NEUTROSOPHIC FUZZY TRANSLATIONS OF COMMUTATIVE IDEALS AND NEUTROSOPHIC FUZZY TRANSLATIONS OF N-FOLD (H AND IMPLICATIVE) IDEALS OF BCK-ALGEBRA

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Abstract:

This paper explores the application of Neutrosophic fuzzy translation to Neutrosophic fuzzy commutative ideals in BCK-algebras, examining various properties associated with these translations. Builds on Jun et.al., work on fuzzy translations (FT) and introduces neutrosophic fuzzy translations (NFT) as a new concept of neutrosophic fuzzy translations of n-fold H-ideal and neutrosophic fuzzy translations of n-fold implicative ideal in BCK-algebra, examining their properties and potential applications.

Keywords: *Fuzzy translation, Neutrosophic fuzzy Sub-Algebra, Neutrosophic fuzzy ideal, Neutrosophic fuzzy commutative ideal, and Neutrosophic fuzzy translation, Neutrosophic Fuzzy H-ideal, Neutrosophic Fuzzy n-fold H-ideal, Neutrosophic fuzzy β-translations, Neutrosophic Fuzzy Implicative Ideal, Neutrosophic Fuzzy n-fold Implicative Ideal.*

1. Introduction

BCK-algebras developed by Imai et. al., in 966 [3, 4] and Iseki and Tanaka introduced an ideal theory of BCK-algebras. Numerous researchers are doing work on this area. In 1965 Zadeh [24] introduced fuzzy set which is generalization of Crisp set and also which contains truth membership degree. In 1991, Xi's [23] introduced fuzzy (ideals) BCK-algebras. Mostafa and Jun et. al.,[8, 12, 13] applied fuzzy set theory to implicative ideals in BCK-algebras. Atanassov [1] introduced a novel concept of intuitionistic fuzzy set which is generalization of ordinary fuzzy sets. In [9] Jun et.al., studied an intuitionistic fuzzy ideals of BCK-algebras in 2000. Satyanarayana et.al., [15] generalized some results on intuitionistic fuzzy ideals of BCK-algebras. In 2009, Lee et.al.,[11] develops a concept of fuzzy translations and fuzzy multiplications of BCK/BCI-algebras. In [10] Jun, introduced a new concept of fuzzy which is a Translations of fuzzy ideals in BCK/BCI-algebras. Zhan and Tan [6], introduced the novel concepts characterisations of doubt fuzzy H-ideals in BCK-algebras. Satyanarayana et.al., [19] introduced the notion of an intuitionistic fuzzy H-ideals of BCK-algebras after that Senapati et.al., [21, 22] study the concept of fuzzy translations of fuzzy H-ideals and generalizing this concept to Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy H-ideals in BCK/BCI-algebras. Hung and Chen [2] introduced the concepts of n-fold (implicative, (weak), commutative, positive implicative) ideals. Jun et.al.,[7], introduced the notions of n-fold fuzzy positive implicative ideals in BCK-algebras and some results are investigated. Satyanarayana et.al., [14, 18] introduced the notion of foldness of intuitionistic fuzzy H-ideals in BCK-algebras, and also studied foldness of intuitionistic fuzzy (implicative & commutative) ideals of BCK-algebras and their properties are discussed. Authors [17] are generalized to neutrosophic fuzzy n-fold H-ideal within the BCK-algebras. Recently, Satyanarayana et.al., [16, 20] studied on intuitionistic Fuzzy translations of implicative ideals and intuitionistic fuzzy translations of n-fold H-ideals of BCK-algebras and also their properties are discussed. In this research we are generalizing to neutrosophic fuzzy translations of commutative ideal and neutrosophic fuzzy translations of n-fold (H & Implicative) ideals of BCK-algebra. For the purpose of this paper, we define the following terms and abbreviations:

- $\acute{G}$: BCK-algebra
- FHI : Fuzzy H-ideal
- $\mathcal{FCI}$: Fuzzy Commutative Ideal

- $\mathcal{FSA}$: Fuzzy Subalgebra
- nHI : n- fold H-ideal
- $\mathcal{NFS}$: Neutrosophic fuzzy set
- NFnHI : Neutrosophic fuzzy n-fold H-ideal
- NF β- T : Neutrosophic fuzzy β- translation
- $\mathcal{NFCI}$: Neutrosophic Fuzzy Commutative Ideal
- NFHI : Neutrosophic fuzzy H-ideal
- $\mathcal{NFSA}$: Neutrosophic fuzzy subalgebra
- nII : n- fold implicative-ideal
- $\mathcal{NFI}$: Neutrosophic fuzzy ideal
- $NFII$: Neutrosophic fuzzy implicative -ideal
- $NFnII$: Neutrosophic fuzzy n-fold implicative ideal
- NF_{T} : Neutrosophic fuzzy translation

2. Preliminaries

Definition 2.1. A $\mathcal{BCK}$ -Algebra is categorized as an algebraic system characterized by a (2, 0) type specification if it adheres to the subsequent principles, $\forall$ ѣ, ɰ, ɣ ∈ Ǵ.

($\mathcal{BCK}$-1) $((ѣ * ɰ) * (ѣ * ɣ)) * (ɣ * ɰ) = 0$

($\mathcal{BCK}$ -2) $(ѣ * (ѣ * ɰ)) * ɰ = 0$

($\mathcal{BCK}$ -3) $ѣ * ѣ = 0$

($\mathcal{BCK}$ -4) $0 * ѣ = 0$

($\mathcal{BCK}$ -5) $ѣ * ɰ = 0$ and $ɰ * ѣ = 0$ implies $ѣ = ɰ$.

