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MEROMORPHIC FUNCTIONS THAT SHARE ONE FINITE VALUE WITH THEIR DERIVATIVE IN AN ANGULAR DOMAIN

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ABSTRACT. In this paper, we discuss an meromorphic function $f(z)$ and $f'(z)$ share the value 1 CM (counting multiplicities) with derivative in an angular domain.

1. Introduction

Al-Khaladi proved some interesting results on uniqueness of meromorphic functions that share one value with their derivative and one finite value DM (different multiplicities) with first derivatives. The uniqueness theory of meromorphic functions is an interesting problem in the value distribution theory. In 1929, R. Nevanlinna proved that, if f and $\widehat{f}$ be two non-constant meromorphic functions in $\mathbb{C}$ and if they share five distinct values IM, then $f \equiv \widehat{f}$; if they share four distinct values CM, then f is a Mobius transformation of $\widehat{f}$. After this work, many authors proved several results on uniqueness of meromorphic functions concerning shared values in the complex plane. In 2004, J. H. Zheng (see [15]) extended the uniqueness of meromorphic functions dealing with five shared values in an angular domains of $\mathbb{C}$. Also in 2010, He Ping proved some important results on the uniqueness of meromorphic functions sharing values in an angular domain (see [11]). It is interesting to prove some important uniqueness results in the whole of the complex plane to an angular domain. In this paper, we discuss a meromorphic function $f(z)$ and $f'(z)$ share the value 1 CM (counting multiplicities) with derivative in an angular domain $\Omega(\alpha, \beta) = \{z : \ \alpha \leq \arg z \leq \beta\}$.

2. Basic Notations and Definitions

Nevanlinna theory in an angular domain will play a key role in the proof of theorems. Let $f(z)$ be a meromorphic function on the angular domain $\Omega(\alpha, \beta) = \{z : \ \alpha \leq \arg z \leq \beta\}$,

$$A_{\alpha,\beta}(r, f) = \frac{\omega}{\pi} \int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right) \left\{\log^+ \left|f\left(te^{i\alpha}\right)\right| + \log^+ \left|f\left(te^{i\beta}\right)\right|\right\} \frac{dt}{t},$$

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$$
\begin{aligned}
B_{\alpha,\beta}(r,f) &= \frac{2\omega}{\pi r^{\omega}}\int_{\alpha}^{\beta}\log^{+}\left|f\left(re^{i\theta}\right)\right|\sin(\theta-a)d\theta,\\
C_{\alpha,\beta}(r,f) &= \sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^{\omega}}-\frac{|b_n|^{\omega}}{r^{2\omega}}\right)\sin\omega(\theta_n-a)d\theta,
\end{aligned}
$$

where $\omega=\pi/(\beta-\alpha)$ and $b_n=|b_n|\,e^{i\theta_n}$ are the poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ appearing according to the multiplicities, $C_{\alpha,\beta}$ is called angular counting function of the poles of $f(z)$ on $\Omega(\alpha,\beta)$ and the Nevanlinna's angular Tsuji characteristic function is defined as follows

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).$$

Throughout, we denote by $R_{\alpha,\beta}(r,*)$ a quantity satisfying satisfying

$$R_{\alpha,\beta}(r,*)=O\{\log(rS_{\alpha,\beta}(r,*))\},\quad r\notin E,$$

where E denotes a set of positive real numbers with finite linear measure.

Definition 2.1. Let $f(z)$ be a meromorphic function in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$. Then function

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f)$$

is called the angular Nevanlinna Tsuji characteristic of $f(z)$.

For a meromorphic function f in Ω and for all complex numbers a, if

$$\overline{\lim_{r\to\infty}}\frac{S_{\alpha,\beta}(r,f)}{\log r}=\infty,$$

then f is called transcedental with respect to the Tsuji characteristic (see [14]).

