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ON THE VERTEX CONNECTIVITY INDEX OF GRAPHS

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ABSTRACT. In this paper, we introduce the vertex connectivity index of a simple finite connected graph. Moreover, we determine the vertex connectivity index of some graphs and analyse some of its properties concerning the wiener index of a graph.

1. Introduction

In our day-to-day life, there may usually arise situations in which time-bound visits have to be made at different places in a single journey. All the intermediate places between the source and the final destination can be equally important so that none of them can be avoided. The common practice in such situations is to use the shortest routes to reach the intermediate places between the source and the destination. But there may arise situations in which the shortest routes cause trouble due to potholes, road maintenance works, heavy traffic, and waiting time in signals. To avoid these, the shortest route between different destinations cannot always be preferable, and we prefer routes that optimize time and cost of fuel. This is possible only if an alternate path exists between the source and the destination connecting all the intermediate places to achieve our purpose, even if the shortest route is not accessible.

Graph theoretically speaking, if we consider the places to be visited as the vertices of a graph and the shortest distance between the places as the edges of a graph, then we aim to search for those vertex pairs which do not alter the connectivity of the graph. More precisely, the vertex pairs do not disconnect the graph even if the shortest path between them is deleted.

The authors have already introduced the vertex connectivity polynomial [1] of a simple finite connected graph which explicitly reveals the number of vertex pairs that disconnect a graph. In this paper, we confine to study those vertex pairs in a graph that does not alter their connectivity. For that, the metric nature of the graph is also incorporated to formulate an index that gives an idea of both the metric and the connectivity relations in a graph. Throughout this paper, G denotes a finite simple connected graph with vertex set and edge set denoted by $V(G)$ and $E(G)$ respectively.

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2. Main Results

Definition 2.1. ([4]) Let $G = (V, E)$ be a graph. Then the *Wiener index* of G denoted by $W(G)$ is defined as the sum of all distances between the vertex pairs of G. That is,

$$W(G) = \frac{1}{2} \sum_{u,v \in V} d(u, v).$$

Definition 2.2. ([1]) Let G be a graph of order n. Then the *vertex connectivity polynomial* of G , denoted by $V[G; x]$ is defined by

$$V[G; x] = \sum_{i=1}^{\text{diam(G)}+1} |D_i| x^i,$$

where D_i is the set of all vertex pairs which disconnects G into i components.

Definition 2.3. Let $G = (V, E)$ be a graph. Then the vertices u and v of G are said to be *vertex-connected* if the deletion of none of the shortest paths connecting u and v disconnects the graph G. The graph G is *vertex-connected* if each of its vertex pairs are *vertex-connected.*

Definition 2.4. Let $G = (V, E)$ be a graph. Then a pair of vertices $(u, v) \in V \times V$ at a distance $d > 0$ are said to be *d-metrically connective* if it disconnects the graph G into d components. The graph G is *metrically connective* if every pair of its vertices are *d-metrically connective* for some $d > 0$.

Definition 2.5. Let $G = (V, E)$ be a graph. Then the *vertex connectivity index* $V_c[G]$ of G is defined as the sum of all distances between vertex-connected vertices of G. If G is vertex-connected, then the vertex connectivity index of G coincides with the wiener index of G and $V_c[G] \geq V_c[K_n] \forall n$.

Theorem 2.6. *For a graph G, the following holds:*

(1) $W[G] \geq V_c[G]$, where $W[G]$ is the Wiener index of G. In particular, if G has cut edges, then $W[G] > V_c[G]$ and equality holds if the vertex connectivity polynomial $V[G; x]$ of G is linear;

(2) G is a tree iff $V_c[G] = 0$;

(3) If G is metrically connective of order n and size m , then $V_c[G]= m$. In particular, if G is also vertex-connected, then G is isomorphic to the complete graph K_n and $V_c[G]= \binom{n}{2}$.

Proof. (1) Since all the vertices of G need not be vertex-connected, it follows from the definition that $W[G] \geq V_c[G]$. Now if G has at least one cut edge, the end vertices of cut edges are not vertex-connected. Thus all vertex pairs of G are not vertex-connected and hence $W[G] > V_c[G]$. The linearity of $V[G; x]$ implies that all pairs of vertices of G are vertex-connected so that $W[G] = V_c[G]$ holds in this case.

