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NUMERICAL SOLUTIONS OF TWO-POINT NONLINEAR BOUNDARY VALUE PROBLEM VIA TAYLOR DECOMPOSITION SHOOTING METHOD

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Abstract

This research aims to present a hybrid approach method to find approximate solutions to the boundary value problems of ordinary differential equations (ODEs). specifically, the class of two-point boundary value problems. This method based of shooting method combination with the Taylor Series Expansion. The proposed method examined several boundary value problems for second and third order nonlinear ODEs. For validating the suggested methods, the obtained results are compared with the exact solution and other approximation methods. It has been found that the convergence of the proposed methods to the exact solution is so high. The proposed methods provide the most accurate numerical results for boundary value problems, in accordance with this study's findings.

***Keywords**: Boundary Value Problems (BVPs), Taylor Series Expansion, Adomian decomposition method (ADM), Shooting Method.*

INTRODUCTION

A lot of models derived in real life situations from different fields of mathematics and other related areas such engineering, biology, biochemistry, physics, biotechnology and even biomathematics often degenerate into linear or nonlinear differential equations. These equations may either be Ordinary Differential Equations (ODEs) or Partial Differential Equations (PDEs). The solutions of most linear ODEs and PDEs can be obtained by direct integration, separation of variable methods, Laplace transformation method, Fourier transformation method etc. [1]-[3]. However, most nonlinear differential equations be it ODEs or PDEs are not that easy to solve for their exact solutions. Hence, many researchers have developed different methods of solutions including analytical and numerical methods [4]-[14]. Numerical methods are used to obtain approximate solutions to any given problem. As often observed, most of these models are very difficult to solve for their exact solutions hence this has necessitated the use of approximated technique to obtain approximated solutions. Some of these methods include Adomian decomposition method [4], [5], differential transformation method [6], [7], the Taylor series approximation method [8], Fourier spectral method [9], Gamma function method [10], perturbation method [11], He frequency formulation and the dimensional method [12]-[14], homotopy perturbation method [15]-[21], the ancient Chinese algorithm [22] etc. The area of exact solutions to nonlinear differential equations has become very popular in recent decades when the development of personal computers enabled more efficient work with known algorithms. The Adomian decomposition method [23–28] in the matter of fact was developed to find approximated solutions to differential equations, but in many publications [28,29] we can find interesting examples where the obtained power series were actually summable to exact solutions. A typical way to obtain such solutions is to sum up certain Taylor series. In our paper, we are going to present some situations when it seems reasonable to use modified techniques to obtain the Taylor series. In mathematical physics, we often must deal with scalar partial differential equations (PDEs) of space and time variables $x, t \in R$. We will show that for non-autonomous equations of this type, the method could be easily modified to get Taylor series directly. To explain our idea, let us first briefly present the classical method, so it would be easier to show the differences.

In this regard, a lot of mathematicians have endeavored to propose various efficient numerical approaches that rapidly converge to the available exact solution with an utmost level of exactitude. In fact, we mention the shooting methods [30] as one of such vibrant numerical approaches in this circle. Besides, as the method possesses so many benefits, nevertheless, the method is found to require an enormous computational space, as a drawback in obtaining perfect approximation, more particularly, with regard to nonlinear models.

In [31], Al-Zaid et al. applied the method to third-order linear equations and obtained good results. The present paper deploys the mixture of the shooting method and the ADM [32-38] to deeply examine the class of boundary value models accompanied by two-point boundary data. In order to solve boundary value problems (BVP), Sandoval-Hernandez et al. In [39] they combined the Taylor series method (TSM) with imaging methodology. In this paper, our aim is to study the shooting method for the general form of the second and third order BVPs for nonlinear case with the standard ADM and the normal form of the inverse operator which reduce the computational processes. Therefore, as the coupling between the Taylor Expansion and shooting decomposition method is named as Taylor Decomposition Shooting Method (TDSM) in this study, the coupled TDSM method will be proposed for the nonlinear models. Also, we shall be demonstrating the method on some test models, and further establish a comparative study between the approximate solutions posed by the proposed method and those of the exact analytical solutions, together with others from the available open literature. Lastly, we will give some concluding notes about the performance of the proposed method.

TAYLOR DECOMPOSITION SHOOTING METHOD (TDSM)

Adomian in 1980 introduced the famous ADM for the solution of functional equations. The method is highly advantageous as the governing unknown function of the model is sought as a sum of an infinite series of rapidly convergent components. Moreover, the method competently and effectively solves several real-life applications; thereby boldly writing its name in the field of numerical methods for functional equations. Besides, let us first give a brief outline of the implantation of the standard ADM [5-8].