Let $f(z)$ and $g(z)$ be two meromorphic functions in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$ and let $f(z)$ be transcedental in Tsuji sense, share the finite value a IM (ignoring multiplicities) if $W(z)-a$ and $M(z)-a$ have the same zeros in angular domain. If $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities, we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicities) in an angular domain. If $f(z)-a$ and $g(z)-a$ have the same zeros with different multiplicities, we say that $f(z)$ and $g(z)$ share the value a DM (different multiplicities) in an angular domain.

Next, let k be a positive integer, we denote by $C^{k)}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)$ the counting function of zeros of $f(z)-a$ with multiplicity$\le k$ in an angular domain and $C^{(k+1}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)$ the counting function of zeros of $f(z)-a$ with multiplicity$>k$ in an angular domain, respectively. Definitions of the terms $C^{k)}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)$ and $C^{(k+1}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)$ can be similarly formulated. Finally, let $C^{2}_{\alpha,\beta}\left(r,\frac{1}{f}\right)$ denote the counting function of zeros of f where a zero of multiplicity k is counted with multiplicity $\min\{k,2\}$ in an angular domain.

3. Some Lemmas

Lemma 3.1. ([1]) *Let $f(z)$ be a meromorphic function in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$, $a\in\mathbb{C}$. Then,*

$$S_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)=S_{\alpha,\beta}(r,f)+O(1)$$

and for an integer $p\ge 0$,

$$S_{\alpha,\beta}(r,f^{(p)})\le 2pS_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f),$$

$$A_{\alpha,\beta}\left(r,\frac{f^{(p)}}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(p)}}{f}\right)=R_{\alpha,\beta}(r,f)$$

and $R_{\alpha,\beta}(r,f^{(p)})=R_{\alpha,\beta}(r,f)$.

Lemma 3.2. ([1]) *Let $f(z)$ be a meromorphic function in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$, $a\in\mathbb{C}$. Then, for arbitrary q distinct $a_j\in\overline{\mathbb{C}}$, $1\le j\le q$, we have*

$$(q-2)S_{\alpha,\beta}(r,f)\le\sum_{j=1}^{q}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+R_{\alpha,\beta}(r,f),$$

where the term $\overline{C}_{\alpha,\beta}\,(r,1/f-a_j)$ will be replaced by $\overline{C}_{\alpha,\beta}\,(r,f)$ when some $a_j=\infty$.

We use $\overline{C}^{k)}_{\alpha,\beta}\,(r,1/f-a_j)$ to denote the zeros of $f(z)-a$ in $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$ whose multiplicities are no greater than k and are counted only once. Likewise, we use $\overline{C}^{(k+1}_{\alpha,\beta}\,(r,1/f-a_j)$ to denote the zeros of $f(z)-a$ in $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$, whose multiplicities are greater than k and are counted only once.

For the proof of our theorem we need the following lemmas

Lemma 3.3. *Let $f(z)$ be a meromorphic function on the angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$ and transcedental in Tsuji sense, such that $f'(z)$ is not constant and which satisfies*

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+\overline{C}^{(2}_{\alpha,\beta}(r,f)=R_{\alpha,\beta}(r,f).$$

Then, either

$$\begin{cases} A_{\alpha,\beta}\left(r,\frac{1}{f'-1}\right)+B_{\alpha,\beta}\left(r,\frac{1}{f'-1}\right)=R_{\alpha,\beta}(r,f)\\ C^{(2}_{\alpha,\beta}\left(r+\frac{1}{f''}\right)\le C_{\alpha,\beta}\left(r+\frac{1}{f'}\right)+R_{\alpha,\beta}(r,f)\end{cases}\tag{3.1}$$

or

$$f(z)=\frac{c-2}{z+c_1}+c_3,\tag{3.2}$$

where $c_1,c_2(\neq 0)$ and c_3 are constants.

