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ODD HARMONIOUS LABELING OF LINE AND DISJOINT UNION OF GRAPHS

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Abstract. A graph $G(p,q)$ is said to be odd harmonious if there exists an injection $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ such that the induced function $f^* : E(G) \to \{1,3,\cdots,2q-1\}$ defined by $f^*(uv) = f(u)+f(v)$ is a bijection. In this paper we prove that disjoint union of $C_4 \cup C_{bn}$, $C_n \cup P_3 (n \equiv 0(mod\, 4))$, $C_n \cup S_m (n \equiv 0(mod\, 4))$, $C_n \cup K_{p,q} (n \equiv 0(mod\, 4))$, $C_n \cup H_{m,m} (n \equiv 0(mod\, 4))$, $K_{m,n} \cup K_{p,q}$, line graph of P_n, $C_n (n \equiv 0(mod\, 4))$, $K_{2,n}$, C_{b2}, $H_{2,2}$ are odd harmonious. Also the line graph of $K_{m,n} (m \geq 2, n > 2)$, $H_{n,n} (n > 2)$ and $C_{bn} (n > 2)$ are not odd harmonious.

1. Introduction

Throughout this paper by a graph we mean a finite, simple and undirected one. For standard terminology and notation we follow Harary [4]. A graph $G = (V, E)$ with p vertices and q edges is called a (p, q) – graph. The graph labeling is an assignment of integers to the set of vertices or edges or both, subject to certain conditions. An extensive survey of various graph labeling problems is available in [2]. Labeled graphs serve as useful mathematical models for many applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encoding of integers. Graham and Sloane [3] introduced harmonious labeling during their study of modular versions of additive bases problems stemming from error correcting codes. A graph G is said to be harmonious if there exists an injection $f : V(G) \to Z_q$ such that the induced function $f^* : E(G) \to Z_q$ defined by $f^*(uv) = (f(u) + f(v)) \pmod q$ is a bijection and f is called harmonious labeling of G. The concept of an odd harmonious labeling was due to Liang and Bai [15]. A graph G is said to be odd harmonious if there exists an injection $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ such that the induced function $f^* : E(G) \to \{1,3,\cdots,2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. If $f(V(G)) = \{0,1,2,\cdots,q\}$ then f is called as strongly odd harmonious labeling and G is called as strongly odd harmonious graph.The odd harmoniousness of graph is useful for the solution of undetermined equations. Several results have been published on odd harmonious labeling and an interested reader can refer to [5] to [19].

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We use the following definitions in the subsequent section.

Definition 1.1. The graph $H_{n,n}$ has the vertex set $V(H_{n,n}) = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and the edge set $E(H_{n,n}) = \{v_i u_j : 1 \leq i \leq n, n-i+1 \leq j \leq n\}$.

Definition 1.2. Let G be a graph on p vertices $v_1, v_2, \cdots, v_p$ and $H_1, H_2, \cdots, H_p$ be p graphs isomorphic to a graph H with n vertices. The corona graph $G \odot H$ is obtained by joining each vertex v_i of G with every vertex of H_i for $1 \leq i \leq p$ and $1 \leq j \leq n$. The comb graph $P_n \odot K_1$ denoted by C_{bn}.

Definition 1.3. Given a graph G, its line graph $L(G)$ is a graph such that (i) each vertex of $L(G)$ represent an edge of G and (ii) two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common end point in G, that is , it is the intersection graph of the edge of G, representing each edge by the set of its two end points.

2. Main Results

In this section, we prove that disjoint union of $C_4 \cup C_{bn}$, $C_n \cup P_3 (n \equiv 0 (mod\ 4))$, $C_n \cup S_m (n \equiv 0 (mod\ 4))$, $C_n \cup K_{p,q} (n \equiv 0 (mod\ 4))$, $C_n \cup H_{m,m} (n \equiv 0 (mod\ 4))$, $K_{m,n} \cup K_{p,q}$, line graph of P_n, $C_n (n \equiv 0 (mod\ 4))$, $K_{2,n}$, C_{b2}, $H_{2,2}$ are odd harmonious. Also, the line graph of $S_n (n \geq 3)$, $K_{m,n} (m \geq 2, n > 2)$, $H_{n,n} (n > 2)$ and $C_{bn} (n > 2)$ are not odd harmonious.