(2) If G is a tree, then all its edges are cut edges so that none of the vertex pairs are vertex-connected. Therefore $V_c[G] = 0$. Conversely, if G is not a tree, then there exists at least one edge of G as a part of some cycle. Since it is not a

cut edge its deletion doesn't disconnect the graph. Thus the end vertices of this edge are vertex-connected so that $V_c[G] \geq 1$. That is, $V_c[G] \neq 0$.

(3) Since G is metrically connective, the only vertex-connected vertices are those which are adjacent. Therefore $V_c[G]= m$. If G is also vertex-connected, then we get that all the $\binom{n}{2}$ vertex-connected vertices are adjacent. That is, $G = K_n$ and hence $V_c[G]= \binom{n}{2}$.

This completes the proof. □

Theorem 2.7. *The following are the vertex connectivity index of some important graphs:*

(1) *For* $n \geq 3$, $V_c[C_n] = n$;
(2) *For* $m, n \geq 2$ *and* $m \leq n$,

$$V_c[K_{m,n}] = \begin{cases} 4 & \text{if } \ m = n = 2, \\ n(n+1), & \text{if } \ m = 2, n > 2, \\ mn + m(m-1) + n(n-1), & \text{if } \ m > 2; \end{cases}$$

(3) *Let* G_1, G_2 *be two graphs and let* $G_1 + G_2$ *be their join. Then,* $V_c[G_1 + G_2] = W[G_1 + G_2]$.

Proof. (1) Since C_n is metrically connective, it follows from Theorem 2.6 that $V_c[C_n] = \text{number of edges of } C_n = n$.

(2) Let M and N be the bipartite sets of vertices of $K_{m,n}$ with cardinalities m and n respectively. We consider three cases:

Case 1. $m, n = 2$.

$K_{2,2}$ is C_4 and hence from (1) the result follows trivially.

Case 2. $m = 2, \ n > 2$.

Since the linear coefficient of the vertex connectivity polynomial of $K_{2,n}$ is $\binom{n+2}{2}-1$ and since the unique vertex pair in the partite set M is not vertex-connected, it follows that there are $2n$ vertex pairs at unit distance and $\binom{n}{2}$ vertex pairs at twice the unit distance which are vertex-connected. Therefore,

$$V_c[K_{2,n}] = 2n \times 1 + \binom{n}{2} \times 2 = n(n+1).$$

Case 3. $m > 2$.

In this case, the linear coefficient of the vertex connectivity polynomial is $\binom{m+n}{2}$ so that all the vertex pairs are vertex-connected. It can be observed that there are mn vertex pairs at unit distance and $\binom{m}{2} + \binom{n}{2}$ vertex pairs at twice the unit distance in $K_{m,n}$. Therefore,

$$\begin{aligned} V_c[K_{m,n}] &= mn \times 1 + 2\left[\binom{m}{2} + \binom{n}{2}\right] \\ &= mn + m(m-1) + n(n-1). \end{aligned}$$

(3) It directly follows from the fact that [1] $V[G_1 + G_2; x]$ is linear.

This completes the proof. □

Note that for a cut edge free graph G, $V_c[G]$ and $W[G]$ need not be equal. For example, not every vertex pair of the cycle graph C_n are vertex-connected for $n \geq 4$ so that $V_c[C_n] \neq W[C_n]$.

Theorem 2.8. *For $k \geq 3$, let G be a cut edge free k-regular graph. Then, $V_c[G] = W[G]$.*

Proof. Since every edge of G is part of some cycle and since the degree of every vertex of G is greater than or equal to three, the deletion of any of the paths connecting vertex pairs in G isolates none of its vertices. Therefore, G is vertex-connected and hence $V_c[G] = W[G]$. This completes the proof. □

Theorem 2.9. *Let $G = (V, E)$ be a graph and let $G^{'} = (V^{'}, E^{'})$ be the graph obtained by adjoining pendent vertices to the vertices of G through bridges. Then $V_c[G^{'}] = V_c[G]$.*

Proof. Since the new pendent vertices of $G^{'}$ are adjoined to the vertices of G through bridges, the deletion of any of these bridges makes the graph disconnected. Hence the newly adjoined vertices are neither mutually vertex-connected nor vertex-connected with any of the vertices of G so that $V_c[G^{'}] = V_c[G]$. This completes the proof. □

