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Linguistic Hedge Based Constrained Representation of Interval Type-2 Fuzzy Sets

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Abstract **- In general representation, a type-2 fuzzy set is defined as being composed of all its embedded type-1 fuzzy sets (ET1-FSs) which can be non-convex, sub-normal, and/or discontinuous. However, interval type-2 fuzzy sets (IT2-FSs) are constructed by blurring a baseline type-1 membership function form in many applications, and thus, uncertainties are actually modeled by using only ET1-FSs which preserve the similar meaningful functional form. Therefore, the derived results can be too generic in such cases when all ET1-FSs are included in type-2 fuzzy computations. In this study, a new constrained representation of IT2-FSs is proposed for the above mentioned formation. In this representation, IT2-FSs are defined as being composed of only convex, normal and continuous ET1-FSs by using linguistic hedges. This constrained representation provides ease of computation and more precise results in interval type-2 fuzzy computations. The proposed constrained representation is applied to the centroid computation which is the most important uncertainty measure of IT2-FSs and also an important task in interval type-2 fuzzy logic systems (IT2-FLSs). It is shown that the centroid of the proposed constrained IT2-FS is calculated in closed form without the need of any iterative algorithm. In this way, the computational burden of the centroid type reduction process is removed. The effectiveness of the proposed constrained representation is shown through an illustrative example.**

Index Terms - Centroid type reduction, constrained type-2 fuzzy set, footprint of uncertainty, linguistic hedges, representation theorem.

INTRODUCTION

Type-2 fuzzy sets (T2-FSs) have more capability for handling uncertainties than the conventional type-1 fuzzy sets (T1-FSs) [1,2]. Interval type-2 fuzzy sets (IT2-FSs) are a subset of T2-FSs and their effectiveness is shown in many type-2 fuzzy logic system (T2-FLS) applications [3,4].

There are various ways to represent IT2-FSs [1]. In Mendel-John Representation Theorem [5], the footprint of uncertainty (FOU) of an IT2-FS is represented as the combination of all its embedded type-1 fuzzy sets (ET1-FSs). Since there is no constraint on ET1-FSs; non-convex, sub-normal, and/or discontinuous ET1-FSs which may not correspond to real uncertainty situation on a FS are considered in type-2 fuzzy logic computations.

There are several ways for the formation of T2-FSs in literature [6-8]. One method which is mostly used is to take a baseline type-1 membership function (T1-MF) form and model the uncertainty (FOU) by varying its parameters. This is a natural way for the human perception when modeling uncertainty on a linguistic term [1]. For example, a Gaussian IT2-FS is formed by using Gaussian baseline T1-MF with uncertain standard deviation or mean when the linguistic term is described according to data obtained by polling a group of experts without certain agreement. Similarly, a triangular IT2-FS is formed by using experimentally derived triangular T1-MF distribution of the described linguistic term. Therefore, in such cases, it is actually assumed that all ET1-FSs of the IT2-FS are in the form of the chosen baseline T1-MF with varying parameters and thus, preserve the similar semantic meaning of the baseline T1-FS. However, the derived results can be too generic based on the conventional unconstrained representation of IT2-FSs in such cases since all ET1-FSs are considered in the computations regardless of their forms. In order to illustrate this situation, consider an IT2-FS $\tilde{A}$ given in Figure 1 where the FOU is formed by varying parameters of the baseline T1-MF. The ET1-FSs shown in Figure 1a are convex, normal and continuous FSs that preserve the similar meaningful functional form. On the other hand, the ET1-FS shown in Figure 1b is non-convex, sub-normal, and discontinuous. This ET1-FS is mathematically possible but meaningless and thus, it can cause inappropriate results in T2-FS computations [9]. For example, ET1-FSs which give minimum centroids of $\tilde{A}$ based on unconstrained representation and the actual uncertainty situation are shown in Figure 1c as A_{uc} and A_c, respectively.

As it is seen from Figure 1c, the ET1-FS A_{uc} is discontinuous and thus, hard to be interpreted with respect to the baseline meaning of the FS $\tilde{A}$. On the other hand, A_c can easily be interpreted since it preserves the meaningful triangular form. Moreover, c_{uc} the centroid of A_{uc} is smaller than c_c the centroid of A_c and thus, c_{uc} contains the uncertainties which are not related to actual uncertainty situation.

Although the actual uncertainty situation can be described more properly by considering only meaningful ET1-FSs in the above mentioned cases, in literature, a limited number of studies has been presented for the constrained T2-FSs [9-13]. In [10], the concept of constrained embedded membership function (E-MF) is proposed for T2-FSs formed by taking baseline T1-MF form and allowing a parameter to vary. In this paper, it is shown that constraining ET1-FSs can make computations easier and the results narrower in T2-FLSs. Especially in the type reduction (TR) process, the centroid result computed by using constrained E-MFs is strictly narrower and also more descriptive of the actual situation than the centroid result calculated in the conventional unconstrained case since infeasible E-MFs are eliminated from consideration. Similarly in [9], the concept of constrained T2-FS is introduced by using an alternative constraining mechanism. A constrained T2-FS is constructed by altering the position of a baseline T1-FS on the *x*-axis. Therefore, all ET1-FSs of a constrained T2-FS are in the same form which represents the chosen baseline T1-FS. However, this type of formation of constrained T2-FSs reduces uncertainty modeling capabilities of T2-FSs due to using only ET1-FSs in exactly the same form where only their positions are altered. In addition, although the importance of constraining ET1-FSs is investigated in detail, the TR of proposed constrained T2-FSs is not investigated in this study. In [11], a Constrained Representation Theorem is proposed for IT2-FSs by using only normal and convex ET1-FSs. Additionally, this constrained representation theorem is applied to the calculation of the constrained centroid of IT2-FSs. However; although the convexity and normality of ET1-FSs are preserved, other important properties of ET1-FSs such as meaningfulness, interpretability, and continuity are not considered in this representation.