Theorem 2.1. *The graph $C_4 \cup C_{bn}$ is odd harmonious.*

Proof. Let v_1, v_2, v_3, v_4 be the vertices of the cycle C_4 and let $u_1, u_2, \cdots, u_n$ and $w_1, w_2, \cdots, w_n$ be the vertices of the comb C_{bn}.

Then $|V(G)| = 4 + 2n$ and $|E(G)| = 3 + 2n$.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(3+2n) - 1\}$ as follows:

$f(v_i) = i - 1$ for $1 \leq i \leq \dfrac{n}{2}$;

$f(v_i) = \begin{cases} i+1 & \text{if } i \text{ is odd} \\ i-1 & \text{if } i \text{ is even.} \end{cases}, \dfrac{n}{2} + 1 \leq i \leq n.;$

$f(u_i) = 2i$ for integers $1 \leq i \leq n$ and if i is odd;

$f(u_i) = 3 + 2i$ for integers $1 \leq i \leq n$ and if i is even.

Case 1. n is even.

$f(w_i) = 2i + 7$ for $1 \leq i \leq n - 1$ if i is odd;

$f(w_i) = 2i + 4$ for $1 \leq i \leq n - 2$ if i is even;

$f(w_n) = 2n + 2$.

Case 2. n is odd.

$f(w_i) = 2i + 7$ for $1 \leq i \leq n - 2$ if i is odd;

$f(w_i) = 2i + 4$ for $1 \leq i \leq n - 1$ if i is even;

$f(w_n) = 2n + 5$.

The induced edge labels are

$f^*(v_1 v_2) = 1$;

$f^*(v_iv_{i+1}) = 2i+1$ if $\frac{n}{2} \leq i \leq n-1$;

$f^*(v_1v_4) = 3$;

$f^*(u_iw_i) = 4i+7$ if $1 \leq i \leq n-1$;

$f^*(u_nw_n) = 4n+5$.

In view of the above defined labeling pattern, we therefore know that $C_4 \cup C_{bn}$ is an odd harmonious graph. □

An odd harmonious labeling of $C_4 \cup C_{b3}$ is shown in Figure 1.

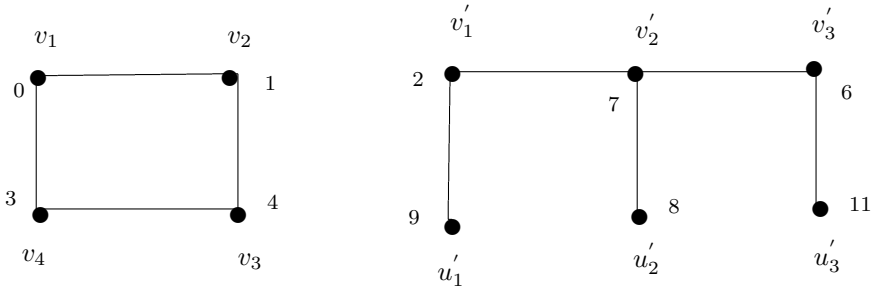

Figure 1 An odd harmonious labeling of $C_4 \cup C_{b3}$

Theorem 2.2. *The graph $C_n \cup P_3$ is odd harmonious.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_n and let u_1, u_2, u_3 be the vertices of the path P_3. Then $|V(G)| = n+3$ and $|E(G)| = n+2$.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(n+2)-1\}$ as follows:

$f(v_i) = i-1$ for integers $1 \leq i \leq \frac{n}{2}$;

$f(v_i) = \begin{cases} i+1 & \text{if } i \text{ is odd} \\ i-1 & \text{if } i \text{ is even.} \end{cases}, \frac{n}{2}+1 \leq i \leq n.$

$f(u_i) = 2n+i-\frac{n}{2}$ for $i = 1, 3$;

$f(u_2) = \frac{n}{2}$.