Corollary 2.10. *Let $G = (V, E)$ be a graph and let $G^{'} = (V^{'}, E^{'})$ be the graph obtained by adjoining acyclic graphs to the vertices of G through bridges. Then $V_c[G^{'}] = V_c[G]$.*

Proof. The proof follows immediately from the fact that none of the newly adjoined acyclic graphs increases the number of vertex-connected vertices in $G^{'}$ from that of G. □

Corollary 2.11. *For $m \geq 3$, let $G_m = P_2 \times C_m$ and $G = G_m \odot K_{1,n}$ be the generalized n-crown obtained by introducing n new pendent edges at each vertex of the outermost C_m in G_m. Then,*

$$V_c[G] = \begin{cases} \frac{m(m^2+2m-1)}{2}, & \textit{if } m \textit{ is odd}, \\ \frac{m^2(m+2)}{2}, & \textit{if } m \textit{ is even}. \end{cases}$$

Proof. Since G is constructed by adjoining n new pendent vertices at each vertex of the outermost C_m in G_m, it follows from Theorem 2.9 that $V_c[G] = V_c[G_m]$. But since G_m being a $3-regular$ cut edge free graph, it is vertex-connected by Theorem 2.8 so that $V_c[G_m] = W[G_m]$. In [3], it has been proved that,

$$W[G_m] = \begin{cases} \frac{m(m^2+2m-1)}{2}, & \text{if } m \textit{ is odd}, \\ \frac{m^2(m+2)}{2}, & \text{if } m \textit{ is even}. \end{cases}$$

This completes the proof. □

Theorem 2.12. *Let G be a graph of diameter 2 with n vertices and m edges among which n_1 pairs of vertices are vertex-connected and m_1 are cut edges. Then,*

$$V_c[G] = 2n_1 - m + m_1.$$

Proof. Out of the n_1 pairs of vertex-connected vertices, there are $m - m_1$ edges linking adjacent vertex-connected vertices and the remaining are at two unit distance. Therefore $V_c[G] = m - m_1 + 2[n_1 - (m - m_1)]$. This completes the proof. $\square$

Theorem 2.13. *Let $G_1, G_2, \cdots G_n$ be connected graphs and let G be the connected graph obtained by adjoining G_i and G_{i+1} through a bridge, where $i = 1, \cdots, n-1$. Then,*

$$V_c[G] = \sum_{i=1}^{n} V_c[G_i].$$

Proof. For $i = 1, \cdots, n-1$ and $j = 1, 2, \cdots, n-i$, it can be observed that every path linking the vertices of G_i and G_{i+j} incorporates j bridges mentioned in the construction of G. Thus the addition of bridges neither alters the vertex-connected vertices of G_i $\forall i$ nor produces new vertex-connected vertices in G. Therefore, $V_c[G] = \sum_{i=1}^{n} V_c[G_i]$. This completes the proof. $\square$

The *corona* of two graphs G_1 and G_2 has been defined by Frucht and Harary in [2] to be the graph G formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2 and is denoted by $G = G_1 \circ G_2$.

Theorem 2.14. *Let G_1 and G_2 be two nontrivial graphs of orders n_1, n_2 and sizes m_1, m_2 respectively. If G_1 is vertex-connected, then $G_1 \circ G_2$ is vertex-connected and*

$$V_c[G_1 \circ G_2] = (n_2+1)^2 V_c[G_1] + n_1(n_2^2 - m_2) + n_1 n_2 (n_1 - 1)(n_2 + 1).$$

Proof. Let $G = G_1 \circ G_2$. Since G_1 is vertex-connected, the vertex pairs in G belonging to G_1 remains vertex-connected. The i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2 and since G_2 is a nontrivial connected graph, the deletion of the edge between the i^{th} vertex of G_1 and the k^{th} vertex of G_2 in the i^{th} copy of G_2 does not make G disconnected, where $i \in \{1, \cdots, n_1\}$ and $k \in \{1, \cdots, n_2\}$. The shortest path linking the i^{th} vertex of G_1 and the k^{th} vertex of G_2 in the j^{th} copy of G_2 is through the i-j path in G_1 followed by the j-k edge, where $i, j \in \{1, \cdots, n_1\}, i \neq j$ and $k \in \{1, \cdots, n_2\}$. Since G_1 is vertex-connected, we get that the i^{th} vertex of G_1 and the k^{th} vertex of G_2 in the j^{th} copy of G_2 are also vertex-connected. Now since every vertex pair belonging to the i^{th} copy of G_2 in G is connected through the i^{th} vertex of G_1 and also through the vertices of G_2, they also remain vertex-connected. Thus G is vertex-connected and hence it follows that $V_c[G] = W[G]$. In [5], the wiener index of corona of two graphs has been proved as $W[G] = (n_2+1)^2 W[G_1] + n_1(n_2^2 - m_2) + n_1 n_2 (n_1 - 1)(n_2 + 1)$. This completes the proof. $\square$