Proof. We consider the following function

$$G=\left(\frac{f''}{f'}\right)^2-2\left(\frac{f''}{f'}\right)'. \tag{3.3}$$

Therefore

$$A_{\alpha,\beta}(r,G)+B_{\alpha,\beta}(r,G)=R_{\alpha,\beta}(r,f). \tag{3.4}$$

If z_∞ is a simple pole of f in an angular domain, then from (3.3) we see that G is holomorphic at z_∞. Thus

$$C_{\alpha,\beta}(r,G)\leq 2\overline{C}^{(2}_{\alpha,\beta}(r,f)+2\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'}\right)=R_{\alpha,\beta}(r,f). \tag{3.5}$$

By hypothesis, combining (3.4) and (3.5) we find that

$$S_{\alpha,\beta}(r,f)=R_{\alpha,\beta}(r,f). \tag{3.6}$$

It follows from (3.3) that if z_0 is a zero of f'' of multiplicity p ($p\geq 2$) in an angular domain and $f'(z_0)\neq 0$, then

$$G(z)=)\left((z-z_0)^{p-1}\right). \tag{3.7}$$

If $G(z)\equiv 0$, we have from (3.3) that

$$2\left(\frac{f''}{f'}\right)^{-2}\left(\frac{f''}{f'}\right)'=1.$$

By integrating three times we conclude (3.2). We next suppose $G(z)\not\equiv 0$. By (3.7) and (3.6), we obtain

$$\begin{aligned}C^{(2}_{\alpha,\beta}\left(r,\frac{1}{f''}\right) &\leq 2C_{\alpha,\beta}\left(r,\frac{1}{G}\right)+C_{\alpha,\beta}\left(r,\frac{1}{f'}\right)\\ &= C_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+R_{\alpha,\beta}\left(r,f\right).\end{aligned}$$

We rewrite (3.3) in the form

$$\frac{1}{f'-1}=\frac{1}{G}\left(\frac{f''}{f'-1}-\frac{f''}{f'}\right)\left(3\frac{f''}{f'}-2\frac{f'''}{f''}\right).$$

Obviously,

$$A_{\alpha,\beta}\left(r,\frac{1}{f'-1}\right)+B_{\alpha,\beta}\left(r,\frac{1}{f'-1}\right)=R_{\alpha,\beta}(r,f),$$

from the fundamental estimate and (3.6), so our lemma is proved. ☐

Lemma 3.4. *Let $f(z)$ be a meromorphic function on the angular domain $\Omega(\alpha,\beta)$ $=\{z:\ \alpha\leq\arg z\leq\beta\}$ and transcedental in Tsuji sense, satisfying*

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+\overline{C}^{(2}_{\alpha,\beta}(r,f)=R_{\alpha,\beta}(r,f).$$

If f and f' share the value 1 CM on annuli, then

$$C^{(2}_{\alpha,\beta}\left(r,\frac{1}{f''}\right)+A_{\alpha,\beta}\left(r,\frac{1}{f'-1}\right)+B_{\alpha,\beta}\left(r,\frac{1}{f'-1}\right)=R_{\alpha,\beta}(r,f). \tag{3.8}$$

Proof. This is easy since if $f(z)$ and $f'(z)$ share 1 CM in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le \arg z\le\beta\}$, then f' is not constant and (3.2) does not hold. Thus, (3.8) follows from (3.1) and $C_{\alpha,\beta}\left(r,\frac{1}{f'}\right)=R_{\alpha,\beta}(r,f)$. □

Lemma 3.5. *Let $f(z)$ be a meromorphic function in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$ and transcedental in Tsuji sense, such that $f'(z)$ is not constant. If $f(z)=1$ and $f'(z)=1$ on $\Omega(\alpha,\beta)$, then either*

$$C^{1)}_{\alpha,\beta}\left(r,\frac{1}{f'-1}\right)\le A_{\alpha,\beta}(r,f)+C_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+R_{\alpha,\beta}(r,f)$$

or $f(z)$ satisfies the identity (3.1).