We have the ability to establish a binary relation $\leq$ on Ǵ by $ѣ \leq ɰ \Leftrightarrow ѣ * ɰ = 0$. Under those circumstances (Ǵ, ≤) Constitutes a partially ordered set featuring a minimal member 0. Furthermore, (Ǵ, ∗, 0) constitutes a Ǵ iff it adheres to the following rules $\forall$ ѣ, ɰ, ɣ ∈ Ǵ.

i) $((ѣ * ɰ) * (ѣ * ɣ)) \leq (ɣ * ɰ)$

ii) $(ѣ * (ѣ * ɰ)) \leq ɰ$

iii) $ѣ \leq ѣ$

iv) $0 \leq ѣ$

v) $ѣ \leq ɰ$ and $ɰ \leq ѣ$ implies $ѣ = ɰ$, $\forall$ ѣ, ɰ, ɣ ∈ Ǵ.

Ǵ is distinguished by the following attributes:

($\mathcal{P}$-1) $ѣ * 0 = ѣ$

($\mathcal{P}$-2) $ƀ * щ \le ƀ$

($\mathcal{P}$-3) $(ƀ * щ) * ɣ = (ƀ * ɣ) * щ$

($\mathcal{P}$-4) $(ƀ * ɣ) * (щ * ɣ) \le (ƀ * щ)$

($\mathcal{P}$-5) $ƀ * (ƀ * (ƀ * щ)) = ƀ * щ$

($\mathcal{P}$-6) $ƀ \le щ \Rightarrow ƀ * ɣ \le щ * ɣ$ and $ɣ * щ \le ɣ * ƀ$

($\mathcal{P}$-7) $ƀ * щ \le ɣ \Rightarrow ƀ * ɣ \le щ$ $\forall ƀ, щ, ɣ \in Ǵ$.

- $Ǵ$ is considered to be commutative when $ƀ * (щ * ƀ) = щ * (ƀ * щ)$, $\forall ƀ, щ \in Ǵ$.
- An ideal of $Ǵ$ if (I-1) $0 \in 𝔜$, (I-2) $ƀ * щ$ and $щ \in 𝔜$ implies $ƀ \in 𝔜$ $\forall ƀ, щ \in Ǵ$.
- A $\mathcal{BCK}$ CI if (I-1), (I-3) $(ƀ * щ) * ɣ$ and $ɣ \in 𝔜$ implies $ƀ * (щ * (щ * ɣ)) \in 𝔜$, for any $ƀ, щ, ɣ \in Ǵ$.
- A subset $𝔜$ of a $\mathcal{BCK}$ -algebra $Ǵ$ qualifies as a H-ideal if it meets the following criteria: (HI-1) $0 \in 𝔜$, (HI-2) $ƀ * (щ * ɣ) \in 𝔜$ and $щ \in 𝔜 \Rightarrow ƀ * ɣ \in 𝔜$, $\forall ƀ, щ, ɣ \in Ǵ$.
- A non-empty subset $𝔜$ of $Ǵ$ is termed an implicative ideal if it satisfies, ($II-1$) $0 \in 𝔜$, ($II-2$) $(ƀ * (щ * ƀ)) * ɣ \in 𝔜$ and $ɣ \in 𝔜 \Rightarrow ƀ \in 𝔜$ $\forall ƀ, щ, ɣ \in Ǵ$.
- A $\mathcal{BCK}$ -algebra $Ǵ$ is said to be an implicative $\mathcal{BCK}$ -algebra if it satisfies, $ƀ = ƀ * (щ * ƀ)$ $\forall ƀ, щ \in Ǵ$.
- (i) A non-empty subset $𝔜$ of a $Ǵ$ is called n-fold $\mathcal{BCK}$ -ideal of $Ǵ$ if (I-1) and (I-4) and there exists a fixed $n \in Ǵ$ such that $(ƀ * щ^{n+1}) * ɣ \in 𝔜$ and $ɣ \in 𝔜 \Rightarrow ƀ * щ^{n} \in 𝔜$ $\forall ƀ, щ, ɣ \in Ǵ$.

(ii) For any elements $ƀ$ and $щ$ of $Ǵ$, $ƀ * щ^{n}$ denotes $(\ldots\ldots\ldots((ƀ * щ) * щ) * \ldots\ldots) * щ$ in which '$щ$' occurs n -times.

Definition 2.2: A non empty subset $𝔜$ of a $\mathcal{BCK}$ -algebra $Ǵ$ is termed as nHI of $Ǵ$, if

(nHI-1) $0 \in 𝔜$

(nHI-2) For all $ƀ, щ, ɣ \in Ǵ$ there exists a fixed $n \in Ǵ$ such that $ƀ * (щ * ɣ^{n}) \in 𝔜$ and $щ \in 𝔜 \Rightarrow ƀ * ɣ^{n} \in 𝔜$.

Definition 2.3: A non empty subset $𝔜$ of a $\mathcal{BCK}$ -algebra $Ǵ$ is termed as nII of $Ǵ$, if

(nII-1) $0 \in 𝔜$

(nII-2) For all $ƀ, щ, ɣ \in Ǵ$ there exists a fixed $n \in Ǵ$ such that $(ƀ * (щ * ƀ^{n})) * ɣ \in 𝔜$ and $ɣ \in 𝔜 \Rightarrow ƀ \in 𝔜$.