Consider the following nth-order nonlinear two-point BVP

$$y^{(n)} = f\left(x, y, y', \dots, y^{(n-1)}\right), \quad a \le x \le b, \tag{1}$$

coupled with the two-point boundary data as follows

$$y^{(i)}(a) = \alpha_i, \quad y(b) = \beta, \quad i = 0,1 \dots n-2, \tag{2}$$

where β andα_i, for $i = 0,1 \dots n-2$ are prescribed real constants (in this regard, $n = 2$ or 3).

Additionally, since the ADM normally expresses the aiming equation in an operator form, we therefore re-express (1) in the following format

$$Ly + Ry + Ny = r(x), \tag{3}$$

together with the following initial conditions

$$y^{(i)}(a) = \alpha_i, \quad y^{(n-1)}(a) = t, \; i = 0,1 \dots n-2, \tag{4}$$

where L represents the highest-order derivative of the second or third-order with the initial conditions (4), R is also a linear differential operator, but of order less than that of L; Ny denotes the nonlinear differential operator, and $r(x)$ is a given source function. Lastly, α_i for $i = 0,1 \dots n-2$, and t are given constants (real). Additionally, let us consider the differential operator for the differential equation (3), as below

$$L(.) = \frac{d^n y}{dx^n}(.), \tag{5}$$

coupled with its corresponding n-fold integral operator, the inverse operator L^{-1} to (5) as follows

$$L^{-1}(.) = \int_a^x \int_a^x \dots \int_a^x (.)\, dx\, dx \dots dx. \tag{6}$$

Moreover, ADM proceeds by suggesting that the unknown function $y(x)$ of the governing functional equation be expressed as a sum of infinite series of components expressed as

$$y(x) = \sum_{m=0}^{\infty} y_m(x), \tag{7}$$

while the nonlinear term Ny is obtained with the aid of the following recurrent formula

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[N\left(\sum_{i=0}^{m}\lambda^i y_i\right)\right]_{\lambda=0}, \quad m = 0, 1, 2, \dots,$$

through using ADM, the recursive relation can be determined as

$$y_0 = \phi(x) + L^{-1}\big(r(x)\big), \tag{8}$$

$$y_{m+1} = -L^{-1}(Ry_m) - L^{-1}(A_m), \quad m \geq 0.$$

where the function $\phi(x)$ represents the terms arising from integrate Ly in (3) and from using the given conditions (4), so $L\phi(x) = 0$. After that, we use the following relationship, $y_{M+1} = \sum_{m=0}^{M} y_m(x)$, to get the approximate solution.

Furthermore, to derive the Taylor Decomposition Method (TDM) based on the ADM and Taylor Series Expansion, in Eq. (3) the functions Ny or $r(x)$ or both can be rewritten in terms of the Taylor Series. Therefore, from Eq. (8), we, thus obtain the following (TDM) recurrent relation as follows

$$y_0(x) = \phi(x) + \mathrm{T}\left(L^{-1}\big(r(x)\big)\right), \tag{9}$$

$$y_{m+1}(x) = -L^{-1}(Ry_m) - L^{-1}\big(\mathrm{T}(A_m)\big), m \geq 0.$$

The shooting method is an iterative method that is used for solving BVP by recasting it to a system of initial-value problems (IVPs) with specified initial conditions. Even though, none of these IVPs is tackled exactly; so, the solution will be approximated using the one-step methods or multistep methods or even directly using ADM. However, in this paper, we will make use of the modification directly to solve nonlinear IVPs of the second and third orders. In fact, we will be coupling the modification for ADM with the shooting method [30-31] and termed this procedure as TDSM, to examine the said two-point BVPs.

Firstly, the solution to the boundary value problem in equation (1) with the boundary conditions (2) is approximated through the solutions to a sequence of IVPs of the parameter t. So, we will convert the nth-order BVP into IVPs, where we replace the boundary condition with specific initial conditions. Indeed, the problem has the following format

$$y^{(n)} = f\big(x, y, y', \dots, y^{(n-1)}\big), \quad a \leq x \leq b, \tag{10}$$

with the initial conditions given by

$$y^{(i)}(a) = \alpha_i, \quad y^{(n-1)}(a) = t, \quad i = 0,1 \dots n-2. \tag{11}$$