The induced edge labels are

$f^*(v_iv_{i+1}) = 2i-1$ for integers $1 \leq i \leq \frac{n}{2}-1$;

$f^*(v_iv_{i+1}) = 2i+1$ for integers $\frac{n}{2} \leq i \leq n-1$;

$f^*(v_1v_n) = n-1$;

$f^*(u_iu_2) = 2n+i$ for $i = 1, 3$.

In view of the above defined labeling pattern, we know that $C_n \cup P_3$ is an odd harmonious graph. □

An odd harmonious labeling of $C_8 \cup P_3$ is shown in Figure 2.

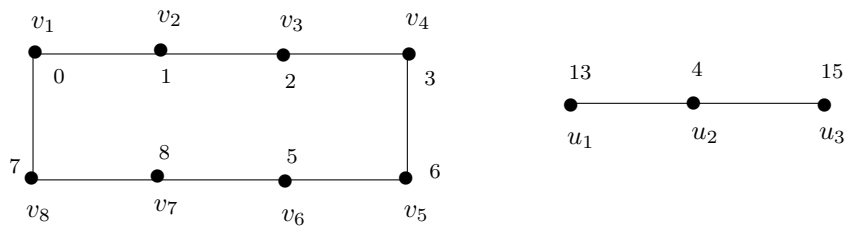

Figure 2 An odd harmonious labeling of $C_8 \cup P_3$

Theorem 2.3. *The graph $C_n \cup K_{p,q}$ is odd harmonious.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_n and let $u_1, u_2, \cdots, u_p$ and $w_1, w_2, \cdots, w_q$ be the vertices of the complete bipartite graph $K_{p,q}$. Then $|V(G)| = n + p + q$ and $|E(G)| = n + pq$.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(n + pq) - 1\}$ as follows:

$f(v_i) = i - 1$ for integers $1 \leq i \leq \dfrac{n}{2}$;

$f(v_i) = \begin{cases} i+1 & \text{if } i \text{ is odd} \\ i-1 & \text{if } i \text{ is even.} \end{cases}, \quad \dfrac{n}{2} + 1 \leq i \leq n.;$

$f(u_i) = \dfrac{n}{2} + (i-1)2q$ for integers $1 \leq i \leq p$;

$f(w_i) = 2n + 1 - \dfrac{n}{2} + 2(i-1)$ for integers $1 \leq i \leq q$.

The induced edge labels are

$f^*(v_i v_{i+1}) = 2i - 1$ for integers $1 \leq i \leq \dfrac{n}{2} - 1$;

$f^*(v_i v_{i+1}) = 2i + 1$ for integers $\dfrac{n}{2} \leq i \leq n - 1$;

$f^*(u_i w_j) = (i-1)2q + 2n + 1 + 2(j-1)$ for integers $1 \leq i \leq p$ and $1 \leq j \leq q$.

Thus, $C_n \cup K_{p,q}$ is an odd harmonious graph. □

An odd harmonious labeling of $C_8 \cup K_{2,4}$ is shown in Figure 3.

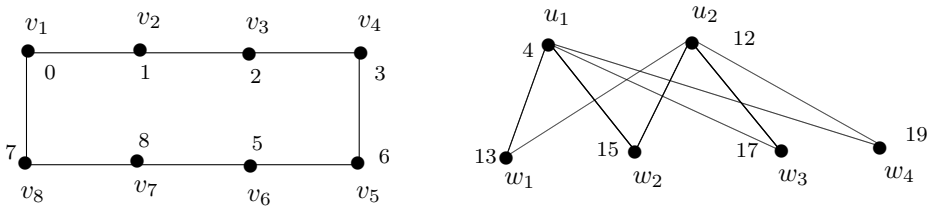

Figure 3 An odd harmonious labeling of $C_8 \cup K_{2,4}$

Theorem 2.4. *The graph $C_n \cup H_{m,m}$, $m > \dfrac{n}{4} + 1$ is odd harmonious.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_n and let $u_1, u_2, \cdots, u_m$ and $w_1, w_2, \cdots, w_m$ be the vertices of the graph $H_{m,m}$. Then $|V(G)| = n + 2m$ and $|E(G)| = n + \dfrac{m(m+1)}{2}$.