The tensor product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G = G_1 \wedge G_2$ with vertex set $V = \{v_1, \cdots, v_n, u_1, \cdots, u_m\}$ and edge set $E = \{w_1 w_2 \mid u_1 u_2 \in E_1$ and $v_1 v_2 \in E_2\}$ where $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$. The tensor product of P_3 and C_n is a connected (2,4) biregular graph with $3n$ vertices when n is odd whereas it is a disconnected graph with both the components (2,4) biregular graphs with $\frac{3n}{2}$ vertices when n is even [3].

Theorem 2.15. *For odd $n \geq 3$, the vertex connectivity index of the graph $P_3 \wedge C_n$ is $14n$.*

Proof. The biregular graph $P_3 \wedge C_n$ is a connection of n cycles with 4 vertices and each cycle has one common vertex with the other cycle. That is it has an outer cycle of length $2n$ and the remaining n vertices together with n alternate vertices of the outer cycle again constitutes a cycle of length $2n$. Thus all the vertices of $P_3 \wedge C_n$ except those belonging to both the $2n$ cycles are of degree 2. Now consider the outer cycle of $P_3 \wedge C_n$. Every vertex of degree 2 along with any other vertex at a distance greater than 2 on the outer cycle disconnects the graph and the vertices of degree 4 are vertex-connected only with its adjacent vertices on the outer cycle. Thus there are $2n$ vertex-connected vertex pairs at a unit distance and n vertex-connected vertex pairs at twice the unit distance on the outer cycle.

Therefore on the outer cycle, sum of all distances between the vertex-connected vertices is given by

$$S_o = 2n \times 1 + n \times 2 = 4n.$$

Now, every vertex of degree 2 on the outer cycle is vertex-connected to exactly 3 vertices at twice the unit distance on the inner cycle whereas every vertex of degree 4 on the outer cycle is vertex-connected to exactly 2 vertices at a unit distance on the inner cycle. Thus, sum of all distances between the vertex-connected vertices on the outer cycle and the inner cycle ,

$$S_{oi} = 3 \times 2 \times n + 2 \times 1 \times n = 8n.$$

Now, all the n vertices of degree 2 on the inner cycle are vertex-connected with the vertices located at twice the unit distance on the inner cycle. Therefore, the sum of all distances between the vertex-connected vertices on the inner cycle is given by

$$S_i = 2 \times n = 2n.$$

Thus,

$$V_c[P_3 \wedge C_n] = S_o + S_{oi} + S_i = 4n + 8n + 2n = 14n.$$

This completes the proof. □

Theorem 2.16. *For even $n > 4$, let G be one of the components of the graph $P_3 \wedge C_n$. Then, $V_c[G] = 6n$.*

Proof. In this case, both the components of $P_3 \wedge C_n$ are $(2, 4)$ biregular with $\frac{3n}{2}$ vertices among which $\frac{n}{2}$ of degree 4 and n are of degree 2. For vertices having degree 4, it can be observed that only the adjacent vertices are vertex-connected and for degree 2 vertices vertex-connected vertices are those located at at most 2 unit distance. Therefore,

$$V_c[G] = \frac{1}{2}\left[\frac{n}{2} \times 1 \times 4 + n \times 1 \times 2 + n \times 2 \times 4\right] = 6n.$$

This completes the proof. □

Let G_1 and G_2 be simple graphs such that $u = (u_1, u_2)$ and $v = (v_1, v_2) \in V(G_1) \times V(G_2)$. Then, the *Cartesian product* $G_1 \times G_2$ is defined in [6] as follows:

The vertices u and v are adjacent in $G_1 \times G_2$ whenever $u_1 = v_1$ and u_2 adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 adjacent to v_1 in G_1.