Definition 2.4: A fuzzy set $ℙ$ in $Ǵ$ qualifies as a fuzzy commutative ideal if it adheres to

($\mathcal{FCI}$-1) $ℙ(0) \ge ℙ(ƀ)$

($\mathcal{FCI}$-2) $\mathbb{P}\left(ƀ * \left(ɰ * (ɰ * ɣ)\right)\right) \geq min\{\mathbb{P}((ƀ * ɰ) * ɣ), \mathbb{P}(ɣ)\}$, $\forall$ $ƀ, ɰ, ɣ \in Ǵ$.

Example 2.5: Let $Ǵ = \{0, g_\alpha, k_\beta, r_\gamma\}$ in which $*$ is defined in the following table.

$*$	0	g_α	k_β	r_γ
0	0	0	0	0
g_α	g_α	0	0	g_α
k_β	k_β	g_α	0	k_β
r_γ	r_γ	r_γ	r_γ	0

Then $(Ǵ, *, 0)$ is a $\mathcal{BCK}$ -algebra.

Define a FS $\mathbb{P}$ in $Ǵ$ by $\mathbb{P}_{A}(0) = x_0, \mathbb{P}_{A}(1) = x_1$, $\mathbb{P}_{A}(2) = \mathbb{P}_{A}(3) = x_2$ where $x_0, x_1, x_2 \in [0,1]$ such that $x_0 > x_1 > x_2$. A routine calculation gives that $\mathbb{P}$ is a $\mathcal{FCI}$ of $Ǵ$.

Definition 2.6: A fuzzy set $\mathbb{P}$ in $Ǵ$ is called $\mathcal{FSA}$ (fuzzy subalgebra) of $Ǵ$, if

$$\mathbb{P}(ƀ * ɰ) \geq min\{\mathbb{P}(ƀ), \mathbb{P}(ɰ)\}, \ \forall\ ƀ, ɰ \in Ǵ.$$

Definition 2.7: A fuzzy subset $\mathbb{P}$ in a $\mathcal{BCK}$ -algebra $Ǵ$ qualifies as a Fuzzy H-Ideal (FHI) if it satisfies the following properties:

(FHI-1) $\mathbb{P}(0) \geq \mathbb{P}(ƀ)$

(FHI-2) $\mathbb{P}(ƀ * ɣ) \geq min\{\mathbb{P}(ƀ * (ɰ * ɣ)), \mathbb{P}(ɰ)\}$, $\forall$ $ƀ, ɰ, ɣ \in Ǵ$.

Definition 2.8: A fuzzy set $\mathbb{P}$ in a $\mathcal{BCK}$ -algebra $Ǵ$ qualifies as a Fuzzy Implicative ideal (FII) if it satisfies the following properties:

(FII-1) $\mathbb{P}(0) \geq \mathbb{P}(ƀ)$

(FII-2) $\mathbb{P}(ƀ) \geq min\left\{\mathbb{P}\left(\left(ƀ * (ɰ * ƀ)\right) * ɣ\right), \mathbb{P}(ɣ)\right\}$, $\forall$ $ƀ, ɰ, ɣ \in Ǵ$.

Definition 2.9: A fuzzy subset $\mathbb{P}$ in a $\mathcal{BCK}$ -algebra $Ǵ$ characterized as a FnHI of $Ǵ$ if it fulfills the following properties:

(FnHI-1) $\mathbb{P}(0) \geq \mathbb{P}(ƀ)$

(FnHI-2) There exists a fixed $n \in Ǵ$ such that

$\mathbb{P}(ƀ * ɣ^n) \geq min\{\mathbb{P}(ƀ * (ɰ * ɣ^n)), \mathbb{P}(ɰ)\}$, $\forall$ $ƀ, ɰ, ɣ \in Ǵ$.

Definition 2.10: A fuzzy subset $\mathbb{P}$ in a $\mathcal{BCK}$ -algebra $Ǵ$ characterized as a $FnII$ of $Ǵ$ if it fulfills the following properties:

($FnII$-1) $\mathbb{P}(0) \geq \mathbb{P}(ƀ)$

($FnII$-2) There exists a fixed $n \in Ɠ$ such that

$$\mathbb{P}(ƀ) \geq min\left\{\mathbb{P}\left(\left(ƀ * (ɰ * ƀ^n)\right) * ɣ\right), \mathbb{P}(ɣ)\right\}, \forall\, ƀ, ɰ, ɣ \in Ɠ.$$

Definition 2.11: An Intuitionistic fuzzy set $Ⱥ = (Ɠ, \mathbb{P}_{Ⱥ}, ч_{Ⱥ})$ in $Ɠ$ is deemed an Intuitionistic Fuzzy n H-Ideal (IFnHI) of $Ɠ$ if it fulfills the designated criteria:

(IFnHI-1) $\mathbb{P}_{Ⱥ}(0) \geq \mathbb{P}_{Ⱥ}(ƀ)$ and $ч_{Ⱥ}(0) \leq ч_{Ⱥ}(ƀ)$

(IFnHI-2)There exists a fixed $n \in Ɠ$ such that

$$\mathbb{P}_{Ⱥ}(ƀ * ɣ^n) \geq min\{\mathbb{P}_{Ⱥ}(ƀ * (ɰ * ɣ^n)), \mathbb{P}_{Ⱥ}(ɰ)\}$$