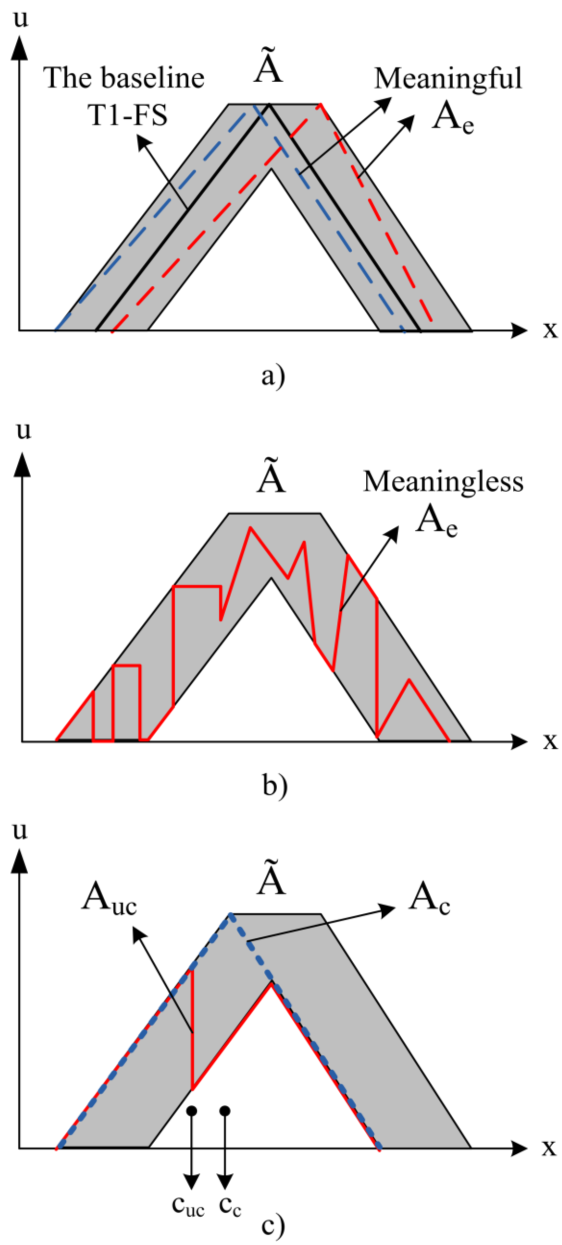

FIGURE 1
ILLUSTRATION OF A) THE MEANINGFUL ET1-FSS AND B) THE MEANINGLESS ET1-FS C) THE CONSTRAINED AND UNCONSTRAINED ET1-FSS GIVING THE MINIMUM CENTROID OF $\tilde{A}$

In this study, a new constrained representation of IT2-FSs is proposed by utilizing the linguistic hedge (LH) concept. One of the important properties of LHs is to preserve the main properties of a FS to which they are applied such as convexity, normality and continuity on the resulting FS. Based on this property of LHs, IT2-FSs which are formed by blurring a baseline type-1 membership function form are represented as the union of ET1-FSs which are obtained by applying LHs to a baseline T1-FS. In this manner, it is provided that an IT2-FS is composed of only convex, normal and continuous ET1-FSs representing the similar semantic meaning of the baseline T1-FS as it actually is. Constraining an IT2-FS in this way describes the actual situation more properly. This constrained representation provides ease of computations and more precise results in interval type-2 fuzzy computations. Although there are various types of membership function forms and LH definitions in literature [13-17], the triangular membership functions and the classical LH definitions proposed by Zadeh [17] are used in this study. The centroid computation of the proposed constrained IT2-FS is also investigated. As an important result, since only constrained ET1-FSs are used, ET1-FSs which give the minimum and maximum centroids of an IT2-FS are determined a priori without any calculation. Thus, the centroid of the IT2-FS is calculated in closed form without the need for any iterative algorithm. In this way, the computational burden of the TR process is removed. The effectiveness of the proposed constrained representation is shown through an illustrative example.

The organization the paper is as follows: Preliminaries are presented in Section 2. The proposed constrained representation is presented in Section 3. In Section 4, the centroid computation of the proposed constrained IT2-FS is investigated. In section 5, an illustrative example is given for the proposed constrained representation. Finally, in section 6, conclusions are outlined.

PRELIMINARIES

I. Interval Type-2 Fuzzy Sets

An IT2-FS $\tilde{A}$ is defined by an IT2-MF as [1]

$$\tilde{A} = \int_{x\in X} \int_{u\in J_x \subseteq [0,1]} 1/(x,u) = \int_{x\in X} \left[\int_{u\in J_x \subseteq [0,1]} 1/u \right] \Big/ x \tag{1}$$

The FOU of $\tilde{A}$ can be defined as the combination of all primary memberships [5]

$$FOU(\tilde{A}) = \bigcup_{x\in X} J_x = \left\{ (x,u) : u \in J_x \subseteq [0,1] \right\} \tag{2}$$

As shown in Figure 2, two T1-MFs limit FOU of $\tilde{A}$. The upper bound of $FOU(\tilde{A})$ is named as Upper MF (UMF), and represented by $\bar{\mu}_{\tilde{A}}(x)$. The lower bound of $FOU(\tilde{A})$ is called as Lower MF (LMF), and represented by $\underline{\mu}_{\tilde{A}}(x)$. Their definitions are given as

$$\bar{\mu}_{\tilde{A}}(x) = \overline{FOU(\tilde{A})} \qquad \forall x \in X \tag{3}$$

$$\underline{\mu}_{\tilde{A}}(x) = \underline{FOU(\tilde{A})} \qquad \forall x \in X \tag{4}$$