We will then use the TDSM directly to solve the IVP expressed in (10) - (11); in fact, we do this by choosing the parameters $t = t_s$ in a manner to ensure that:

$$\lim_{s\to\infty} y(b, t_s) = y(b) = \beta,$$

where $y(x, t_s)$ represents the solution to the IVP (10) - (11), while $t = t_s$, and $y(x)$ represents the solution to the BVP (1) - (2). More so, the solution of the first IVP in the required sequence will be determined by imposing initial guess $t_0 = \frac{\beta-\alpha}{b-a}$. Then, we make use of the Newton's method to determine the value t_1 as follows

$$t_1 = t_0 - \frac{y(b,t_0)-\beta}{\frac{dy}{dt}(b,t_0)}.$$

Moreover, to determine the solutions to remaining sequence, the following guess points t_s, $s = 2, 3, \dots$, with nonlinear function $y(b,t) - \beta = 0$ are found with the help of the Secant iterative procedure, which takes the following expression

$$t_s = t_{s-1} - \frac{(y(b,t_{s-1})-\beta)(t_{s-1}-t_{s-2})}{y(b,t_{s-1})-y(b,t_{s-2})}, \quad s = 2,3, \dots.$$

More so, the computational process will be stopped when

$$|y(b, t_s) - \beta| \leq tolerance.$$

NUMERICAL EXAMPLES

This section examines the proposed methodology for the nonlinear case of the second and third order BVPs by demonstrating its application on some selected test problems. The new method is also compared with the shooting method with the Runge-Kutta method of order-fourth (SRKM4) to further assess the performance of the proposed TDSM. Similarly, we will compare the proposed results with others.

Further, we present certain supportive Tables 1-6 and Figures 1-3 reporting the absolute error difference between the exact analytical solution and, on the other hand, the obtained numerical results using the TDSM, further validated with SRKM4.

Example 1: Consider the following second-order nonlinear BVP [40-42]

$$y''(x) = -y^2(x) + \sin^2(\pi x) - \pi^2 \sin(\pi x), \quad y(0) = y(1) = 0.$$

Then, the actual analytical solution of the BVP is expressed as $y(x) = sin(\pi x)$.
Using 7 iterations and $M = 16$, then $y(x, t_s)$ represents the solution of the second order BVP with $t = t_7$; see Table 1 and 2 for the numerical results.

Table 1: The absolute error for SRKM4, and TDSM when $h = 1 / 16$.

x	SRKM4	TDSM
0	0	0
2 / 16	9.4×10^{-7}	6.6×10^{-14}
4 / 16	1.8×10^{-6}	1.3×10^{-13}
6 / 16	2.3×10^{-6}	1.9×10^{-13}
8 / 16	2.5×10^{-6}	2.5×10^{-13}
10 / 16	2.2×10^{-6}	3.0×10^{-13}
12 / 16	1.7×10^{-6}	3.4×10^{-13}
14 / 16	8.7×10^{-7}	3.7×10^{-13}
1	3.0×10^{-13}	4.0×10^{-13}

Table 2: Comparison between different methods and different value of M.

Numerical Methods	M	Maximum Error
TDSM	16	4.0×10^{-13}
SRKM4	16	2.5×10^{-6}
MLAM [40]	255	5.4×10^{-5}
2P1BVS [41]	57	1.1×10^{-10}
3SAM [42]	45	6.1×10^{-12}

In Table 1, we report the absolute error difference between the actual analytical solution, and the proposed solution **TDSM** and further validate with **SRKM4**. From Table 2, we can see that **TDSM** is the most competent technique for solving the governing model when a smaller number $M = 16$ is used in comparison with the **SRKM4**, and the methods used in [40-42]. Again, we portray the actual analytical and the contending approximate solutions in Figure 1, where one would notice an ideal agreement between the solutions.