(IFnHI-3) $ч_{Ⱥ}(ƀ * ɣ^n) \leq max\{ч_{Ⱥ}(ƀ * (ɰ * ɣ^n)), ч_{Ⱥ}(ɰ)\}, \forall\, ƀ, ɰ, ɣ \in Ɠ.$

Definition 2.12: An Intuitionistic fuzzy set $Ⱥ = (Ɠ, \mathbb{P}_{Ⱥ}, ч_{Ⱥ})$ in $Ɠ$ is deemed an Intuitionistic Fuzzy n Implicative-Ideal ($IFnII$) of $Ɠ$ if it fulfills the designated criteria:

($IFnII$-1) $\mathbb{P}_{Ⱥ}(0) \geq \mathbb{P}_{Ⱥ}(ƀ)$ and $ч_{Ⱥ}(0) \leq ч_{Ⱥ}(ƀ)$ there exists a fixed $n \in Ɠ$ such that

($IFnII$-2) $\mathbb{P}_{Ⱥ}(ƀ) \geq min\left\{\mathbb{P}_{Ⱥ}\left(\left(ƀ * (ɰ * ƀ^n)\right) * ɣ\right), \mathbb{P}_{Ⱥ}(ɣ)\right\}$

($IFnII$-3) $ч_{Ⱥ}(ƀ) \leq max\left\{ч_{Ⱥ}\left(\left(ƀ * (ɰ * ƀ^n)\right) * ɣ\right), ч_{Ⱥ}(ɣ)\right\} \forall\, ƀ, ɰ, ɣ \in Ɠ.$

Definition 2.13: A $\mathcal{NFS}\ Ⱥ = (\mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, ч_{Ⱥ})$ is $\mathcal{NFSA}$ of $Ɠ$, if it satisfies

($\mathcal{NFSA}$-1) $\mathbb{P}_{Ⱥ}(ƀ * ɰ) \geq min\{\mathbb{P}_{Ⱥ}(ƀ), \mathbb{P}_{Ⱥ}(ɰ)\}$

($\mathcal{NFSA}$-2) $\sigma_{Ⱥ}(ƀ * ɰ) \geq min\,\{\sigma_{Ⱥ}(ƀ), \sigma_{Ⱥ}(ɰ)\}$

($\mathcal{NFSA}$-3) $ч_{Ⱥ}(ƀ * ɰ) \leq max\,\{ч_{Ⱥ}(ƀ), ч_{Ⱥ}(ɰ)\}$, $\forall\, ƀ, ɰ \in Ɠ.$

Example 2.14: Let $Ɠ = \{0, ꞡ_\alpha, ꝁ_\beta, ꞧ_\gamma\}$ in which $*$ is defined in the following table.

$*$	0	$ꞡ_\alpha$	$ꝁ_\beta$	$ꞧ_\gamma$
0	0	0	0	0
$ꞡ_\alpha$	$ꞡ_\alpha$	0	0	$ꞡ_\alpha$
$ꝁ_\beta$	$ꝁ_\beta$	$ꞡ_\alpha$	0	$ꝁ_\beta$
$ꞧ_\gamma$	$ꞧ_\gamma$	$ꞧ_\gamma$	$ꞧ_\gamma$	0

Then $(Ɠ, *, 0)$ is a $\mathcal{BCK}$ -algebra.

Define a $\mathcal{NFS}\ Ⱥ = (\mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, ч_{Ⱥ})$ in $Ɠ$ by $\mathbb{P}_{Ⱥ}(0) = \mathbb{P}_{Ⱥ}(ꞡ_\alpha) = \mathbb{P}_{Ⱥ}(ꝁ_\beta) = 0.08, \mathbb{P}_{Ⱥ}(ꞧ_\gamma) = 0.04$ $\sigma_{Ⱥ}(0) = \sigma_{Ⱥ}(ꞡ_\alpha) = \sigma_{Ⱥ}(ꝁ_\beta) = 0.08,\quad \sigma_{Ⱥ}(ꞧ_\gamma) = 0.04$ and $ч_{Ⱥ}(0) = ч_{Ⱥ}(ꞡ_\alpha) = ч_{Ⱥ}(ꝁ_\beta) = 0.03,$ $ч_{Ⱥ}(ꞧ_\gamma) = 0.06$. Then $Ⱥ = (\mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, ч_{Ⱥ})$ is $\mathcal{NFSA}$ of $Ɠ$.

Definition 2.15: A $\mathcal{NFS}$ $\mathbb{A} = (\mathbb{P}_{\mathbb{A}}, \sigma_{\mathbb{A}}, \mathcal{U}_{\mathbb{A}})$ in $\mathbb{G}$ is called $\mathcal{NFI}$ (Neutrosophic fuzzy ideal) of $\mathbb{G}$ if it satisfies:

($\mathcal{NFI}$-1) $\mathbb{P}_{\mathbb{A}}(0) \geq \mathbb{P}_{\mathbb{A}}(b)$, $\sigma_{\mathbb{A}}(0) \geq \sigma_{\mathbb{A}}(b)$ and $\mathcal{U}_{\mathbb{A}}(0) \leq \mathcal{U}_{\mathbb{A}}(b)$

($\mathcal{NFI}$-2) $\mathbb{P}_{\mathbb{A}}(b) \geq min\{\mathbb{P}_{\mathbb{A}}(b * u), \mathbb{P}_{\mathbb{A}}(u)\}$

($\mathcal{NFI}$-3) $\sigma_{\mathbb{A}}(b) \geq min\ \{\sigma_{\mathbb{A}}(b * u), \sigma_{\mathbb{A}}(u)\}$

($\mathcal{NFI}$-4) $\mathcal{U}_{\mathbb{A}}(b) \leq max\ \{\mathcal{U}_{\mathbb{A}}(b * u), \mathcal{U}_{\mathbb{A}}(u)\}$ $\forall$ $b, u \in \mathbb{G}$.

