropriate to refer to design graphs with parameter c = 1 as extremal design graphs. There are some examples of extremal design graphs among the graphs that we already know. One is the cycle C6. The other, more interesting, is the Heawood graph. In the next section, we will construct more design graphs, some of them extremal, by using finite fields. A partial design graph is a regular bipartite graph with the property that there are only two possible values for the number of neighbours shared by any two distinct vertices of the same colour.

The parameters of a partial design graph are denoted m, d, respectively $c1$ and $c2$. Note that $c1$ 6= $c2$, and that the roles of $c1$ and $c2$ are interchangeable.

Example 2. The bipartite double of a (non-bipartite) strongly regular graph is a design graph or a partial design graph.

Example 3. Since the cube graph Qn is a partial design graph, its parameters are c1 = 0 and c2 = 2. To illustrate this, consider the bipartition provided by weight parity: Fix two different strings with the same weight parity. If they differ in two slots, they have two common neighbours; if not, they have no common neighbours.

Example 4. Consider the complete graph K6. It has 15 edges. A matching is a choice of three edges with distinct endpoints, i.e., a partition of the six vertices into two-element subsets. There are 15 matchings, as well. Define a bipartite graph by using the edges and the matchings as vertices, and connecting matchings to the edges they contain. This is the Tutte - Coxeter graph, drawn in Figure 1 below.

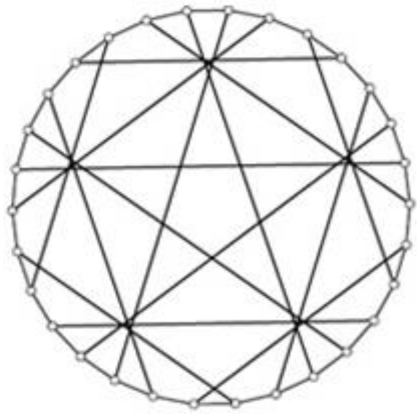

Figure 1: The Tutte - Coxeter graph.

In fact, a matching is adjacent to three edges, and an edge is adjacent to three matchings. The joint parameters are c1 = 0, c2 = 1. Two distinct edges are adjacent to a unique matching if they have disjoint endpoints, and to no matching otherwise. The Tutte-Coxeter graph is a partial design graph with a half-size of m = 15 and a degree of d = 3. Two distinct matchings share no more than one edge.

A further property of the Tutte - Coxeter graph is that it has diameter 4 and girth. The verification is left to the reader.

Finite Fields

In this section, and in the next one, the focus is on finite fields. Among other things, we construct some interesting families of graphs, and we acquire some of the tools that later on will prove useful for studying these graphs. Let us start by recalling some fundamental facts about finite fields. A finite field has q = pd elements for some prime p, the characteristic of the field, and some positive integer d, the dimension of the field over its prime subfield. For each prime power q there exists a field with q elements, which is furthermore unique up to isomorphism. We think of Zp = Z/pZ as 'the' field with p elements.

In general, 'the'field with $q = pd$ elements can be realized as a quotient Zp[X]/(f (X)),

where the irreducible polynomial of degree d is f (X) = Zp[X]. There is a polynomial f for every given d, but there is no known general formula for generating one.

The multiplicative group of a finite field is cyclic. Once more, this is a non-constructive existence: there is no known formula for creating a multiplicative generator given a field. We now discuss finite field extensions. In this regard, the outcome that follows is crucial.

Theorem. Let K be a field with qn elements. Then the map φ: K→K, given by

$\varphi(a) = aq$, has the following properties:

(i) φ is an automorphism of K of order n;

(ii) F = {$a \in$ K: φ (a) = a} is a field with q elements, and φ is an F-linear isomorphism when K is viewed as a linear space over F.

