Research article

Pure bending analysis of thin rectangular SSSS plate Using Taylor-Mclaurin series

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ABSTRACT

The bending properties of a plate depend greatly on its thickness as compared with its other dimensions. Previous studies have also made distinction between bending in; thin plates with small deflections, thin plates with large deflections and thick plates. Earlier studies on pure bending of thin rectangular plates were, however, centered on the use of trigonometric series and approximate methods in form of numerical and energy methods. In the present study, an in-depth bending analysis of transversely loaded thin rectangular SSSS plate was carried out. The analysis was accomplished through a theoretical formulation of shape function based on Taylor-McLaurin's series and subsequently used the shape function on Ritz energy method. The Taylor-McLaurin's series shape function was truncated at the fifth terms in x and y before substituting into a total potential energy functional, which was formulated based on Ritz energy approach. Boundary conditions for SSSS plate was first applied to the shape function in order to determine the constants J_m and K_n before substitution into the potential energy functional. The resulting functional equation was minimized and simplified to obtain α. Values of α from the present study were compared with those obtained by other researchers in previous studies. The average percentage difference is about 3.63 and this indicates that the Taylor-McLaurin's shape function is very close to exact displacement shape function of SSSS plate. Therefore, we conclude that the Taylor-McLaurin's shape function is a close approximation of the exact displacement function of a deflected simply supported rectangular thin plate.

Keywords: Bending, boundary conditions, functional, potential energy, taylor-mclaurin series and thin-walled plate vibration.

Notation

w is the shape function (deflection function).
Wmax is the maximum deflection.
"a" and "b" are plate dimensions (lengths) in x and y directions.
R and Q are dimensionless quantities defined as $R = \frac{x}{a}$; $Q = \frac{y}{b}$; $0 \leq R \leq 1$; $0 \leq Q \leq 1$.
First partial derivative of deflection with respect to R(or x)is: $\frac{\partial w}{\partial x} = \frac{1}{a}\frac{\partial w}{\partial R} = \frac{1}{a}w'^{R}$.
Second partial derivative of deflection with respect to R(or x)is:
$\frac{\partial^2 w}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 w}{\partial R^2} = \frac{1}{a^2}w''^{R}$
First partial derivative of deflection with respect to Q(or y)is: $\frac{\partial w}{\partial y} = \frac{1}{b}\frac{\partial w}{\partial Q} = \frac{1}{b}w'^{Q}$.
Second partial derivative of deflection with respect to Q(or y) is: $\frac{\partial^2 w}{\partial y^2} = \frac{1}{b^2}\frac{\partial^2 w}{\partial Q^2} = \frac{1}{b^2}w''^{Q}$.

Second partial derivative of deflection with respect to R(or x) and Q(or y) is:
$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{ab}\frac{\partial^2 w}{\partial R \partial Q} = \frac{1}{ab} w''^{RQ}.$$
J_m is constants in x direction.
k_n is constants in y direction.
A is the product of J_4 and K_4. It is the amplitude of deflection function.
P is the aspect ratio ($P = a/b$).
D is flexural rigidity.
E is Young's modulus of elasticity.
t is the plate thickness.
µ is the Poisson's ratio.
U_{max} is Maximum strain energy functional.
E_{max} is the maximum external work functional.
q is the lateral load applied uniformly on the plate.
Π is the total potential energy functional.
α is the dimensionles maximum deflection parameter with no consideration for Poisson's ratio.
α1 is α from Present study; α2 is α from Timoshenko & Woinowsky-Krieger (1959); α3 is α from Ventsel and Krauthammer (2001).
β is the dimensionles maximum deflection parameter for Poisson's ratio of 0.3.

1. Introduction

Pure bending of thin rectangular plate with all the four edges simply supported (SSSS plate) had been handled by Navier (1823) and Levy (1899). They did this by direct integration of the governing differential equation. It was expected that both solutions should be exact since direct integration of the governing differential equation was done. However, there were about 2.4% average percentage difference in the two solutions. However, many scholars were of the opinion that Levy's solution produced the exact solution since he used infinite single trigonometric series, while Navier used double Fourier (trigonometric) series (Timoshenko and Woinowsky-Krieger, 1959; Iyenger, 1988; Ye, 1994; Ugural, 1999).

Many scholars had gone ahead using energy and numerical approaches (approximate methods) to obtain solutions of pure bending of thin rectangular SSSS plates (Timoshenko & Woinowsky-Krieger, 1959; Ventsel , 1997; Ventsel and Krauthammer, 2001; Tottenham, 1979; Hartmann, 1991; Brebbia et al 1984; Banerjee and Butterfield, 1981; Ugural, 1999). One thing is common to all the solutions. They made use of trigonometric functions mainly. None of them used Taylor-MacLaurin's series to represent the deformed shape of the SSSS plate. The reason why the later scholars adopted approximate approach to the analysis is because of the difficulty involved in direct integration of governing differential equation. The difference between the solutions of Navier and Levy (exact methods) was another reason for using approximate methods.