Definition 2.16: A $\mathcal{NFS}$ $\mathbb{A} = (\mathbb{P}_{\mathbb{A}}, \sigma_{\mathbb{A}}, \mathcal{U}_{\mathbb{A}})$ in $\mathbb{G}$ is called $\mathcal{NFCI}$ (Neutrosophic fuzzy Commutative ideal) of $\mathbb{G}$ if it satisfies:

($\mathcal{NFCI}$-1) $\mathbb{P}_{\mathbb{A}}(0) \geq \mathbb{P}_{\mathbb{A}}(b)$, $\sigma_{\mathbb{A}}(0) \geq \sigma_{\mathbb{A}}(b)$ and $\mathcal{U}_{\mathbb{A}}(0) \leq \mathcal{U}_{\mathbb{A}}(b)$

($\mathcal{NFCI}$-2) $\mathbb{P}_{\mathbb{A}}\big(b * (u * (u * b))\big) \geq min\{\mathbb{P}_{\mathbb{A}}((b * u) * y), \mathbb{P}_{\mathbb{A}}(y)\}$

($\mathcal{NFCI}$-3) $\sigma_{\mathbb{A}}\big(b * (u * (u * b))\big) \geq min\{\sigma_{\mathbb{A}}((b * u) * y), \sigma_{\mathbb{A}}(y)\}$

($\mathcal{NFCI}$-4) $\mathcal{U}_{\mathbb{A}}\big(b * (u * (u * b))\big) \leq max\{\mathcal{U}_{\mathbb{A}}((b * u) * y), \mathcal{U}_{\mathbb{A}}(y)\}$ $\forall$ $b, u, y \in \mathbb{G}$.

Example 2.17: Let $\mathbb{G} = \{0, f, w, v\}$ be a $\mathcal{BCK}$ -algebra with the given table.

$*$	0	f	w	v
0	0	0	0	0
f	f	0	0	f
w	w	f	0	w
v	v	v	v	0

Then $(\mathbb{G}, *, 0)$ is a $\mathcal{BCK}$ -algebra. Define a $\mathcal{NFS}$ $\mathbb{A}$ in $\mathbb{G}$ by $\mathbb{P}_{\mathbb{A}}(0) = 0.08$, $\mathbb{P}_{\mathbb{A}}(f) = 0.06$, $\mathbb{P}_{\mathbb{A}}(w) = \mathbb{P}_{\mathbb{A}}(v) = 0.03$, $\sigma_{\mathbb{A}}(0) = 0.08$, $\sigma_{\mathbb{A}}(f) = 0.06$, $\sigma_{\mathbb{A}}(w) = \sigma_{\mathbb{A}}(v) = 0.03$ and $\mathcal{U}_{\mathbb{A}}(0) = 0.03$, $\mathcal{U}_{\mathbb{A}}(f) = 0.06$, $\mathcal{U}_{\mathbb{A}}(w) = \mathcal{U}_{\mathbb{A}}(v) = 0.08$ where $0.03, 0.06\ and\ 0.08 \in [0,1]$ and $0.08 > 0.06 > 0.03$, and $0.03 < 0.06 < 0.08$. By usual calculations one can easily check that $\mathbb{A} = (\mathbb{P}_{\mathbb{A}}, \sigma_{\mathbb{A}}, \mathcal{U}_{\mathbb{A}})$ is $\mathcal{NFCI}$ of $\mathbb{G}$.

Definition 2.18: A Neutrosophic Fuzzy set $\mathbb{A} = (\mathbb{P}_{\mathbb{A}}, \sigma_{\mathbb{A}}, \mathcal{U}_{\mathbb{A}})$ in $\mathbb{G}$ is deemed a Neutrosophic Fuzzy n H-Ideal (NFnHI) of $\mathbb{G}$ if it fulfills certain requirements:

(NFnHI-1) $\mathbb{P}_{\mathbb{A}}(0) \geq \mathbb{P}_{\mathbb{A}}(b)$, $\sigma_{\mathbb{A}}(0) \geq \sigma_{\mathbb{A}}(b)$ and $\mathcal{U}_{\mathbb{A}}(0) \leq \mathcal{U}_{\mathbb{A}}(b)$ there exists a fixed $n \in \mathbb{G}$ such that

(NFnHI-2) $\mathbb{P}_{\mathbb{A}}(b * y^n) \geq min\{\mathbb{P}_{\mathbb{A}}(b * (u * y^n)), \mathbb{P}_{\mathbb{A}}(u)\}$

(NFnHI-3) $\sigma_{\mathbb{A}}(b * y^n) \geq min\{\sigma_{\mathbb{A}}(b * (u * y^n)), \sigma_{\mathbb{A}}(u)\}$

(NFnHI-4) $\mathcal{U}_{\mathbb{A}}(b * y^n) \leq max\{\mathcal{U}_{\mathbb{A}}(b * (u * y^n)), \mathcal{U}_{\mathbb{A}}(u)\}$ $\forall$ $b, u, y \in \mathbb{G}$.