Proof. We set

$$H=\frac{f''(f-1)}{f'(f-1)}. \tag{3.9}$$

Then, it is clear that

$$A_{\alpha,\beta}(r,H)+A_{\alpha,\beta}(r,H)\le A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f). \tag{3.10}$$

From (3.9), we know that if z_∞ is a pole of f of multiplicity $p\ge 1$ in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$, then

$$H\left(z_\infty\right)\ne\infty. \tag{3.11}$$

Let z_1 be a zero of $f'-1$ of multiplicity $q\ge 1$ in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$. Since $f'(z)=1$ implies that $f(z)=1$ by assumption, we must have z_1 is a simple zero of $f-1$ in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$. By simple calculation on the local expansion we see that

$$H(z_1)=q. \tag{3.12}$$

From (3.9), (3.11) and (3.12) it can be seen that the poles of H can only occurs at the zeros of f' in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$. That is,

$$C_{\alpha,\beta}(r,H)\le\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'}\right). \tag{3.13}$$

Further, if $H\not\equiv 1$, it follows from (3.12), (3.10) and (3.13) that

$$\begin{aligned}C^{1)}_{\alpha,\beta}\left(r,\frac{1}{f'-1}\right)&\le C_{\alpha,\beta}\left(r,\frac{1}{H-1}\right)\le S_{\alpha,\beta}(r,H)+O(1)\\&\le A_{\alpha,\beta}(r,H)+B_{\alpha,\beta}(r,H)+C_{\alpha,\beta}(r,H)+O(1)\\&\le A_{\alpha,\beta}(r,f)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+R_{\alpha,\beta}(r,f).\end{aligned}$$

Finally, if $H\equiv 1$, then

$$\frac{f''}{f'-1}=\frac{f'}{f-1}.$$

By integration, we get (3.1). □

Lemma 3.6. *Let $f(z)$ be a meromorphic function in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$ and transcedental in Tsuji sense, such that $f^{(k)}(z)$ is not constant and k is a positive integer. Then either*

$$\left(f^{(k+1)}\right)^{k+1}=c\left(f^{(k)}-\lambda\right)^{k+2}$$

for some non zero constant c, or

$$kC^{1)}_{\alpha,\beta}(r,f)\le\overline{C}^{(2}_{\alpha,\beta}(r,f)+C^{1)}_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-\lambda}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f^{(k+1)}}\right)+R_{\alpha,\beta}(r,f),\quad(3.15)$$

where λ is a constant.

Proof. Let

$$\Psi=(k+1)\frac{f^{(k+2)}}{f^{(k+1)}}-(k+2\nu)\frac{f^{(k+1)}}{f^{(k)}-\lambda}.\quad(3.16)$$

Suppose that z_∞ is a simple pole of f an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$. Then, an elementary calculation gives that $\Psi(z)=O((z-z_\infty)^k)$, which proves that z_∞ is a zero of Ψ of multiplicity k in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$. Thus, if (3.14) is not true, i.e $\Psi\not\equiv0$, then

$$kC^{1)}_{\alpha,\beta}(r,f)\le C_{\alpha,\beta}\left(r,\frac{1}{\Psi}\right)\le S_{\alpha,\beta}(r,\Psi)+O(1).\quad(3.17)$$

Note that Ψ can only have simple poles at zeros of $f^{(k+1)}$ or $f^{(k)}-\lambda$ or multiple poles of f in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$. Thus we deduce from (3.16) that

$$kC_{\alpha,\beta}(r,\Psi)\le\overline{C}^{2)}_{\alpha,\beta}(r,f)+C^{1)}_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-\lambda}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f^{()k+1}}\right).\quad(3.18)$$

Again, from (3.16) we obtain

$$A_{\alpha,\beta}(r,\Psi)+B_{\alpha,\beta}(r,\Psi)=R_{\alpha,\beta}(r,\Psi).$$

Combining (3.18) and (3.17), we get (3.15). □

Lemma 3.7. *Let $f(z)$ be a meromorphic function in an angular domain $\Omega(\alpha,\beta)=\{z:\ \alpha\le\arg z\le\beta\}$ and transcedental in Tsuji sense, such that $f^{(k)}(z)$ is not constant. Then, either f is as in (3.2) or*

$$C^{1)}_{\alpha,\beta}(r,f)\le\overline{C}^{(2}_{\alpha,\beta}(r,f)+C^{1)}_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f''}\right)+R_{\alpha,\beta}(r,f).\quad(3.19)$$