Proof. (i) Clearly, φ is injective, multiplicative, and φ(1)= 1. To see that φ is additive, we iterate the basic identity (a+b)p = ap +bp, where p is the characteristic of K, up to (a+b)q = aq +bq. Thus φ is automorphism of K. Each a ∈K∗ satisfies the relation aqn−1 = 1, so aqn = a for all a K. Thus φn is the identity map on K. Assuming that φt is the identity map on K for some 0 < t < n, we would get that Xqt = X has qn solutions in K, a contradiction.

(ii) F is a subfield since φ is an automorphism. As K is cyclic of order $qn - 1$, and $q - 1$ divides $qn - 1$, there are precisely $q - 1$ elements a K satisfying $aq{-}1 = 1$.

Therefore F has q elements. Finally, note that $\varphi(ab) = \varphi(a)\,\varphi(b) = a\varphi(b)$ whenever a = F and b = K.

Turning the above theorem on its head, we can start with a field F with q elements, and then consider a field K with qn elements as an overfield of F. Then K is an n-dimensional linear space over F, and we say that K is an extension of F of degree n. The map φ is called the Frobenius automorphism of K over F.

Exercise 2. Let F consist of q items. Prove that an element in the generic linear group GLn(F) has a maximal order of qn −1.

Let F be a field with q elements, and let K be an extension of F of degree n. The trace and the norm of an element a =K are defined as follows:

$$\mathrm{Tr}(a) = \sum_{k=0}^{n-1} \phi^k(a) = a + a^q + \ldots + a^{q^{n-1}}, \qquad \mathrm{N}(a) = \prod_{k=0}^{n-1} \phi^k(a) = a \cdot a^q \cdot \ldots \cdot a^{q^{n-1}}$$

One might think of the trace and the norm as the additive, respectively the multiplicative content of a Frobenius orbit. The importance of the trace and the norm stems from the following properties.

Theorem. The trace is additive, in fact F-linear, while the norm is multiplicative. The trace and the norm map K onto F.

Proof. Let's move on to the second statement, which is self-evident: the trace and the norm are Frobenius-invariant, φ(Tr(a)) = Tr(a) and φ(N(a)) =N(a), so they are F-valued. Tr:K→F is an additive homomorphism; its kernel size is at most qn−1, meaning that the size of its image is at least qn/qn−1 = q. As a result, Tr is onto. Similarly, N: K∗ →F∗ is a multiplicative homomorphism; its kernel size is at most (qn −1)/(q−1), meaning that the size of its image is at least q−1.

Projective combinatorics.

Let F be a field with q elements, and consider a linear space V of dimension n over F. We think of V as an ambient space, and we investigate the geometry and combinatorics of its subspaces. A k-dimensional subspace of V is called a k-space in what follows.

Proposition. The number of *k*-spaces is given by the *q*-binomial coefficients:

$$\binom{n}{k}_q := \frac{(q^n-1)\ldots(q^n-q^{k-1})}{(q^k-1)\ldots(q^k-q^{k-1})} = \frac{(q^n-1)\ldots(q^{n-k+1}-1)}{(q^k-1)\ldots(q-1)}$$

Proof. There exist (qn −1) . . . (qn − qk−1) ordered ways of choosing k linearly independent vectors. Some of these selections span one and the same k-space. In particular, the bases of a k-space are (qk−1)... (qk − qk−1) ordered. This is equivalent to the previous count's scenario where n = k.

We obtain the following practical result when we apply the previous proposition to a quotient space of V.

Incidence graphs

Let *V* be an ambient linear space of dimension $n \geq 3$ over a finite field F with q elements. The *incidence graph In* (q) is the bipartite graph whose vertices are the 1-spaces respectively the (n−1)-spaces, and whose edges connect 1-spaces to (n−1)- spaces containing them. Note that the construction only depends on the dimension of *V*, and not on *V* itself; hence the notation.