All the solutions from both exact approach and approximate (numerical and energy) approach varied from one another. Though, the variations are marginal for maximum deflection, they are significant when stresses are computed from their results. Due to the shortcomings of the previous solutions, this present study used Taylor-MacLaurin's series in Ritz method (energy approach) to obtain solutions for pure bending of thin rectangular SSSS plate. Thin rectangular SSSS plate is schematically represented on figure 1.

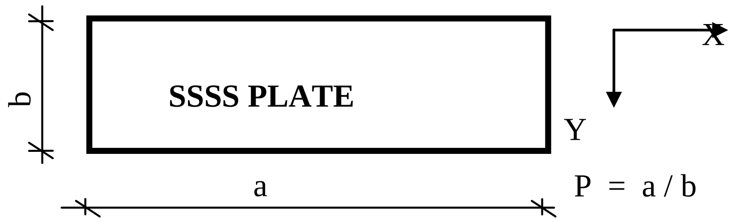

Figure 1: Schematic Representation of SSSS plate under lateral uniform load

2. Application of Ritz Method For Pure Bending of Rectangular Plate

Ibearugbulem (2012) gave the maximum strain energy functional for a thin rectangular isotropic plate subject to pure bending and Taylor McLaurin's shape function respectively as follows:

$$U_{max} = \frac{Da}{2b^3} \iint \left[\frac{1}{P^4}(w''^R)^2 + \frac{2}{P^2}(w''^{RQ})^2 + (w''^Q)^2\right] \partial R \partial Q \qquad (1)$$

$$w = \sum_{m=0}^{4} \sum_{n=0}^{4} J_m K_n \ R^m . Q^n \qquad (2)$$

$$D = \frac{Et^3}{12(1-\mu^2)} \qquad (3)$$

Ventsel and Krauthammer (2001) gave the maximum external work functional as:

$$E_{max} = abq \iint wdR.dR \qquad (4)$$

Adding equations (1) and (4) gave the total potential energy functional of rectangular plate subject to pure bending by uniformly distributed transverse load as:

$$\Pi = \frac{Da}{2b^3} \iint \left[\frac{1}{P^4}(w''^R)^2 + \frac{2}{P^2}(w''^{RQ})^2 + (w''^Q)^2\right] \partial R \, \partial Q - abq \iint wdR.dR \qquad (5)$$

The boundary conditions for SSSS plate are

$$w(R=0) = w''^R(R=0) = 0 \qquad (6)$$
$$w(R=1) = w''^R(R=1) = 0 \qquad (7)$$
$$w(Q=0) = w''^Q(Q=0) = 0 \qquad (8)$$
$$w(Q=1) = w''^Q(Q=1) = 0 \qquad (9)$$

Substituting equations (6)and (8)into equation (2)gave: $J_0 = 0; \ J_2 = 0; \ K_0 = 0; K_2 = 0$
Also, substituting equation (7) into equation (2) and solving the resulting two simultaneous equations gave: $J_1 = J_4; \ J_3 = -2J_4$. Substituting equation (9) into equation (2) and solving the resulting two simultaneous equations gave: $K_1 = K_4; K_3 = -2K_4$

Substituting the values of $J_0, \ J_1, J_2, \ J_3, \ J_4, \ K_0, \ K_1, K_2, \ K_3$ and K_4 into equation (2) gave

$w = (R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)J_4K_4$. That is

$$w = A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4) \qquad (10)$$

Differentiating and integrating equation (10) partially gave the following:

$$\int_0^1 \int_0^1 w\,\partial R\,\partial Q = A(0.2)(0.2) = 0.04A \tag{11}$$

$$\int_0^1 \int_0^1 (w''^{R})^2\,\partial R\,\partial Q = A^2(4.8)(0.04921) = 0.23621A^2 \tag{12}$$

$$\int_0^1 \int_0^1 (w''^{Q})^2\,\partial R\,\partial Q = A^2(0.04921)(4.8) = 0.23621A^2 \tag{13}$$

$$\int_0^1 \int_0^1 (w''^{RQ})^2\,\partial R\,\partial Q = A^2(0.48571)(0.48571) = 0.23591A^2 \tag{14}$$

Substituting equations (11), (12), (13) and (14) into equation (5) gave:

$$\Pi = \frac{DA^2}{2b^2}\left(\frac{0.23621}{P^3} + \frac{0.47182}{P} + 0.23621P\right) - 0.04APb^2q \tag{15}$$

Minimizing equation (15) and solving the resulting eigen-value equation gave:

$$A = \frac{1}{D}\left(\frac{0.04b^4q}{\frac{0.23621}{P^4} + \frac{0.47182}{P^2} + 0.23621}\right) \tag{16}$$