Definition 2.19: A $\mathcal{NFS}\ Ⱥ = (\mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, \mathcal{Y}_{Ⱥ})$ in $Ǵ$ is deemed a Neutrosophic Fuzzy n Implicative-Ideal ($NFnII$) of $Ǵ$ if it fulfills the designated criteria:

($NFnII$-1) $\mathbb{P}_{Ⱥ}(0) \geq \mathbb{P}_{Ⱥ}(ƀ)$, $\sigma_{Ⱥ}(0) \geq \sigma_{Ⱥ}(ƀ)$ and $\mathcal{Y}_{Ⱥ}(0) \leq \mathcal{Y}_{Ⱥ}(ƀ)$ there exists a fixed $n \in Ǵ$ such that

($NFnII$-2) $\mathbb{P}_{Ⱥ}(ƀ) \geq min\left\{\mathbb{P}_{Ⱥ}\left(\left(ƀ * (ɰ * ƀ^n)\right) * ɣ\right), \mathbb{P}_{Ⱥ}(ɣ)\right\}$

($NFnII$-3) $\sigma_{Ⱥ}(ƀ) \geq min\left\{\sigma_{Ⱥ}\left(\left(ƀ * (ɰ * ƀ^n)\right) * ɣ\right), \sigma_{Ⱥ}(ɣ)\right\}$

($NFnII$-4) $\mathcal{Y}_{Ⱥ}(ƀ) \leq max\left\{\mathcal{Y}_{Ⱥ}\left(\left(ƀ * (ɰ * ƀ^n)\right) * ɣ\right), \mathcal{Y}_{Ⱥ}(ɣ)\right\}$ $\forall$ $ƀ, ɰ, ɣ \in Ǵ.$

Example 2.20: Let $Ǵ = \{0, \varpi_{\mathfrak{d}}, \vartheta_{\mathfrak{p}}, \varrho_{\mathfrak{q}}, \varsigma_{\mathfrak{r}}\}$ be a $\mathcal{BCK}$ -algebra with the following Cayley table

$*$	0	$\varpi_{\mathfrak{d}}$	$\vartheta_{\mathfrak{p}}$	$\varrho_{\mathfrak{q}}$	$\varsigma_{\mathfrak{r}}$
0	0	0	0	$\varrho_{\mathfrak{q}}$	$\varsigma_{\mathfrak{r}}$
$\varpi_{\mathfrak{d}}$	$\varpi_{\mathfrak{d}}$	0	$\vartheta_{\mathfrak{p}}$	$\varsigma_{\mathfrak{r}}$	$\varrho_{\mathfrak{q}}$
$\vartheta_{\mathfrak{p}}$	$\vartheta_{\mathfrak{p}}$	$\vartheta_{\mathfrak{p}}$	0	$\varrho_{\mathfrak{q}}$	$\varrho_{\mathfrak{q}}$
$\varrho_{\mathfrak{q}}$	$\varrho_{\mathfrak{q}}$	$\varrho_{\mathfrak{q}}$	$\varrho_{\mathfrak{q}}$	0	0
$\varsigma_{\mathfrak{r}}$	$\varsigma_{\mathfrak{r}}$	$\varrho_{\mathfrak{q}}$	$\varsigma_{\mathfrak{r}}$	$\varpi_{\mathfrak{d}}$	0

Define a $\mathcal{NFS}\ Ⱥ = (\mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, \mathcal{Y}_{Ⱥ})$ in $Ǵ$ by

$\mathbb{P}_{Ⱥ}(0) = 0.08, \mathbb{P}_{Ⱥ}(\varpi_{\mathfrak{d}}) = \mathbb{P}_{Ⱥ}(\varsigma_{\mathfrak{r}}) = 0.06, \mathbb{P}_{Ⱥ}(2) = \mathbb{P}_{Ⱥ}(\varrho_{\mathfrak{q}}) = 0.05$

$\sigma_{Ⱥ}(0) = 0.5, \sigma_{Ⱥ}(\varpi_{\mathfrak{d}}) = \sigma_{Ⱥ}(\varsigma_{\mathfrak{r}}) = 0.4, \sigma_{Ⱥ}(2) = \mathbb{P}_{Ⱥ}(\varrho_{\mathfrak{q}}) = 0.3$ and

$\mathcal{Y}_{Ⱥ}(0) = 0.3, \mathcal{Y}_{Ⱥ}(\varpi_{\mathfrak{d}}) = \mathcal{Y}_{Ⱥ}(\varsigma_{\mathfrak{r}}) = 0.5, \mathcal{Y}_{Ⱥ}(2) = \mathbb{P}_{Ⱥ}(\varrho_{\mathfrak{q}}) = 0.6$.

Then $Ⱥ = (\mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, \mathcal{Y}_{Ⱥ})$ is $NFnII$ of $Ǵ.$

For brevity, we represent a Neutrosophic Fuzzy Set ($\mathcal{NFS}$) $Ⱥ$ as $Ⱥ = (Ǵ, \mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, \mathcal{Y}_{Ⱥ})$ or $Ⱥ = (\mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, \mathcal{Y}_{Ⱥ})$. In this paper, we adopt the convention $C = inf\{\mathcal{Y}_{Ⱥ}(ƀ) | ƀ \in Ǵ\}$ for any $\mathcal{NFS}\ Ⱥ = (\mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, \mathcal{Y}_{Ⱥ})$ of $Ǵ.$

3. Neutrosophic Fuzzy Translations of Commutative Ideal of $\mathcal{BCK}$-Algebra

In this phase, we introduce and practice the idea of Fuzzy Translations (FT) to Neutrosophic fuzzy commutative ideals in $\mathcal{BCK}$ -algebras and few properties are examined.