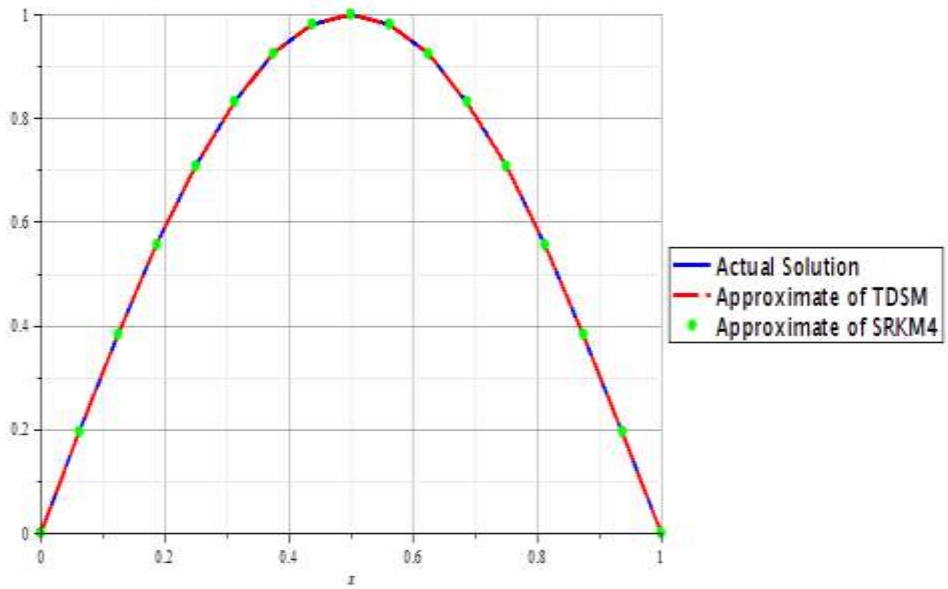

Figure 1. Graphical comparison, depicting the exact and contending approximate solutions with $h = 1 / 16$.

Example 2: Consider the second-order nonlinear BVP as follows [43-44]

$$y''(x) = -\frac{x}{\sqrt{1-y(x)}}\, y'(x) - 4 - 16x^{12}, \qquad y(0) = 0,\ y(1) = -3.$$

The actual analytical solution is given by $y(x) = 1 - (x^2 + 1)^2$.

Using 7 iterations and $M = 8$, then $y(x, t_s)$ represents the solution of the second order BVP with $t = t_7$; see Table 3 and 4 for the numerical results.

Table 3: The absolute error for SRKM4, and TDSM when $h = 1 / 8$.

x	**SRKM4**	**TDSM**
0	0	0
1 / 8	3.4×10^{-7}	1.7×10^{-13}
2 / 8	6.3×10^{-7}	3.4×10^{-13}
3 / 8	8.1×10^{-7}	5.1×10^{-13}
4 / 8	8.5×10^{-7}	6.7×10^{-13}
5 / 8	7.6×10^{-7}	8.2×10^{-13}
6 / 8	5.7×10^{-7}	9.2×10^{-13}
7 / 8	3.1×10^{-7}	7.4×10^{-13}
1	1.3×10^{-39}	2.2×10^{-23}

Table 4: Comparison between different methods when $h = 1 / 8$.

Numerical Methods	**Maximum Error**
TDSM	9.2×10^{-13}

SRKM4	8.5×10^{-7}
RAM [43]	4.5×10^{-4}
EFDM [44]	5.8×10^{-6}

In Table 3, we report the absolute error difference between the actual analytical solution, and the proposed solution **TDSM** and further validate with **SRKM4**. From Table 4, we can see that **TDSM** is the most competent technique for solving the governing model in comparison with the **SRKM4**, and the methods used in [43-44]. Again, we portray the actual analytical and the contending approximate solutions in Figure 2, where one would notice an ideal agreement between the solutions.

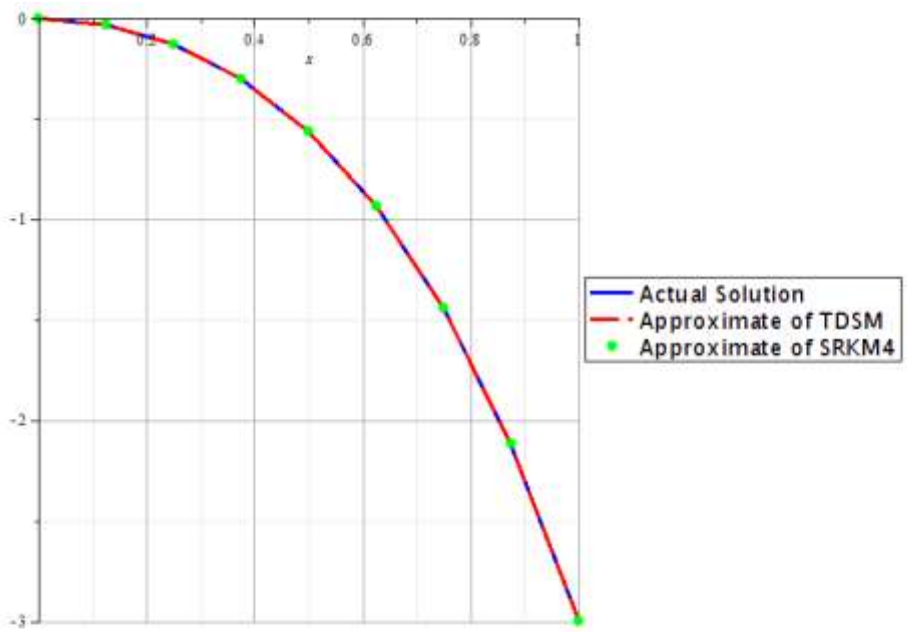

Figure 2. Graphical comparison, depicting the exact and contending approximate solutions with $h = 1/8$.