But maximum deflection for SSSS plate occurs at the point where R = Q = ½. Substituting the values of R = 0.5 and Q = 0.5 and equation (16) into equation (10) gave the maximum deflection as:

$$W_{max} = \frac{1}{D}\left(\frac{0.04b^4q}{\frac{0.23621}{P^4} + \frac{0.47182}{P^2} + 0.23621}\right)(0.3125)(0.3125)$$

$$= \frac{1}{D}\left(\frac{0.016537b^4q}{\frac{1}{P^4} + \frac{1.99746}{P^2} + 1}\right)$$

$$= \frac{1}{D}\left(\frac{0.016537a^4q}{\frac{P^4}{P^4} + \frac{1.99746P^4}{P^2} + P^4}\right)$$

That is:

$$W_{max} = \frac{1}{D}\left(\frac{0.016537a^4q}{1 + 1.99746P^2 + P^4}\right) \tag{17}$$

That is:

$$W_{max} = \frac{\alpha}{D}\,.\,a^4q \tag{18}$$

$$\alpha = \left(\frac{0.016537}{1 + 1.99746P^2 + P^4}\right) \tag{19}$$

With a Poisson's ratio of 0.3, equation (17), upon substitution of equation (3), gave

$$W_{max} = \beta \cdot \frac{a^4 q}{Et^3} \tag{20}$$

$$\beta = \left(\frac{0.180584}{1 + 1.99746P^2 + P^4}\right) \tag{21}$$

3. Results and discussion

Ventsel and Krauthammer (2001) applied double Fourier series in Navier's method and got α as:

$$\alpha = \frac{1}{\pi^4} \frac{P_{mn}}{\left[\left(m/a\right)^2 + \left(n/b\right)^2\right]^2}; \text{Where } P_{mn} = \frac{16}{\pi^2 mn}$$

For a case of first mode of failure; where $m = n = 1$ then the expression for α becomes

$$\alpha = \frac{16}{\pi^6 (1 + 2P^2 + P^4)}$$

From table 1, it would be noticed that where as the values of α from the present study is upper bound to the values from Timoshenko & Woinowsky-Krieger (1959), the values from Ventsel and Krauthammer (2001) are upper bound to the values from the present study.

The implication of this development is that the values from the present study lie between two sets of values, hitherto admitted to be closest to the exact values. It would also be noticed that the percentage differences between the present study and Timoshenko & Woinowsky-Krieger (1959), $100(\alpha_1 - \alpha_2)/\alpha_2$ are all less than 5% with average of 2.89%. The percentage differences between the present study and Ventsel and Krauthammer (2001), $100(\alpha_1 - \alpha_3)/\alpha_3$ are less than 5.2%.

The average is 3.63%. This closeness between the values from this present study and the values from earlier researches as was seen here indicated the closeness of the Taylor-McLaurin shape function to the exact displacement shape function of SSSS plate. The values may not be the exact values, but they are very close to it. Hence, one could conclude that displacement function of a deflected simply supported rectangular thin plate can be approximated using Taylor-McLaurin shape function.

4. Conclusion

It can be concluded here that the deflection function from Taylor-McLaurin series gave a very good approximation of the rectangular isotropic plate. The kinematic and force boundary conditions of SSSS plate were all satisfied. An obvious fact here is that, this deflection function is very amenable to Ritz method of continuum analysis. Thus, we are recommending the use of this deflection function for dynamic and stability analysis of rectangular isotropic and orthotropic plates.

Table 1: Values for α of this present study, Timoshenko & Woinowsky-Krieger (1959) and Ventsel and Krauthammer (2001)

b/a	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
P =a/b	1.0	0.9091	0.8333	0.7692	0.7143	0.6667	0.6250	0.5882	0.5556	0.5263	0.5000
α1	0.00414	0.00496	0.00576	0.00653	0.00725	0.00793	0.00855	0.00913	0.00966	0.01014	0.01058
α2	0.00406	0.00485	0.00564	0.00638	0.00705	0.00772	0.00830	0.00883	0.00931	0.00974	0.01013
α3	0.00416	0.00499	0.00580	0.00657	0.00730	0.00798	0.00861	0.00919	0.00972	0.01021	0.01065
Percentage Difference (α1- α2)	1.82	2.18	2.10	2.27	2.78	2.61	2.95	3.27	3.59	3.95	4.28
Percentage Difference (α1- α3)	-2.479	-2.864	-2.775	-2.960	-3.505	-3.324	-3.687	-4.030	-4.383	-4.779	-5.146
β	0.04517	0.05417	0.06293	0.07132	0.07922	0.08660	0.09343	0.09972	0.10550	0.11078	0.11562

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