Theorem 2.17. *For $n > 1$,*

$$V_c[P_2 \times P_n] = \begin{cases} \frac{4n^3-6n^2-4n+30}{6}, & \text{if } n \text{ is odd} \\ \frac{4n^3-6n^2-4n+24}{6}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. In $P_2 \times P_n$, there are 4 vertices of degree 2 and $2n-4$ vertices of degree 3. The vertices of degree 2 are vertex-connected with those vertices of $P_2 \times P_n$ which lie in the same row or column. Therefore, the sum of all distances between the vertices which are vertex-connected with degree 2 vertices is given by

$$\begin{aligned} S_2 &= 4(1+1+2+\ldots+n-1) \\ &= 2n^2 - 2n + 4. \end{aligned}$$

The vertices of degree 3 are vertex-connected with all other vertices of $P_2 \times P_n$ except those vertices of degree 2 which does not lie in the same row. If m is even, then the sum of all distances between the vertices which are vertex-connected with degree 3 vertices is given by

$$\begin{aligned} S_3 &= 8\sum_{i=2}^{\frac{n}{2}}\sum_{j=1}^{n-i} j + 4\left[1+1+2(2)+1+2(2+3)+\cdots 1+2\left(2+\cdots+\frac{n}{2}-1\right)\right] \\ &= 8\sum_{i=2}^{\frac{n}{2}}\sum_{j=1}^{n-i} j + 4\left[\left(\frac{n}{2}-1\right)+2\left\{2\left(\frac{n}{2}-2\right)+\ldots+(\frac{n}{2}-1)(\frac{n}{2}-(\frac{n}{2}-1))\right\}\right] \\ &= 8\sum_{i=2}^{\frac{n}{2}}\sum_{j=1}^{n-i} j + 4\left[\left(\frac{n}{2}-1\right)+2\left\{\frac{n}{2}(2+\cdots+\frac{n}{2}-1)-(2^2+\cdots+(\frac{n}{2}-1)^2)\right\}\right] \\ &= 8\sum_{i=2}^{\frac{n}{2}}\sum_{j=1}^{n-i} j + 4\left[\left(\frac{n}{2}-1\right)+n\left\{\frac{n(n-2)}{8}-1\right\}-\left\{\frac{n(n-1)(n-2)}{12}-2\right\}\right] \\ &= 8\sum_{i=2}^{\frac{n}{2}}\sum_{j=1}^{n-i} j + 4\left[\left(\frac{n}{2}-1\right)+\left\{\frac{3n^3-6n^2-24n}{24}\right\}-\left\{\frac{2n^3-6n^2+4n-48}{24}\right\}\right] \\ &= n^3-4n^2+4n+4\left[\frac{(\frac{n}{2}-1)(\frac{n}{2})(\frac{n}{2}+1)}{3}\right]+\left[\frac{n^3-16n+24}{6}\right] \\ &= n^3-4n^2+4n+\left[\frac{n^3-4n}{6}\right]+\left[\frac{n^3-16n+24}{6}\right] = \frac{4n^3-12n^2+2n+12}{3}. \end{aligned}$$

Therefore,

$$\begin{aligned} V_c[P_2 \times P_n] &= \frac{1}{2}[S_2+S_3] \\ &= \frac{1}{2}\left[(2n^2-2n+4)+\frac{4n^3-12n^2+2n+12}{3}\right] \\ &= \frac{4n^3-6n^2-4n+24}{6}. \end{aligned}$$

If m is odd, then the sum of all distances between the vertices which are vertex-connected with degree 3 vertices is given by

$$S_3 = 8\sum_{i=2}^{\frac{n+1}{2}}\sum_{j=1}^{n-i} j + \left[\frac{n^3-3n^2-13n+39}{6}\right]$$
$$= \frac{4n^3-12n^2+2n+18}{3}.$$

Therefore,

$$V_c[P_2\times P_n] = \frac{1}{2}[S_2+S_3]$$
$$= \frac{1}{2}\left[2n^2-2n+4+\frac{4n^3-12n^2+2n+18}{3}\right]$$
$$= \frac{4n^3-6n^2-4n+30}{6}.$$

This completes the proof. □

Theorem 2.18. *For integers $m, n > 2$,*

$$V_c[P_m\times P_n] = W[P_m\times P_n] - (2m^2n+2mn^2-4mn-2m^2-2n^2+4).$$

Proof. In $P_m\times P_n$, there are 4 vertices of degree 2, $2(m+n-4)$ vertices of degree 3 and $(m-2)(n-2)$ vertices of degree 4. Every vertex of degree 4 is vertex-connected with all other vertices of $P_m\times P_n$.