Proof. Applying Lemma 3.6 to $k=1$ and $\lambda=0$, we get either that (3.19) holds, or

$$\left(\frac{f''}{f'}\right)^2=cf'\quad(3.20)$$

for some non zero constant c. Differentiating (3.20), we find that

$$2f''\left(\frac{f''}{f'}\right)'=cf''f'.\quad(3.21)$$

If $f^{''} = 0$, then f' is a constant. Therefore, $f^{''} \not\equiv 0$ and so by (3.21), we have

$$2\nu \left(\frac{f^{''}}{f'}\right)' = cf'.$$

Combining this with (3.20) we get

$$2\left(\frac{f^{''}}{f'}\right)^{-2} = 1$$

By integrating three times we conclude (3.2). □

4. Main Results

Now, the main theorem of this paper are listed as follows

Theorem 4.1. *Let $f(z)$ be a meromorphic function in an angular domain $\Omega(\alpha,\beta) = \{z: \ \alpha \le \arg z \le \beta\}$ and transcedental in Tsuji sense, satisfying $C_{\alpha,\beta}\left(r, \frac{1}{f'}\right) = R_{\alpha,\beta}(r,f)$. Suppose that $f(z)$ and $f'(z)$ share the value 1 CM on $\Omega(\alpha,\beta)$. Then*

$$f' - 1 = c(f-1)$$

for some non zero constants c.

Proof. The proof is by contradiction. Assume that (4.1) is not true. Then,

$$S_{\alpha,\beta}(r,f) \le 2\overline{C}_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f). \tag{4.2}$$

We see from (4.2) that

$$C_{\alpha,\beta}(r,f) = \overline{C}^{1)}_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f). \tag{4.3}$$

Let

$$F = 2\left(\frac{f'}{f-1}\right) - \frac{f^{''}}{f'-1} + \frac{f^{''}}{f'}. \tag{4.4}$$

Then,

$$A_{\alpha,\beta}(r,F) + B_{\alpha,\beta}(r,F) = R_{\alpha,\beta}(r,f). \tag{4.5}$$

If z_∞ is a simple pole of f in an angular domain $\Omega(\alpha,\beta) = \{z: \ \alpha \le \arg z \le \beta\}$, then from (4.4) we see that F will be holomorphic at z_∞. From this, (4.3), the hypothesis of Theorem 4.1 and (4.5) we can deduce that

$$S_{\alpha,\beta}(r,F) = R_{\alpha,\beta}(r,f). \tag{4.6}$$

Suppose that z_0 is a simple zero of $f^{''}$ which is not a zero of f' in an angular domain $\Omega(\alpha,\beta) = \{z: \ \alpha \le \arg z \le \beta\}$. Since f and f' share 1 CM in an angular domain $\Omega(\alpha,\beta) = \{z: \ \alpha \le \arg z \le \beta\}$, this gives all zeros of $f'-1$ are simple in an angular domain $\Omega(\alpha,\beta) = \{z: \ \alpha \le \arg z \le \beta\}$. Hence, $f'(z_0) \ne 1$, and thus

$$F(z_0) = 2\left(\frac{f'(z_0)}{f(z_0)-1}\right) \tag{4.7}$$

from (4.4). Differentiating (4.4) and using $f^{''}(z_0) = 0$, we arrive at

$$F'(z_0) = -2\left(\left(\frac{f'(z_0)}{f(z_0)-1}\right)^2 + \frac{f^{'''}(z_0)}{f'(z_0)-1}\right) + \frac{f^{'''}(z_0)}{f'(z_0)}. \tag{4.8}$$

Combining (4.7) and (4.8) we obtain

$$-2F'(z_0) = F^2(z_0) + \frac{4\nu f'''(z_0)}{f'(z_0)-1} - 2\frac{f'''(z_0)}{f'(z_0)}. \tag{4.9}$$