Using the LMF and UMF definitions, $FOU(\tilde{A})$ is given as follows

$$FOU(\tilde{A}) = \bigcup_{x\in X} \left[\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x) \right] \tag{5}$$

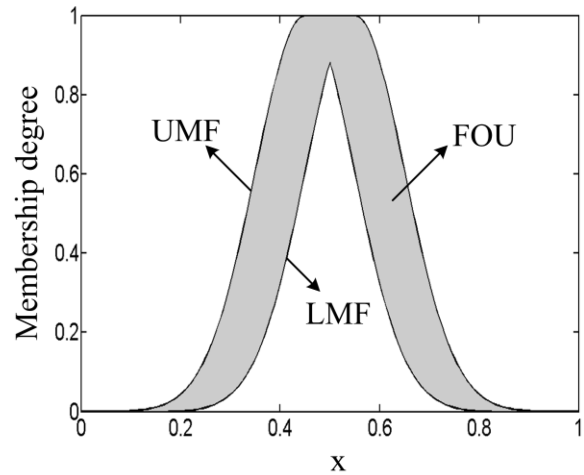

FIGURE 2
IT2-MF

II. Embedded T1-FSs

The infinite number of T1-FSs can be defined in FOU of a T2-FS $\tilde{A}$ [5]. Any defined T1-FS is named as an embedded T1-FS of $\tilde{A}$. For a discrete universe of discourse $X = \{x_1, x_2, \ldots, x_N\}$ and discrete J_x, an ET1-FS A_e has N elements.

$$A_e = \sum_{i=1}^{N} u_i / x_i \qquad u_i \in J_x \subseteq [0,1] \tag{6}$$

Example ET1-FSs are shown in Figure 1.

FOU of $\tilde{A}$ can be represented as the combination of its all ET1-FSs according to Mendel-John Representation Theorem [5] as

$$FOU(\tilde{A}) = \bigcup_{j=1}^{n_A} A_e^j \tag{7}$$

Here $n_A = \prod_{i=1}^{N} M_i$ is the number of ET1-FSs with respect to the discretization level M_i of J_{x_i}. In this general representation, since there is no constraint on ET1-FSs, non-convex, sub-normal, and/or discontinuous ET1-FSs which may cause inappropriate results are also included in the computations as shown in Figure 1b. Additionally, due to the number of discretization, the number of ET1-FSs increases astronomically. For example, if X and J_x are discretized into 100 units, the number of ET1-FSs becomes 100^{100} which is not easy to be evaluated.

III. Linguistic Hedges

The LHs are particular linguistic terms such as *very*, *absolutely*, *more or less*, which are used to change the meaning of FSs by modifying the shape of their membership functions. LH operations can be summarized within two groups; concentration and dilation [17].

The concentration hedge operation of *A*, $h_{con(A)}(x)$, is denoted as

$$h_{con(A)}(x) = \mu_A(x)^{\alpha} \qquad \alpha > 1 \tag{8}$$

where α determines the strength of the concentration. For the concentration type hedge operation, various related hedge operators can be defined such as *absolutely* ($\alpha = 4$), *very* ($\alpha = 2$), *much more* ($\alpha = 1.75$), *more* ($\alpha = 1.5$), *plus* ($\alpha = 1.25$).

The dilation hedge operation of *A*, $h_{dil(A)}(x)$, is denoted as follows

$$h_{dil(A)}(x) = \mu_A(x)^{\alpha} \qquad 0 < \alpha < 1 \tag{9}$$

Similarly, some related dilation type hedge operators can be denoted as *minus* ($\alpha = 0.75$), *more or less* ($\alpha = 0.5$), *slightly* ($\alpha = 0.25$).

In Figure 3, the effects of two of well-known modifiers, "very" from the concentration type and "more or less" from the dilation type LHs are demonstrated on the triangular FS *A*.

It is clearly seen that when the LH "more or less" is applied to the FS, the grades of membership increase and the meaning of the FS is weakened. On the other hand, the LH "very" decreases the grades of membership and strengthens the meaning of the FS.

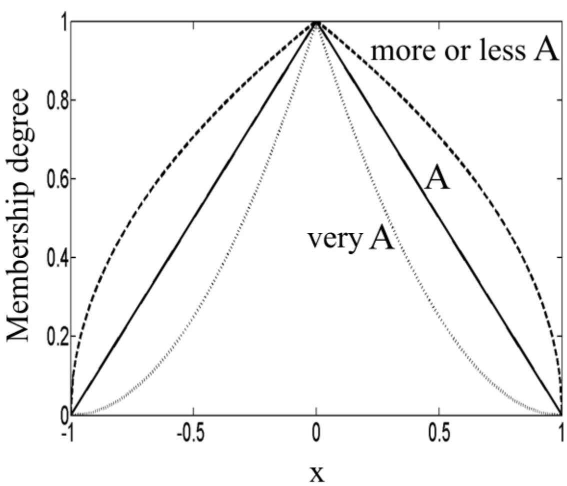

FIGURE 3
THE EFFECTS OF "VERY" AND "MORE OR LESS" LINGUISTIC HEDGES ON THE FUZZY SET A.

IV. Centroid of an IT2-FS

Since there is no uncertainty on the MF of a T1-FS, a T1-FS has only a single centroid value. On the other hand, the centroid of an IT2-FS denotes an interval value (interval T1-FS), because the centroid of an IT2-FS consists of the collection of the centroids of its all ET1-FSs [1].