Theorem. The incidence graph *In* (*q*) is a design graph, with parameters

$$m = \binom{n}{1}_q = \frac{q^n - 1}{q - 1}, \qquad d = \binom{n-1}{1}_q = \frac{q^{n-1} - 1}{q - 1}, \qquad c = \binom{n-2}{1}_q = \frac{q^{n-2} - 1}{q - 1}.$$

Proof. By definition, the incidence graph *In*(*q*) is bipartite. The half-size *m*, as well as the degree *d* for each vertex, are given by Proposition 5.6, Corollary 5.7 and the symmetry property of the *q*-binomial coefficients. To check the design property, we use the dimensional formula:

Dim ($W+W'$) + dim ($W \cap W'$) = dim W + dim W'

where W and W' are subspaces of V.

Let W and W′ be distinct (n−1)-spaces. The common neighbours of W and W′ are the 1 spaces contained in W∩W′. Now W+W′ has dimension n, so W∩W′ is a (n−2)-space. The number of 1-spaces contained in a (n−2)-space is (1n-2)q.

Similiarly, let W and W' be distinct 1-spaces. The common neighbours of W and W' are the (n−1)-spaces containing $W + W'$. This is a 2-space, since $W \cap W'$is 0- dimensional. The number of (n−1)-spaces containing a given 2-space is $(1^{n-2})_q$.

Example 1. The incidence graph I3(q) is an extremal design graph. Its half-size is m= q2 + q+1, and it is regular of degree d = q+1. For q = 2, this is the Heawood graph. Assume now that V is endowed with a scalar product. Proposition 5.8 provides a bijection between the (n−1)-spaces and the 1-spaces. Using this bijection, we can give an alternate, and somewhat simpler, description of In(q). Take two copies of the set of 1-spaces, that is, lines through the origin, and join a 'black' line to a 'white' line whenever they are orthogonal. Note that the orthogonal picture is independent of the choice of scalar product, for it agrees with the original incidence picture.

If you consider Fn as the subspace of Fn+1 with a vanishing final coordinate, then the lines of Fn are also lines in Fn+1, and they are orthogonal in Fn if and only if they are orthogonal in Fn+1. This orthogonal picture makes it clear right away that In(q) is an induced subgraph of In+1(q).

The incidence graph *In*(*q*) enjoys a regularity property that is not obvious at first sight. Here, the orthogonal picture turns out to be very useful.

Around Squares

Given by x 7→ x2, the squaring homomorphism F∗ →F∗ is two-to-one: x2 = y2 if and only if x = ±y. Therefore, half of F∗'s elements are squares, whereas the other half are not. The following provides the quadratic signature σ: F∗→{±1}:

$\sigma(a)$ = {1 if a is a square in F∗, { −1 if a is not a square in F∗.

Theorem. The quadratic signature σ is multiplicative on F∗, and it is explicitly given by the 'Euler formula' $\sigma(a) = a^{(q-1)/2}$.

Proof. For a ∈ F∗, let τ(a) = a(q−1)/2. Because τ(a)2 = a(q−1) = 1, take note of τ(a) = ±1. Additionally, take note that τ does accept −1; otherwise, there would be an excessive number of solutions to the equation X(q−1)/2 = 1 in F. τ: F∗ → {±1} is an onto homomorphism, hence. The size of its kernel is 1/2 (q −1), and the number of non-zero squares in it is likewise 1/2 (q −1).

As a result, the non-zero squares make up the exact kernel of τ. This means that τ = σ, where τ(a) = 1 if an is a square in F∗ and τ(a) = −1 if an is not a square in F∗.

CONCLUSION

This study delves into the algebraic structure and properties of design graphs, highlighting their combinatorial significance and structural efficiency. Design graphs, characterized by parameters such as half-size (mmm), degree (ddd), and shared neighbors (ccc), exhibit a unique regularity that satisfies the fundamental relation c(m−1)=d(d−1)c(m-1) = d(d-1)c(m−1)=d(d−1). This algebraic relationship serves as the backbone for analyzing their symmetry and extremal properties.