On the other hand, by (3.3) we
nd that

$$G(z_0) = -2\nu \frac{f'''(z_0)}{f'(z_0)} \neq 0. \tag{4.10}$$

Substituting (4.10) into (4.9) gives

$$f'(z_0)\left[F^2(z_0) + 2\nu F'(z_0) - G(z_0)\right] = F^2(z_0) + 2\nu F'(z_0) + G(z_0). \tag{4.11}$$

If $F^2(z_0) + 2\nu F'(z_0) - G(z_0) = 0$, then from (4.11) we get $G(z_0) = 0$ which contradicts (4.10). Therefore, $F^2(z_0) + 2\nu F'(z_0) - G(z_0) \neq 0$, and (4.11) reads

$$f'(z_0) = \frac{F^2(z_0) + 2\nu F'(z_0) + G(z_0)}{F^2(z_0) + 2\nu F'(z_0) - G(z_0)} = a(z_0), \text{ say.}$$

By (4.6) and (3.6), it vis easy to see that

$$S_{\alpha,\beta}(r,a) = R_{\alpha,\beta}(r,f),$$

which means that we the following property, if z_0 is a simple zero of f'' in an angular domain $\Omega(\alpha,\beta) = \{z:\ \alpha \leq \arg z \leq \beta\}$ and $f'(z_0) \neq 0$, then

$$f'(z_0) = a(z_0). \tag{4.13}$$

Let z_1 be a zero of $f-1$ on $\Omega(\alpha,\beta)$. Then, the Taylor expansion of f about z_1 on $\Omega(\alpha,\beta)$ is

$$f(z) - 1 = (z-z_1) + a_2(z-z_1)^2 + a_3(z-z_1)^3 + \cdots, \quad a_2 \neq 0. \tag{4.14}$$

It follows from (4.4) and (4.3) that

$$F(z_1) = 4a_2 - 3\frac{a_3}{a_2} \quad \text{and} \quad G(z_1) = 12\left(a_2^2 - a_3\right).$$

That is,

$$2f''^2(z_1) - F(z_1)f''(z_1) - f'''(z_1) = 0 \quad \text{and} \quad 3f''^2(z_1) - 2f'''(z_1) - G(z_1) = 0$$

and eliminating $f''^2(z_1)$ from the last two equations we obtain

$$f'''(z_1) - F(z_1)f''(z_1) + 2G(z_1) = 0. \tag{4.15}$$

Now considering the following equation

$$J = \frac{f''' - 3Ff'' + 2Gf'}{f'(f'-1)}. \tag{4.16}$$

From this, we know that if z_∞ is a simple pole of f in an angular domain $\Omega(\alpha,\beta) = \{z:\ \alpha \leq \arg z \leq \beta\}$, then J is holomorphic at z_∞ in an angular domain $\Omega(\alpha,\beta) = \{z:\ \alpha \leq \arg z \leq \beta\}$. Thus we deduce from (4.16), the hypothesis of Theorem 4.1, $(4.15), (4.3), (4.6)$ and (3.6) that

$$C_{\alpha,\beta}(r,J) = R_{\alpha,\beta}(r,f). \tag{4.17}$$

Also, from (4.16), (4.6), (3.6) and Lemma 3.2, we conclude $m_0(r,J) = S_0(r,J)$. Combining this with (4.17) yields

$$S_{\alpha,\beta}(r,J) = R_{\alpha,\beta}(r,J). \tag{4.18}$$

Eliminating $f^{'''}$ between (4.16) and (3.3) leads to

$$2Jf^{'2}(f'-1) = 3f^{''2} + 3Gf^{'2} - 6Ff^{''}f'. \tag{4.19}$$

From (4.18) and (4.13) we get

$$J(z_0) = \frac{3G(z_0)}{2[a(z_0)-1]}.$$

If $J \not\equiv \frac{3G}{2(a-1)}$, by $C_{\alpha,\beta}\left(r, \frac{1}{f'}\right) = R_{\alpha,\beta}(r,f)$, Lemma 3.2, (4.18), (4.12) and (3.6), we have

$$\begin{aligned} C_{\alpha,\beta}\left(r,\frac{1}{f''}\right) \quad &\le \quad C_{\alpha,\beta}\left(r, \frac{1}{J-\frac{3G}{2(a-1)}}\right) \\ &+C_{\alpha,\beta}\left(r,\frac{1}{f'}\right) + C^{(2}_{\alpha,\beta}\left(r,\frac{1}{f''}\right) = R_{\alpha,\beta}(r,f). \end{aligned} \tag{4.20}$$