The centroid, $C(\tilde{A})$, is denoted as

$$C(\tilde{A}) = \bigcup_{\forall A_e} c(A_e) = \left[c_l(\tilde{A}), c_r(\tilde{A})\right] \tag{10}$$

Here $\bigcup$ represents a union operator and the minimum centroid and the maximum centroid of $\tilde{A}$ are denoted as

$$c_l(\tilde{A}) = \min_{\forall A_e} c(A_e) \tag{11}$$

$$c_r(\tilde{A}) = \max_{\forall A_e} c(A_e) \tag{12}$$

c_l and c_r can be calculated as follows [20]

$$c_r(\tilde{A}) = \frac{\int_{-\infty}^{R} x\underline{\mu}_{\tilde{A}}(x)dx + \int_{R}^{\infty} x\overline{\mu}_{\tilde{A}}(x)dx}{\int_{-\infty}^{R} \underline{\mu}_{\tilde{A}}(x)dx + \int_{R}^{\infty} \overline{\mu}_{\tilde{A}}(x)dx} \tag{13}$$

$$c_l(\tilde{A}) = \frac{\int_{-\infty}^{L} x\overline{\mu}_{\tilde{A}}(x)dx + \int_{L}^{\infty} x\underline{\mu}_{\tilde{A}}(x)dx}{\int_{-\infty}^{L} \overline{\mu}_{\tilde{A}}(x)dx + \int_{L}^{\infty} \underline{\mu}_{\tilde{A}}(x)dx} \tag{14}$$

Here *L* and *R* represent the switch points. These switch points must be known in order to calculate (13) and (14). These points are generally computed by utilizing iterative algorithms [1,19,20] since the closed form formulation of the centroid computation is available for only special cases. Among these iterative algorithms, Karnik-Mendel algorithm is the most popular one [20].

THE PROPOSED CONSTRAINED REPRESENTATION OF IT2-FSS

The uncertainty on membership degrees of a FS directly denotes the uncertainty on the meaning of the FS. LHs strengthen or weaken the meaning of the FS by modifying its membership function. Therefore, this semantic uncertainty of a FS can be modeled by using LHs. For instance, the uncertainty on the meaning of the FS "*tall*" can easily be modeled by using LHs as the modifications on its baseline meaning such as "*more or less tall*", "*more tall*", "*slightly tall*", etc.

In many applications, IT2-FSs are formed by taking a baseline type-1 membership function form and the uncertainty (FOU) is modeled by varying its parameters. The one of the important properties of LHs is that useful properties of the FS to which they are applied such as convexity, normality and continuity are preserved on the resulting FS. Therefore, in this study, a new constrained representation of IT2-FSs of which all ET1-FSs are convex, normal and continuous is presented based on this important property of LHs. Although there are various types of membership function forms and LH definition approaches in literature [13-17], triangular form membership functions and the classical LH definitions proposed by Zadeh [17] are considered in this study. Other types of membership function forms and LH definitions can easily be adapted to the proposed constrained representation of IT2-FSs.

I. The Proposed Constrained Representation

An IT2-FS $\tilde{A}$ can be represented as the combination of only convex, normal and continuous ET1-FSs which are formed by applying LHs to the baseline T1-FS *A* as given in Figure 4. By using this constrained representation idea, a constrained IT2-FS $\tilde{A}_c$ is defined as

$$\begin{aligned}\tilde{A}_c &= \int_{x\in X}\int_{u\in J_x} 1/(x,u) \\ &= \int_{x\in X}\left[\int_{h(\mu_A(x))\in J_x\subseteq[0,1]} 1/h(\mu_A(x))\right]/x\end{aligned} \qquad (15)$$

Here, $\mu_A(x)$ is the MF of the baseline T1-FS A and h denotes the LH operator as follows

$$h(x) = x^{\alpha}, \qquad \alpha\in(0,4] \qquad (16)$$

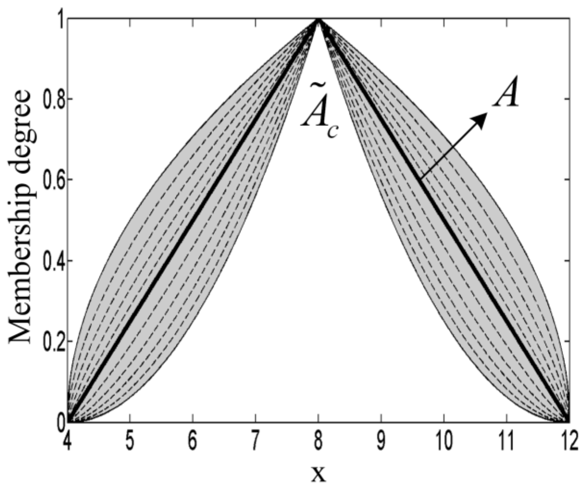

FIGURE 4

THE CONSTRAINED IT2-FS $\tilde{A}_c$ $\left(\bar{\mu}_{\tilde{A}_c} = \mu_A{}^{0.5},\ \underline{\mu}_{\tilde{A}_c} = \mu_A{}^{2}\right)$

All constrained ET1-FSs A_{ce} in the constrained IT2-FS $\tilde{A}_c$ can be given as

$$A_{ce} = \int h(\mu_A(x))/x = \int \mu_A(x)^{\alpha}/x \qquad (17)$$

Here $\alpha\in[\underline{\alpha},\bar{\alpha}]\subseteq(0,4]$. Thus, the FOU of $\tilde{A}_c$ can be defined as the union of all A_{ce} ET1-FSs as

$$FOU(\tilde{A}_c) = \bigcup A_{ce} = \bigcup_{x\in X}\left[\mu_A(x)^{\bar{\alpha}}, \mu_A(x)^{\underline{\alpha}}\right] \qquad (18)$$