Definition 3.1: Let $Ⱥ = (\mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, \mathcal{Y}_{Ⱥ})$ be a $\mathcal{NFS}$ of $Ǵ$ and let $\beta \in [0, C].$ An object having the form $Ⱥ_{\beta}^{T} = \left((\mathbb{P}_{Ⱥ})_{\beta}^{T}, (\sigma_{Ⱥ})_{\beta}^{T}, (\mathcal{Y}_{Ⱥ})_{\beta}^{T}\right)$ is called a $NF^{\beta} - T$ (Neutrosophic fuzzy β-translation) of $Ⱥ$ if $(\mathbb{P}_{Ⱥ})_{\beta}^{T}(ƀ) = \mathbb{P}_{Ⱥ}(ƀ) + \beta, (\sigma_{Ⱥ})_{\beta}^{T}(ƀ) = \sigma_{Ⱥ}(ƀ) + \beta$ and $(\mathcal{Y}_{Ⱥ})_{\beta}^{T}(ƀ) = \mathcal{Y}_{Ⱥ}(ƀ) - \beta$ $\forall$ $ƀ \in Ǵ.$

To maintain simplicity, we adopt the symbol notation $Ⱥ = (\mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, \mathcal{Y}_{Ⱥ}).$

Theorem 3.2 If $\text{Ѧ} = (\mathbb{P}_{\text{Ѧ}}, \sigma_{\text{Ѧ}}, \text{Ч}_{\text{Ѧ}})$ is $\mathcal{NFCI}$ of Ǵ, then the $NF^{\beta} - T\text{Ѧ}_{\beta}^{T} = \left((\mathbb{P}_{\text{Ѧ}})_{\beta}^{T}, (\sigma_{\text{Ѧ}})_{\beta}^{T}, (\text{Ч}_{\text{Ѧ}})_{\beta}^{T}\right)$ of Ѧ is a $\mathcal{NFCI}$ of Ǵ $\forall\ \beta \in [0, \mathcal{C}]$.

Proof: Let Ѧ be a $\mathcal{NFCI}$ of Ǵ and $\beta \in [0, \mathcal{C}]$.

Now $(\mathbb{P}_{\text{Ѧ}})_{\beta}^{T}\left(\text{ƀ} * (\text{щ} * (\text{щ} * \text{ƀ}))\right) = \mathbb{P}_{\text{Ѧ}}\left(\text{ƀ} * (\text{щ} * (\text{щ} * \text{ƀ}))\right) + \beta$

$\geq min\{\mathbb{P}_{\text{Ѧ}}((\text{ƀ} * \text{щ}) * \text{ɣ}), \mathbb{P}_{\text{Ѧ}}(\text{ɣ})\} + \beta$

$= min\{\mathbb{P}_{\text{Ѧ}}((\text{ƀ} * \text{щ}) * \text{ɣ}) + \beta, \mathbb{P}_{\text{Ѧ}}(\text{ɣ}) + \beta\}$

$= min\{(\mathbb{P}_{\text{Ѧ}})_{\beta}^{T}((\text{ƀ} * \text{щ}) * \text{ɣ}), (\mathbb{P}_{\text{Ѧ}})_{\beta}^{T}(\text{ɣ})\}$

$(\sigma_{\text{Ѧ}})_{\beta}^{T}\left(\text{ƀ} * (\text{щ} * (\text{щ} * \text{ƀ}))\right) = \sigma_{\text{Ѧ}}\left(\text{ƀ} * (\text{щ} * (\text{щ} * \text{ƀ}))\right) + \beta$

$\geq min\{\sigma_{\text{Ѧ}}((\text{ƀ} * \text{щ}) * \text{ɣ}), \sigma_{\text{Ѧ}}(\text{ɣ})\} + \beta$

$= min\{\sigma_{\text{Ѧ}}((\text{ƀ} * \text{щ}) * \text{ɣ}) + \beta, \sigma_{\text{Ѧ}}(\text{ɣ}) + \beta\}$

$= min\{(\sigma_{\text{Ѧ}})_{\beta}^{T}((\text{ƀ} * \text{щ}) * \text{ɣ}), (\sigma_{\text{Ѧ}})_{\beta}^{T}(\text{ɣ})\}$ and

$(\text{Ч}_{\text{Ѧ}})_{\beta}^{T}\left(\text{ƀ} * (\text{щ} * (\text{щ} * \text{ƀ}))\right) = \text{Ч}_{\text{Ѧ}}\left(\text{ƀ} * (\text{щ} * (\text{щ} * \text{ƀ}))\right) - \beta$

$\leq max\{\text{Ч}_{\text{Ѧ}}((\text{ƀ} * \text{щ}) * \text{ɣ}), \text{Ч}_{\text{Ѧ}}(\text{ɣ})\} - \beta$

$= max\{\text{Ч}_{\text{Ѧ}}((\text{ƀ} * \text{щ}) * \text{ɣ}) - \beta, \text{Ч}_{\text{Ѧ}}(\text{ɣ}) - \beta\}$

$= max\{(\text{Ч}_{\text{Ѧ}})_{\beta}^{T}((\text{ƀ} * \text{щ}) * \text{ɣ}), (\text{Ч}_{\text{Ѧ}})_{\beta}^{T}(\text{ɣ})\}$