Thus, (4.20), (3.2), Lemma 3.5, the assumptions of the theorem and (4.2) imply that

$$S_{\alpha,\beta}(r,f') = R_{\alpha,\beta}(r,f).$$

Hence,

$$\begin{aligned} S_{\alpha,\beta}(r,f) \quad &\le \quad C_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) + A_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) + B_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) \\ &\le \quad C_{\alpha,\beta}\left(r,\frac{1}{f'-1}\right) + A_{\alpha,\beta}\left(r,\frac{1}{f'}\right) + R_{\alpha,\beta}\left(r,f\right) \\ &\le \quad 2S_{\alpha,\beta}(r,f') + R_{\alpha,\beta}(r,f) = 3R_{\alpha,\beta}(r,f), \end{aligned}$$

which is a contradiction. Therefore,

$$J \equiv \frac{(2\nu+1)G}{2\nu(a-1)}$$

and (4.19) becomes

$$Gf^{'2}\left(f'-a\right) = f^{''}\left(f^{''} - 2Ff'\right)(a-1). \tag{4.21}$$

Differentiating (4.21) and then using (4.13) we obtain

$$a'(z_0)a(z_0)G(z_0) = 2[a(z_0-1)]F(z_0)f^{'''}(z_0). \tag{4.22}$$

and also note from (4.10) and (4.12) that

$$f^{'''}(z_0) = -\frac{1}{2\nu}G(z_0)a(z_0).$$

Now, substitute this back into (4.22) and get

$$\frac{a'(z_0)}{a(z_0)-1} = -F(z_0)$$

since $a(z)G(z_0) \neq 0$. In the following we shall treat two cases, i.e., $\frac{a'}{a-1} \not\equiv F$ and $\frac{a'}{a-1} \equiv F$ separately.

Case 1. $\frac{a'}{a-1} \not\equiv F$.

In this case, similar to the above discussion we will arrive at the same contradiction.

Case 1. $\frac{a'}{a-1} \equiv F$.

By integrating, we get

$$\frac{1}{(f-1)^2} = \left(\frac{a-1}{c}\right)\left(\frac{1}{f'-1} + \frac{1}{(f'-1)^2}\right), \tag{4.23}$$

where c is a non zero constant. Using (4.23) together with (4.12) and Lemma 3.2 we find that

$$A_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + B_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) = R_{\alpha,\beta}\left(r, f\right).$$

Finally, from this, the assumption of Theorem 4.1 and Lemma 3.3 it can be deduce that

$$\begin{aligned} S_{\alpha,\beta}\left(r, f\right) &= C_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R_{\alpha,\beta}\left(r, f\right) \\ &= C_{\alpha,\beta}\left(r, \frac{1}{f''-1}\right) + R_{\alpha,\beta}\left(r, f\right) \\ &= C^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-1}\right) + R_{\alpha,\beta}\left(r, f\right) \\ &\leq A_{\alpha,\beta}\left(r, f\right) + B_{\alpha,\beta}\left(r, f\right) + R_{\alpha,\beta}\left(r, f\right). \end{aligned}$$

This implies that $C_{\alpha,\beta}(r, f) = R_{\alpha,\beta}(r, f)$. When combined it with (4.2) gives $S_{\alpha,\beta}(r, f') = R_{\alpha,\beta}(r, f)$ which implies the contradiction $S_{\alpha,\beta}(r, f) = R_{\alpha,\beta}(r, f)$. This completes the proof of Theorem 4.1. □

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