The LMF and UMF of $\tilde{A}_c$ can be defined as

$$\bar{\mu}_{\tilde{A}_c}(x) \equiv \overline{FOU(\tilde{A}_c)} = \mu_A(x)^{\underline{\alpha}} \qquad \forall x\in X \qquad (19)$$

$$\underline{\mu}_{\tilde{A}_c}(x) \equiv \underline{FOU(\tilde{A}_c)} = \mu_A(x)^{\bar{\alpha}} \qquad \forall x\in X \qquad (20)$$

Considering the formulation of the proposed constrained representation, it is easily derived that if there is no uncertainty on the membership function, that is, $\alpha = 1$, the LMF and UMF become equal, $\underline{\mu}_{\tilde{A}_c} = \bar{\mu}_{\tilde{A}_c}$, and the IT2-FS turns to the baseline T1-FS as in the conventional unconstrained case.

CENTROID OF IT2-FSS

Using constrained IT2-FSs results in fewer computations in centroid computations when compared with the unconstrained case since fewer ET1-FSs are evaluated [9-12]. As an important result of the proposed constrained representation, the centroid of the proposed constrained IT2-FSs is calculated in closed form without the need of any iterative algorithms.

Without loss of generality, let us consider the IT2-FSs $\tilde{A}_1$ and $\tilde{A}_2$ shown in Figure 5a and Figure 5b, respectively, which are formed by using baseline T1-FSs A_1 and A_2 based on the proposed constrained representation. These FSs can be considered as the right and the left linear pieces of a triangular IT2-FS.

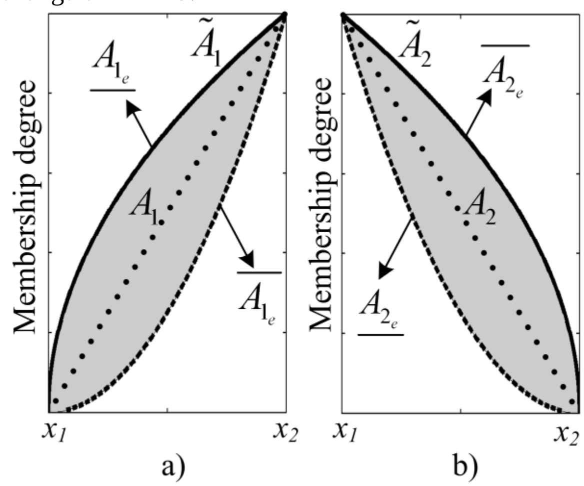

FIGURE 5

THE CONSTRAINED IT2-FSS a) $\tilde{A}_1$ AND b) $\tilde{A}_2$

The membership functions of the baseline T1-FSs A_1 and A_2 can be given as

$$\mu_{A_i}(x) = a_i x + b_i, \quad x\in[x_1, x_2], \ i = 1,2. \qquad (21)$$

So, the MFs of all ET1-FSs A_{i_e} in the IT2-FS $\tilde{A}_i$ are defined as

$$\mu_{A_{i_e}}(x) = (a_i x + b_i)^{\alpha_i}, \qquad \alpha_i = [\underline{\alpha}_i, \bar{\alpha}_i] \qquad (22)$$

Then, the centroids of ET1-FSs of $\tilde{A}_1$ and $\tilde{A}_2$ are calculated as follows

$$c(A_{i_e}) = \frac{\int_{x_1}^{x_2} x\left(\mu_{A_i}(x)\right)^{\alpha_i} dx}{\int_{x_1}^{x_2}\left(\mu_{A_i}(x)\right)^{\alpha_i} dx} = \frac{\int_{x_1}^{x_2} x(a_i x + b_i)^{\alpha_i} dx}{\int_{x_1}^{x_2}(a_i x + b_i)^{\alpha_i} dx} \qquad (23)$$

Here, the integral definitions are calculated as in (24) and (25)

$$\int_{x_1}^{x_2} x(a_i x + b_i)^{\alpha_i} dx = \frac{(a_i x_2 + b_i)(a_i\alpha_i x_2 + a_i x_2 - b_i)e^{\alpha_i \ln(a_i x_2 + b_i)}}{a_i^2(\alpha_i+1)(\alpha_i+2)}$$

$$-\frac{(a_i x_1+b_i)(a_i\alpha_i x_1+a_i x_1-b_i)e^{\alpha_i \ln(a_i x_1+b_i)}}{a_i^2(\alpha_i+1)(\alpha_i+2)} \quad (24)$$

$$\int_{x_1}^{x_2}(a_i x+b_i)^{\alpha_i}\,dx=\frac{(a_i x_2+b_i)^{\alpha_i+1}-(a_i x_1+b_i)^{\alpha_i+1}}{a_i(\alpha_i+1)} \quad (25)$$

The boundary values of membership functions at x_1 and x_2; that is, $\mu_{A_i}(x_1)=a_i x_1+b_i$ and $\mu_{A_i}(x_2)=a_i x_2+b_i$, are known for both IT2-FSs. Therefore, (24) and (25) are simplified for $\tilde{A}_1$ by substituting $\mu_{A_1}(x_1)=a_1x_1+b_1=0$ and $\mu_{A_i}(x_2)=a_i x_2+b_i$ as follows

$$\int_{x_1}^{x_2}x(a_1x+b_1)^{\alpha_1}\,dx=\frac{(a_1\alpha_1x_2+a_1x_2-b_1)}{a_1^2(\alpha_1+1)(\alpha_1+2)} \quad (26)$$

and

$$\int_{x_1}^{x_2}(a_1x+b_1)^{\alpha_1}\,dx=\frac{1}{a_1(\alpha_1+1)} \quad (27)$$