Extremal design graphs, particularly those with diameter 3 and girth 6, minimize vertex count while maintaining high symmetry and structural balance. Classic examples such as the cycle graph (C6C_6C6) and the Heawood graph demonstrate these optimal properties. The study also introduces partial design graphs, which allow two distinct shared neighbor parameters (c1c_1c1and c2c_2c2), exemplified by structures like the cube graph (QnQ_nQn) and the Tutte-Coxeter graph. These graphs exhibit fascinating adjacency properties and serve as benchmarks for studying algebraic graph regularity.

Finite fields (FFF) play a critical role in constructing and analyzing families of design graphs. Finite fields, defined by q=pdq = p^dq=pd, possess cyclic multiplicative groups and automorphisms, such as the Frobenius automorphism, which simplify symmetry analysis. The trace and norm functions further contribute to understanding graph regularity and combinatorial structures within these algebraic systems.

Incidence graphs, derived from finite-dimensional vector spaces over finite fields, provide a rich combinatorial framework. These graphs connect 1-spaces and (n-1)-spaces through well-defined adjacency rules and exhibit design graph properties. For instance, the incidence graph I3(q)I_3(q)I3(q) serves as an extremal design graph with parameters optimized for specific algebraic and combinatorial conditions.

Additionally, the quadratic signature and Euler formula offer powerful tools for analyzing algebraic properties of finite fields and their influence on graph structures. These tools reveal intricate symmetries and optimize adjacency relationships, contributing to a deeper understanding of graph-theoretical problems.

In conclusion, this study demonstrates that algebraic tools, particularly finite fields and their extensions, provide a robust framework for constructing, analyzing, and understanding design graphs. These graphs exhibit remarkable structural regularity, extremal properties, and combinatorial efficiency, making them valuable in fields such as coding theory, cryptography, network optimization, and mathematical combinatorics. Future research may further explore algebraic graph constructions, aiming to uncover new families of design graphs with enhanced properties and practical applications in advanced mathematical and engineering domains.

ACKNOWLEDGEMENT

Author is thankful to the Management, Principal, Teaching and Non Teaching Staff of the Institute for their kind support and motivation.

REFERENCES

1. Orga, C. C., & Ogbo, A. I. Application of Probability Theory in Small Business Management in Nigeria. European Journal of Business and commerce. 2012; 4(12): 72-82.
2. Leontief, W. Input-output analysis, *The new Palgrave. A dictionary of economics*, 2, 1987, 860-64.
3. Karatzas, I., & Shreve, S. E. *Methods of Mathematical Finance.* Springer, 1998 (Vol. 39).
4. Abraham, B. Implementation of Statistics in Business and Industry. Colombian Journal of Statistics, 30, 2007, 1-11.
5. Levine, R., & Zervos, S. Stock markets, banks, and economic growth. American economic review, 1998, 537-558.
6. Nel, C. (2012). Cracking the Vocabulary Code in Mathematics in the Foundation Phase. *South African Journal of Childhood Education*, *2*(2), 15-34.

7. Rosen, K. H. (2011). *Elementary number theory*. London: Pearson Education.
8. Nelson, P., & Weatherald, C. (2014). Cracking the code—An approach to developing professional writing skills. *Social work education*, *33*(1), 105-120.
9. Samanta, S., & Pal, M. (2013). Fuzzy k-competition graphs and p-competition fuzzy graphs. Fuzzy Information and Engineering, 5, 191-204.
10. Rashmanlou, H., Samanta, S., Pal, M., & Borzooei, R. A. (2015). Bipolar fuzzy graphs with categorical properties. International Journal of Computational Intelligence Systems, 8(5), 808-818.
11. Samanta, S., & Pal, M. (2015). Fuzzy planar graphs. IEEE Transactions on Fuzzy Systems, 23(6), 1936-1942.
12. Pramanik, T., Samanta, S., & Pal, M. (2016). Interval-valued fuzzy planar graphs. International journal of machine learning and cybernetics, 7, 653-664.
13. Leontief, W. Input-output analysis, *The new Palgrave. A dictionary of economics, 2, 1987, 860-64.*
14. Levine, R., & Zervos, S. Stock markets, banks, and economic growth. *American economic review, 1998, 537-558.*