Example 3 Consider the third-order nonlinear BVP as follows [45-46]

$$y'''(x) - e^{y(x)}\left(\frac{1}{2} - e^{y(x)}\right) = \frac{-2 - \frac{1}{2}(9998 + x)(10000 + x)}{(10000 + x)^3},$$

$y(0) = -ln(10000),\ y'(0) = \frac{-1}{10000},\ y(1) = -ln(10001),$

that admits the following actual solution $y(x) = ln\left(\frac{1}{x+10000}\right)$.

Using 5 iterations and $M = 5$, then $y(x, t_s)$ represents the solution of the second order BVP with $t = t_5$; see Table 5 and 6 for the numerical results.

Table 5: The absolute error for SRKM4, and TDSM when $h = 1/5$.

x	$E_{\mathbf{SRKM4}}$	$E_{\mathbf{TDSM}}$

0	0	2.0×10^{-39}
1 / 5	1.3×10^{-18}	6.7×10^{-29}
2 / 5	2.0×10^{-18}	2.7×10^{-28}
3 / 5	2.0×10^{-18}	6.0×10^{-28}
4 / 5	1.3×10^{-18}	1.1×10^{-27}
1	2.0×10^{-39}	1.7×10^{-27}

Table 6: Comparison between different methods and different value of M and s.

Numerical Methods	M	s	Maximum Error
TDSM	5	5	1.7×10^{-27}
SRKM4	5	5	2.5×10^{-6}
PGEM [45]	5	--	5.4×10^{-5}
BGERT [46]	10	7	1.1×10^{-10}
LGERT [46]	10	7	6.1×10^{-12}

In Table 5, we report the absolute error difference between the actual analytical solution, and the proposed solution **TDSM** and further validate with **SRKM4**. From Table 6, we can see that **TDSM** is the most competent technique for solving the governing model when a smaller number $M = 5$ and only five iterations are used in comparison with the **SRKM4**, and the methods used in [45-46]. Again, we portray the actual analytical and the contending approximate solutions in Figure 3, where one would notice an ideal agreement between the solutions.

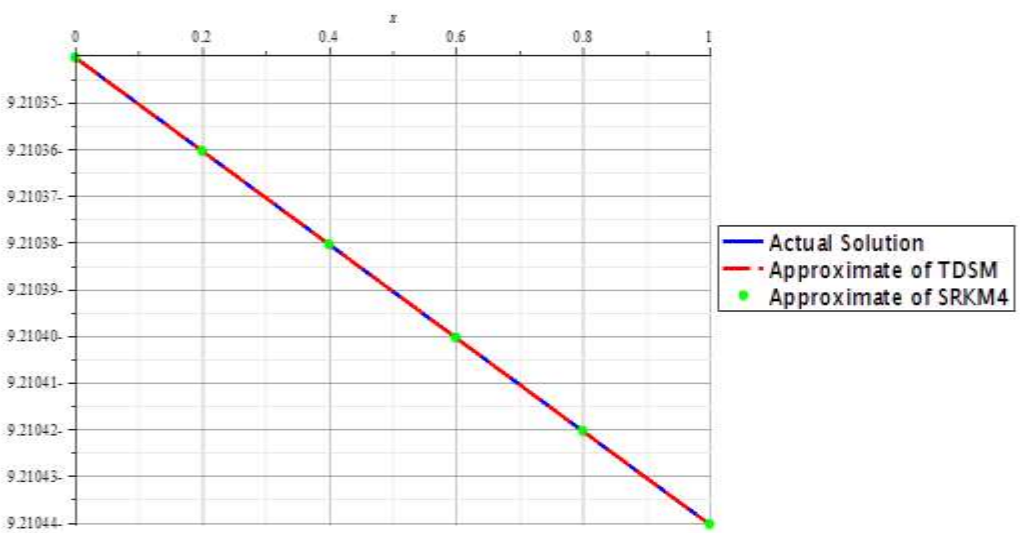

Figure 3. Graphical comparison, depicting the exact and contending approximate solutions with $h = 1 / 5$.

CONCLUSION

In conclusion, modified numerical method has been introduced in the present study to efficiently tackle nonlinear case of the second and third-order two-point BVPs of ordinary differential equations. The proposed method, which is numerically robust and economical, is further applied to several test problems and turned out to outperform SRKM4, and other available methods in the literature. Lastly, we have supported the findings of the present study with some comparison plots and tables – demonstrating the effectiveness of the devised approaches. In addition, the proposed methods can be applied to diverse models of real-life applications in mathematical physics.

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