Similarly, (24) and (25) are simplified for $\tilde{A}_2$ by substituting $\mu_{A_2}(x_1)=a_2x_1+b_2=1$ and $\mu_{A_2}(x_2)=a_2x_2+b_2=0$ as follows

$$\int_{x_1}^{x_2}x(a_2x+b_2)^{\alpha_2}\,dx=\frac{-(a_2\alpha_2x_1+a_2x_1-b_2)}{a_2^2(\alpha_2+1)(\alpha_2+2)} \quad (28)$$

and

$$\int_{x_1}^{x_2}(a_2x+b_2)^{\alpha_2}\,dx=\frac{-1}{a_2(\alpha_2+1)} \quad (29)$$

Thus, after certain rearrangements, the centroid definitions for A_{1_e} and A_{2_e} are obtained as follows

$$c(A_{1_e})=\frac{\int_{x_1}^{x_2}x(a_1x+b_1)^{\alpha_1}\,dx}{\int_{x_1}^{x_2}(a_1x+b_1)^{\alpha_1}\,dx}=\frac{(a_1\alpha_1x_2+a_1x_2-b_1)}{a_1(\alpha_1+2)} \quad (30)$$

$$c(A_{2_e})=\frac{\int_{x_1}^{x_2}x(a_2x+b_2)^{\alpha_2}\,dx}{\int_{x_1}^{x_2}(a_2x+b_2)^{\alpha_2}\,dx}=\frac{(a_2\alpha_2x_1+a_2x_1-b_2)}{a_2(\alpha_2+2)} \quad (31)$$

In order to calculate α_i LH values which provide the extremum points of the centroid of $\tilde{A}_i$, the derivatives of (30) and (31) with respect to α_i should be evaluated

$$\frac{dc(A_{1_e})}{d\alpha_1}=\frac{a_1x_2+b_1}{a_1(\alpha_1+2)^2} \quad (32)$$

$$\frac{dc(A_{2_e})}{d\alpha_2}=\frac{a_2x_1+b_2}{a_2(\alpha_2+2)^2} \quad (33)$$

By substituting $\mu_{A_1}(x_2)=a_1x_2+b_1=1$ and $\mu_{A_2}(x_1)=a_2x_1+b_2=1$ into (32) and (33) respectively, the following derivative equation is obtained

$$\frac{dc(A_{i_e})}{d\alpha_i}=\frac{1}{a_i(\alpha_i+2)^2} \quad (34)$$

As it is seen from (34), since $\alpha_i=[\underline{\alpha}_i,\bar{\alpha}_i]>0$, (30) and (31) are strictly monotonically decreasing or increasing functions depending on the sign of a_i. Since $a_1>0$ for A_1, (32) becomes $\frac{dc(A_{1_e})}{d\alpha_1}>0$ and the centroid function becomes a strictly monotonically increasing function. On the other hand, the centroid function of A_2 is a strictly monotonically decreasing function since $\frac{dc(A_{2_e})}{d\alpha_2}<0$ due to $a_2<0$. Thus, the c_l and c_r of $\tilde{A}_i$ are easily calculated with respect to the corresponding boundary values of $\alpha_i=[\underline{\alpha}_i,\bar{\alpha}_i]$. In this way, the ET1-FSs $\underline{A_{1_e}}$ and $\overline{A_{1_e}}$ giving the minimum centroid and the maximum centroid of $\tilde{A}_1$ are easily be determined by using $\alpha_1=\underline{\alpha}_1$ and $\alpha_1=\bar{\alpha}_1$ respectively as shown in Figure 5a. Similarly, the ET1-FSs $\underline{A_{2_e}}$ and $\overline{A_{2_e}}$ giving the minimum centroid and the maximum centroid of $\tilde{A}_2$ are easily be determined by using $\alpha_2=\bar{\alpha}_2$ and $\alpha_2=\underline{\alpha}_2$ respectively as shown in Figure 5b.

Now let us take into consideration the triangular IT2-FS $\tilde{A}$ shown in Figure 6 and formed by using a baseline T1-FSs *A* based on the proposed constrained representation. The membership functions of the baseline T1-FS *A* is given as follows

$$\mu_A(x)=\begin{cases}0 & \text{if } x<x_1\\ a_1x+b_1 & \text{if } x_1\le x\le x_2\\ a_2x+b_2 & \text{if } x_2<x\le x_3\\ 0 & \text{if } x>x_3\end{cases} \quad (35)$$

So, the MFs of all ET1-FSs A_e in the IT2-FS $\tilde{A}$ are defined as

$$\mu_{A_e}(x)=\begin{cases}0 & \text{if } x<x_1\\ (a_1x+b_1)^{\alpha_1} & \text{if } x_1\le x\le x_2\\ (a_2x+b_2)^{\alpha_2} & \text{if } x_2<x\le x_3\\ 0 & \text{if } x>x_3\end{cases} \quad (36)$$

where $\alpha_i=[\underline{\alpha}_i,\bar{\alpha}_i]$, $i=1,2$.