Let us first consider the vertices of degree 2 in $P_m\times P_n$. A vertex of degree 2 is vertex-connected with all other vertices except those $(m+n-3)$ border vertices which does not belong to its row or column. Therefore, sum of all distance between the vertices which are not vertex-connected with degree 2 vertices is given by

$$S_2^{'} = 4\left[(n+\ldots+n+m-3)+(m+\ldots+m+n-3)+(m+n-2)\right]$$
$$= 4\left[\frac{m-2}{2}(2n+m-3)+\frac{n-2}{2}(2m+n-3)+m+n-2\right]$$
$$= 8mn+2m^2+2n^2-14m-14n+16.$$

The vertices of degree 3 in the first and m^{th} columns are vertex-connected with all other vertices except the border vertices of first and n^{th} rows which does not belong to their respective columns. Similarly, the vertices of degree 3 in the first and n^{th} rows are vertex-connected with all other vertices except the border vertices of first and m^{th} columns which does not belong to their respective rows. Now we consider 4 cases.

Case 1. m and n are even.

The sum of all distance between the vertices which are not vertex-connected with degree 3 vertices is given by

$$S_{31}^{'} = 4\left[\sum_{j=2}^{\frac{n}{2}}\sum_{i=j}^{j+m-2} i + \sum_{j=\frac{n}{2}+1}^{n-1}\sum_{i=j}^{j+m-2} i + \sum_{j=2}^{\frac{m}{2}}\sum_{i=j}^{j+n-2} i + \sum_{j=\frac{m}{2}+1}^{m-1}\sum_{i=j}^{j+n-2} i\right].$$

Observe that,

$$\sum_{j=2}^{\frac{n}{2}} \sum_{i=j}^{j+m-2} i = \frac{(m-1)(n-2)(2m+n)}{8},$$

$$\sum_{j=2}^{\frac{m}{2}} \sum_{i=j}^{j+n-2} i = \frac{(n-1)(m-2)(2n+m)}{8},$$

$$\sum_{j=\frac{n}{2}+1}^{n-1} \sum_{i=j}^{j+m-2} i = \frac{(m-1)(n-2)(2m+3n-4)}{8},$$

$$\sum_{j=\frac{m}{2}+1}^{m-1} \sum_{i=j}^{j+n-2} i = \frac{(n-1)(m-2)(2n+3m-4)}{8}.$$

Therefore,

$$\begin{aligned} S'_{31} &= 4[\frac{(m-1)(n-2)}{8}(4m+4n-4) + \frac{(n-1)(m-2)}{8}(4m+4n-4)] \\ &= 2(m+n-1)(2mn-3m-3n+4) \\ &= 4m^2n + 4n^2m - 16mn - 6m^2 - 6n^2 + 14m + 14n - 8. \end{aligned}$$

Case 2. m is odd and n is even.

The sum of all distance between the vertices which are not vertex-connected with degree 3 vertices is given by

$$\begin{aligned} S'_{32} = 4 &\left[\sum_{j=2}^{\frac{n}{2}} \sum_{i=j}^{j+m-2} i + \sum_{j=\frac{n}{2}+1}^{n-1} \sum_{i=j}^{j+m-2} i + \sum_{j=2}^{\frac{m-1}{2}} \sum_{i=j}^{j+n-2} i \right. \\ &\left. + \sum_{j=\frac{m+1}{2}+1}^{m-1} \sum_{i=j}^{j+n-2} i + \sum_{i=\frac{m+1}{2}}^{\frac{m+1}{2}+n-2} i \right]. \end{aligned}$$

We have,

$$\sum_{j=2}^{\frac{m-1}{2}} \sum_{i=j}^{j+n-2} i = \frac{(n-1)(m-3)(2n+m-1)}{8}.$$

$$\sum_{j=\frac{m+1}{2}+1}^{m-1} \sum_{i=j}^{j+n-2} i = \frac{(n-1)(m-3)(2n+3m-3)}{8}$$

$$\sum_{i=\frac{m+1}{2}}^{\frac{m+1}{2}+n-2} i = \frac{(m+n-1)(n-1)}{2}.$$

Therefore,

$$S'_{32} = 4m^2n + 4mn^2 - 16mn - 6m^2 - 6n^2 + 14m + 14n - 8.$$

Case 3. m is even and n is odd.