We define a labeling $f : V(G) \to \left\{0, 1, 2, \cdots, 2(n + \dfrac{m(m+1)}{2}) - 1\right\}$ as follows:

$f(v_i) = i - 1$ for integers $1 \le i \le \dfrac{n}{2}$;

$f(v_i) = \begin{cases} i+1 & \text{if } i \text{ is odd} \\ i-1 & \text{if } i \text{ is even.} \end{cases}, \dfrac{n}{2} + 1 \le i \le n.$;

$f(u_i) = \dfrac{n}{2} + (m-i)(m+i-1)$ for integers $1 \le i \le m-1$;

$f(w_i) = 2n + 1 - \dfrac{n}{2} + 2(i-1)$ for integers $1 \le i \le m$.

The induced edge labels are

$f^*(v_i v_{i+1}) = 2i - 1$ for integers $1 \le i \le \dfrac{n}{2} - 1$;

$f^*(v_i v_{i+1}) = 2i + 1$ for integers $\dfrac{n}{2} \le i \le n - 1$;

$f^*(u_i w_j) = (m-i)[m+i-1] + 2n + 1 + 2(j-1)$ for integers $1 \le i, j \le m$.

Therefore, $C_n \cup H_{m,m}$ is an odd harmonious graph. □

An odd harmonious labeling of $C_8 \cup H_{4,4}$ is shown in Figure 4.

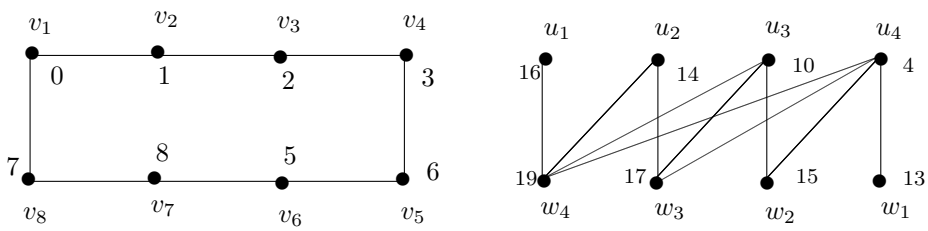

Figure 4 An odd harmonious labeling of $C_8 \cup H_{4,4}$

Theorem 2.5. *The graph $K_{m,n} \cup K_{p,q}$ is odd harmonious.*

Proof. Let $v_1, v_2, \cdots, v_m; u_1, u_2, \cdots, u_n$ be the vertices of $K_{m,n}$ and let $t_1, t_2, \cdots, t_p; w_1, w_2, \cdots, w_q$ be the vertices of $K_{p,q}$. Then $|V(G)| = m + n + p + q$ and $|E(G)| = mn + pq$.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(mn + pq) - 1\}$ as follows:

$f(v_i) = 2n(i-1)$ for integers $1 \le i \le m$;

$f(u_i) = 1 + 2(i-1)$ for integers $1 \le i \le n$;

$f(t_i) = 2m + 2q(i-1)$ for integers $1 \le i \le p$;

$f(w_i) = 2m(n-1) + 1 + 2(i-1)$ for integers $1 \le i \le q$.

The induced edge labels are

$f^*(v_iu_j) = 2n(i-1) + 1 + 2(j-1)$ for integers $1 \le i \le m$ and $1 \le j \le n$;

$f^*(t_iw_j) = 2m + 2q(i-1) + 2m(n-1) + 1 + 2(j-1)$ for integers $1 \le i \le p$ and $1 \le j \le q$.

By definition, $K_{m,n} \cup K_{p,q}$ is an odd harmonious graph. □

An odd harmonious labeling of $K_{2,3} \cup K_{3,3}$ is shown in Figure 5.