ET1-FSs $\underline{A_e}$ and $\overline{A_e}$ giving the minimum and maximum centroids of $\tilde{A}$ are determined a priori by using corresponding boundary values of α_i without the need of any iterative algorithm as follows

$$\mu_{\underline{A}_e}(x)=\begin{cases}0 & \text{if } x<x_1\\ (a_1x+b_1)^{\underline{\alpha}_1} & \text{if } x_1\le x\le x_2\\ (a_2x+b_2)^{\bar{\alpha}_2} & \text{if } x_2<x\le x_3\\ 0 & \text{if } x>x_3\end{cases} \tag{37}$$

and

$$\mu_{\overline{A}_e}(x)=\begin{cases}0 & \text{if } x<x_1\\ (a_1x+b_1)^{\bar{\alpha}_1} & \text{if } x_1\le x\le x_2\\ (a_2x+b_2)^{\underline{\alpha}_2} & \text{if } x_2<x\le x_3\\ 0 & \text{if } x>x_3\end{cases} \tag{38}$$

$\underline{A}_e$ and $\overline{A}_e$ are shown in Figure 6. Then, the minimum centroid and the maximum centroid of $\tilde{A}$ are easily computed as

$$c\left(\underline{A}_e\right)=\frac{\int_{x_1}^{x_2} x\left(a_1x+b_1\right)^{\underline{\alpha}_1}dx+\int_{x_2}^{x_3} x\left(a_2x+b_2\right)^{\bar{\alpha}_2}dx}{\int_{x_1}^{x_2}\left(a_1x+b_1\right)^{\underline{\alpha}_1}dx+\int_{x_2}^{x_3}\left(a_2x+b_2\right)^{\bar{\alpha}_2}dx} \tag{39}$$

$$c\left(\overline{A}_e\right)=\frac{\int_{x_1}^{x_2} x\left(a_1x+b_1\right)^{\bar{\alpha}_1}dx+\int_{x_2}^{x_3} x\left(a_2x+b_2\right)^{\underline{\alpha}_2}dx}{\int_{x_1}^{x_2}\left(a_1x+b_1\right)^{\bar{\alpha}_1}dx+\int_{x_2}^{x_3}\left(a_2x+b_2\right)^{\underline{\alpha}_2}dx} \tag{40}$$

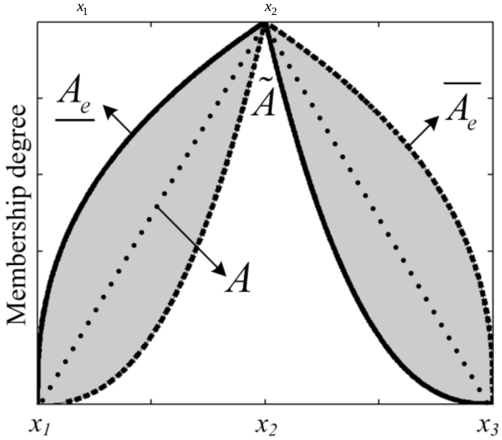

FIGURE 6

THE CONSTRAINED IT2-FS $\tilde{A}$ AND ITS ET1-FS $\underline{A}_e$ AND $\overline{A}_e$

It is important to note that since only convex, normal and continuous ET1-FSs are considered, the constrained centroid result is included in the unconstrained centroid result computed based on the Mendel-John Representation which considers all ET1-FSs without any constraint.

ILLUSTRATIVE EXAMPLE

Let us consider a fuzzy rule inference example where the antecedent and consequent FSs of the rule are formed by blurring a baseline type-1 membership function form. Then, the comparison of the inference results derived based on John-Mendel representation [5], Wu's constrained representation [11] and the proposed constrained representation are presented. When the unconstrained representation is considered, centroid TR is performed by using iterative KM algorithm [20]. In the case of considering Wu's constrained representation, the iterative constrained TR algorithm is used where only convex, normal (if possible) and but also discontinuous ET1-FSs are considered. On the other hand, for the proposed constrained representation, the centroid TR result is calculated in closed form without the need of any iterative algorithm since the ET1-FSs giving the minimum and maximum centroids of the IT2-FS are known a priori. $N=1000$ is chosen for the number of the discretization points.

Consider an IT2-FS $\tilde{A}$ illustrated in Figure 7 which is formed by applying LH operators to a baseline triangular T1-MF of a FS *A*. The baseline T1-MF is defined as

$$\mu_A(x)=\begin{cases}0 & \text{if } x<4\\ 0.25x-1 & \text{if } 4\le x\le 8\\ -0.25x+3 & \text{if } 8<x\le 12\\ 0 & \text{if } x>12\end{cases} \tag{41}$$

and the LMF and UMF of $\tilde{A}$ are given as

$$\underline{\mu}_{\tilde{A}}(x)=\mu_A(x)^{2.5} \tag{42}$$

$$\bar{\mu}_{\tilde{A}}(x)=\mu_A(x)^{0.4} \tag{43}$$

Assume that $\tilde{A}$ is a rule consequent FS and the rule firing interval is $f=[0.7, 0.85]$. In this case, the resulting IT2-FS $\tilde{B}$ is obtained as given in Figure 7. Since $\tilde{B}$ is symmetric, the minimum and maximum centroid of $\tilde{B}$ are also symmetric with respect to the center of $\tilde{B}$. Therefore, only the results for the minimum centroid are presented in the figures.

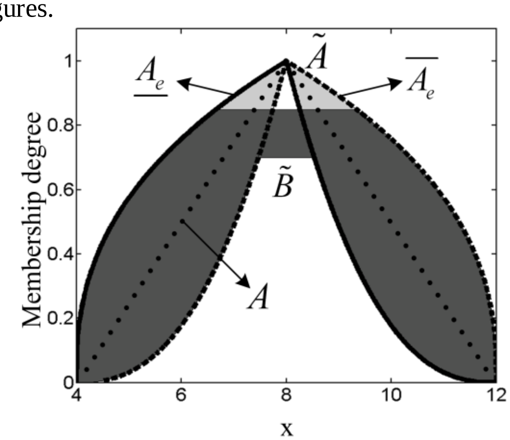

FIGURE 7

THE IT2-FS $\tilde{A}$ AND THE INFERENCE RESULT IT2-FS $\tilde{B}$

If the centroid of $\tilde{B}$ is computed based on the John-Mendel representation by using KM algorithm, the result is obtained as $C_{KM}(\tilde{B})=[7.007, 8.993]$. In this unconstrained case, the ET1-FS $\underline{A}_{e-KM}$ giving the minimum centroid of $\tilde{B}$

is shown in Figure 8. The calculated ET1-FS $\underline{A}_{e-KM}$ is in non-convex and discontinuous form in the unconstrained case as it is seen from Figure 8. Therefore, it is hard to interpret the meaning of this FS with respect to the baseline meaning of the FS $\tilde{A}$.