The sum of all distance between the vertices which are not vertex-connected with degree 3 vertices is given by

$$S_{33}^{'} = 4\left[\sum_{j=2}^{\frac{m}{2}}\sum_{i=j}^{j+n-2} i + \sum_{j=\frac{m}{2}+1}^{m-1}\sum_{i=j}^{j+n-2} i + \sum_{j=2}^{\frac{n-1}{2}}\sum_{i=j}^{j+m-2} i\right.$$
$$\left. + \sum_{j=\frac{n+1}{2}+1}^{n-1}\sum_{i=j}^{j+m-2} i + \sum_{i=\frac{n+1}{2}}^{\frac{n+1}{2}+m-2} i\right].$$

Therefore, similar to Case 2 we obtain

$$S_{33}^{'} = 4n^2m + 4nm^2 - 16mn - 6m^2 - 6n^2 + 14m + 14n - 8.$$

Case 4. m and n are odd.

The sum of all distance between the vertices which are not vertex-connected with degree 3 vertices is given by

$$S_{34}^{'} = 4\left[\sum_{j=2}^{\frac{m-1}{2}}\sum_{i=j}^{j+n-2} i + \sum_{j=\frac{m+1}{2}+1}^{m-1}\sum_{i=j}^{j+n-2} i + + \sum_{j=2}^{\frac{n-1}{2}}\sum_{i=j}^{j+m-2} i\right.$$
$$\left. + \sum_{j=\frac{n+1}{2}+1}^{n-1}\sum_{i=j}^{j+m-2} i + \sum_{i=\frac{n+1}{2}}^{\frac{n+1}{2}+m-2} i + \sum_{i=\frac{m+1}{2}}^{\frac{m+1}{2}+n-2} i\right].$$
$$= 2(m+n-1)[(n-1)(m-3) + (m-1)(n-3) + n - 1 + m - 1]$$
$$= 2(m+n-1)[2mn - 3m - 3n + 4]$$
$$= 4m^2n + 4mn^2 - 16mn - 6m^2 - 6n^2 + 14m + 14n - 8.$$

That is, in all the above cases the sum of all distances between the vertices which are not vertex-connected with degree 3 vertices is the same given by

$$S_3^{'} = 4m^2n + 4mn^2 - 16mn - 6m^2 - 6n^2 + 14m + 14n - 8.$$

Therefore, the sum of all distances between the vertices which are not vertex-connected with either degree 2 vertices or degree 3 vertices is given by

$$S_2^{'} + S_3^{'} = 4m^2n + 4mn^2 - 8mn - 4m^2 - 4n^2 + 8.$$

Hence,

$$V_c[P_m \times P_n] = W[P_m \times P_n] - \frac{1}{2}\left[S_2^{'} + S_3^{'}\right]$$
$$= W[P_m \times P_n] - \frac{1}{2}\left[4m^2n + 4mn^2 - 8mn - 4m^2 - 4n^2 + 8\right]$$
$$= W[P_m \times P_n] - \left[2m^2n + 2mn^2 - 4mn - 2m^2 - 2n^2 + 4\right].$$

This completes the proof. □

References

1. K.Priya and V. Anil Kumar: On the vertex connectivity polynomial of graphs, *Advances and Applications in Discrete Mathematics*, 26(2021), No.2, 133–147.
2. R. Frucht and F. Harary: On the corona of two graphs, *Aequationes Math.* 4 (1970), 322–324.
3. A. Joshi, *Wiener Index and Wiener Polynomial for Graphs*, PhD Thesis, 2009.
4. H. Wiener, Structural determination of paraffin boiling points,*J. Amer. Chem. Soc.*, 69 (1947), 17–20.
5. Yeh, Yeong-Nan, and Ivan Gutman, On the sum of all distances in composite graphs, *Discrete Mathematics*, 135(1994), 359–365.
6. F.Harary, *Graph Theory*, Addison-Wesley, 1969.
7. Balaban, A. T.: Applications of graph theory in chemistry, *Journal of chemical information and computer sciences*, 25(1985), No. 3, 334–343.
8. Thilakam K, *A Study on Wiener Number of a Graph*, PhD Thesis, 2009.

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