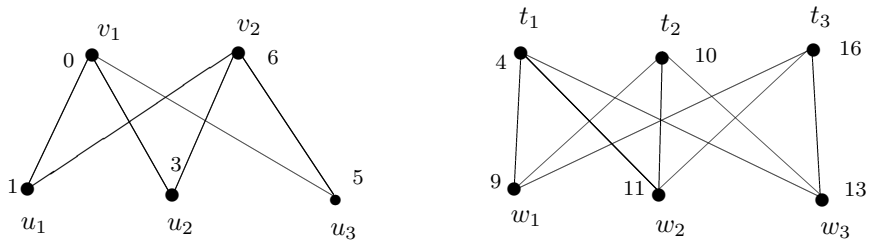

Figure 5 An odd harmonious labeling of $K_{2,3} \cup K_{3,3}$

Theorem 2.6. *The graph $C_n \cup S_m$ is odd harmonious.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of C_n and let $u_0, u_1, u_2, \cdots, u_m$ be the vertices of S_m. Then $|V(G)| = n + m + 1$ and $|E(G)| = n + m$.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(n+m) - 1\}$ as follows:

$f(v_i) = i - 1$ for integers $1 \le i \le \frac{n}{2}$;

$f(v_i) = \begin{cases} i+1 & \text{if } i \text{ is odd} \\ i-1 & \text{if } i \text{ is even.} \end{cases}, \frac{n}{2} + 1 \le i \le n.$;

$f(u_0) = \frac{n}{2}$;

$f(u_i) = 2n + 1 - \frac{n}{2} + 2(i-1)$ for integers $1 \le i \le m$.

The induced edge labels are

$f^*(v_iv_{i+1}) = 2i - 1$ for integers $1 \le i \le \frac{n}{2} - 1$;

$f^*(v_iv_{i+1}) = 2i + 1$ for integers $\frac{n}{2} \le i \le n - 1$;

$f^*(u_0u_i) = 2n + 2i - 1$ for integers $1 \le i \le m$.

In view of the above defined labeling pattern, $C_n \cup S_m$ is an odd harmonious graph. □

An odd harmonious labeling of $C_8 \cup S_5$ is shown in Figure 6.

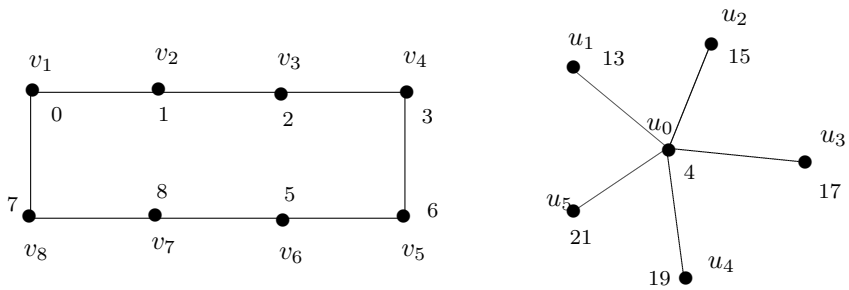

Figure 6 An odd harmonious labeling of $C_8 \cup S_5$

Theorem 2.7. *The line graph of path P_n, $C_n(n \equiv 0(mod\ 4))$, $K_{2,2}$, C_{b2} and $H_{2,2}$ are odd harmonious.*

Proof. Since $L(P_n) = P_{n-1}$, $L(C_n) = C_n, (n \equiv 0(mod\ 4))$, $L(C_{b2}) = P_2$, $L(H_{2,2}) = P_2$ and $L(K_{2,2}) = C_4$ and hence they are odd harmonious. □

Theorem 2.8. *The line graph of $K_{m,n}$, $m, n > 2$, $H_{n,n}$ and C_{bn}, $n > 2$ are not odd harmonious.*

Proof. Since the line graphs $L(K_{m,n})$, $L(H_{n,n})$ and $L(C_{bn})$ contain at least one odd cycle and hence they are not odd harmonious. □

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