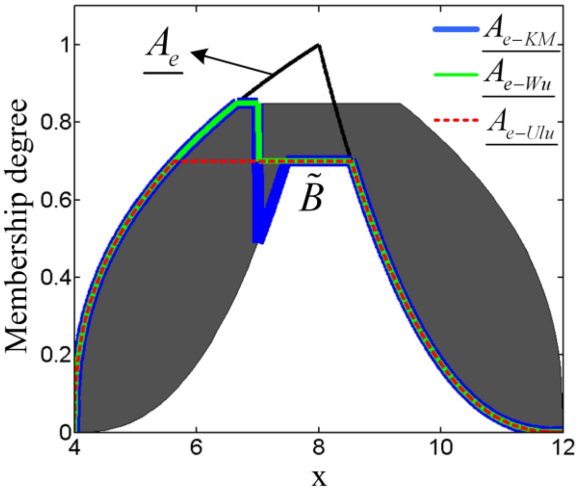

FIGURE 8
THE RESULTING ET1-FSS OBTAINED BY USING KM, WU'S AND THE PROPOSED CONSTRAINED TR APPROACHES

If the constrained TR algorithm based on Wu's constrained representation is utilized for the centroid TR of $\tilde{B}$, the result is computed as $C_{Wu}(\tilde{B}) = [7.009,\ 8.991]$. In this constrained case, the ET1-FS $\underline{A}_{e-Wu}$ giving the minimum centroid of $\tilde{B}$ is shown in Figure 8. As an expected result, since only convex ET1-FSs are considered, Wu's constrained TR algorithm gives the narrower interval result then KM algorithm, that is, $C_{Wu}(\tilde{B}) \subset C_{KM}(\tilde{B})$. However, the obtained ET1-FS $\underline{A}_{e-Wu}$ is discontinuous and thus, hard to be interpreted with respect to the baseline meaning of the FS $\tilde{A}$.

When the proposed constrained representation is considered for the centroid of $\tilde{B}$, the ET1-FS $\underline{A}_e$ giving the minimum centroid of $\tilde{A}$ is determined directly as follows

$$\mu_{\underline{A}_e}(x) = \begin{cases} 0 & \text{if} \quad x < 4 \\ (0.25x-1)^{0.4} & \text{if} \quad 4 \le x \le 8 \\ (-0.25x+3)^{2.5} & \text{if} \quad 8 < x \le 12 \\ 0 & \text{if} \quad x > 12 \end{cases} \tag{44}$$

$\underline{A}_e$ is shown in Figure 8. Then, the ET1-FS $\underline{A}_{e-Ulu}$ giving the minimum centroid of $\tilde{B}$ is easily determined depending on the firing interval $f = [0.7, 0.85]$ on $\underline{A}_e$ as shown in Figure 8. Therefore, the centroid result is calculated directly as $C_{Ulu}(\tilde{B}) = [7.027,\ 8.973]$ without the need of any iterative algorithm. The obtained ET1-FS $\underline{A}_{e-Ulu}$ is convex and continuous and thus, easy to be interpreted with respect to the baseline meaning of the FS $\tilde{A}$.

As it is seen from the results, the narrower centroid result is obtained by using the proposed constrained representation, that is, $C_{Ulu}(\tilde{B}) \subset C_{Wu}(\tilde{B}) \subset C_{KM}(\tilde{B})$. The reason of this result is that the actual situation is described more properly based on the proposed constrained representation than Wu's constrained representation. Because the continuity and interpretability properties of ET1-FSs are also considered additionally in the proposed constrained representation whereas only the convexity property of ET1-FSs are considered in Wu's constrained representation. Therefore, the more precise and appropriate results are derived based on the proposed constrained representation than the other representations if it is know that how the IT2-FS is formed.

CONCLUSION

A new constrained representation is proposed in this study for IT2-FSs which are formed by blurring a baseline type-1 membership function form. In this representation, such IT2-FSs are represented as the union of only convex, normal and continuous ET1-FSs as they actually are. These ET1-FSs are formed by applying LHs to a baseline T1-FS. Due to the nature of LH concept, ET1-FSs of an IT2-FS are constrained not only mathematically but also semantically. In this way, the similar semantic meaning which is consistent with the baseline T1-FS is provided for all ET1-FSs. Therefore, the proposed constrained representation provides a proper and clear connection between T1-FSs and IT2-FSs in this certain context.

By using proposed constrained representation, the centroid of a constrained IT2-FS is calculated in closed form without the need of any iteration. Thus, the computational burden of TR process is removed. The effectiveness of the proposed constrained representation on TR process is shown through an illustrative example. The results show that broader inference interval results are obtained based on the unconstrained representation where the convexity, normality and continuity of ET1-FSs are not considered. On the other hand, more precise and meaningful interval results are obtained by using proposed constrained representation since this representation describes the actual situation more properly.

Various types of IT2-FSs can be represented in a constrained form by using different LH definitions and baseline type-1 membership function forms in the proposed representation. Additionally, many theoretical results for the constrained IT2-FSs can easily be derived based on the proposed representation as they are done based on the unconstrained representation for IT2-FSs.

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