Therefore, $(\mathbb{P}_{\text{Ѧ}})_{\beta}^{T}\left(\text{ƀ} * (\text{щ} * (\text{щ} * \text{ƀ}))\right) \geq min\{(\mathbb{P}_{\text{Ѧ}})_{\beta}^{T}((\text{ƀ} * \text{щ}) * \text{ɣ}), (\mathbb{P}_{\text{Ѧ}})_{\beta}^{T}(\text{ɣ})\}$

$(\sigma_{\text{Ѧ}})_{\beta}^{T}\left(\text{ƀ} * (\text{щ} * (\text{щ} * \text{ƀ}))\right) \geq min\{(\sigma_{\text{Ѧ}})_{\beta}^{T}((\text{ƀ} * \text{щ}) * \text{ɣ}), (\sigma_{\text{Ѧ}})_{\beta}^{T}(\text{ɣ})\}$ and

$(\text{Ч}_{\text{Ѧ}})_{\beta}^{T}\left(\text{ƀ} * (\text{щ} * (\text{щ} * \text{ƀ}))\right) \leq max\{(\text{Ч}_{\text{Ѧ}})_{\beta}^{T}((\text{ƀ} * \text{щ}) * \text{ɣ}), (\text{Ч}_{\text{Ѧ}})_{\beta}^{T}(\text{ɣ})\}$, $\forall\ \text{ƀ}, \text{щ}, \text{ɣ} \in \text{Ǵ}$.

Hence, the $NF^{\beta} - T\text{Ѧ}_{\beta}^{T}$ of Ѧ is $\mathcal{NFCI}$ of Ǵ.

Theorem 3.3. If the $NF^{\beta} - T\text{Ѧ}_{\beta}^{T}$ of Ѧ is a $\mathcal{NFCI}$ of Ǵ $\forall\ \beta \in [0, \mathcal{C}]$ then it is a $\mathcal{NFI}$ of Ǵ.

Proof: Let Ѧ_{β}^{T} of Ѧ be a $\mathcal{NFCI}$ of Ǵ. Put $\text{щ} = 0$ in $\mathcal{NFCI}$-2, 3 and 4

We get

$(\mathbb{P}_{\text{Ѧ}})_{\beta}^{T}(\text{ƀ}) = \mathbb{P}_{\text{Ѧ}}\left(\text{ƀ} * (0 * (0 * \text{ƀ}))\right) \geq min\{\mathbb{P}_{\text{Ѧ}}((\text{ƀ} * 0) * \text{ɣ}), \mathbb{P}_{\text{Ѧ}}(\text{ɣ})\} = min\{\mathbb{P}_{\text{Ѧ}}(\text{ƀ} * \text{ɣ}), \mathbb{P}_{\text{Ѧ}}(\text{ɣ})\}$

$(\sigma_{\text{Ѧ}})_{\beta}^{T}(\text{ƀ}) = \sigma_{\text{Ѧ}}\left(\text{ƀ} * (0 * (0 * \text{ƀ}))\right) \geq min\{\sigma_{\text{Ѧ}}((\text{ƀ} * 0) * \text{ɣ}), \sigma_{\text{Ѧ}}(\text{ɣ})\} = min\{\sigma_{\text{Ѧ}}(\text{ƀ} * \text{ɣ}), \sigma_{\text{Ѧ}}(\text{ɣ})\}$ and

$(\text{Ч}_{\text{Ѧ}})_{\beta}^{T}(\text{ƀ}) = \text{Ч}_{\text{Ѧ}}\left(\text{ƀ} * (0 * (0 * \text{ƀ}))\right) \leq max\{\text{Ч}_{\text{Ѧ}}((\text{ƀ} * 0) * \text{ɣ}), \mathbb{P}_{\text{Ѧ}}(\text{ɣ})\} = max\{\text{Ч}_{\text{Ѧ}}(\text{ƀ} * \text{ɣ}), \text{Ч}_{\text{Ѧ}}(\text{ɣ})\}$,

$\forall\ \text{ƀ}, \text{щ}, \text{ɣ} \in \text{Ǵ}$.

Therefore, Ѧ_{β}^{T} is a $\mathcal{NFI}$ of Ǵ.

Remark 3.4. Converse of the theorem is no longer be genuine it's far proven within example.

Example 3.5. Let $\acute{G} = \{0, \mathfrak{f}, \mathfrak{w}, \mathfrak{o}\}$ be a $\mathcal{BCK}$ -algebra with the given table.

$*$	0	$\mathfrak{f}$	$\mathfrak{w}$	$\mathfrak{o}$
0	0	0	0	0
$\mathfrak{f}$	$\mathfrak{f}$	0	$\mathfrak{f}$	0
$\mathfrak{w}$	$\mathfrak{w}$	$\mathfrak{w}$	0	0
$\mathfrak{o}$	$\mathfrak{o}$	$\mathfrak{o}$	$\mathfrak{o}$	0

Let A be a $\mathcal{NFS}$ of $\acute{G}$.

$\mathbb{P}_{A}(0) = 0.42, \mathbb{P}_{A}(\mathfrak{f}) = 0.33, \mathbb{P}_{A}(\mathfrak{w}) = 0.22, \mathbb{P}_{A}(\mathfrak{o}) = 0.12$

$\sigma_{A}(0) = 0.32, \sigma_{A}(\mathfrak{f}) = 0.21, \sigma_{A}(\mathfrak{w}) = 0.20, \sigma_{A}(\mathfrak{o}) = 0.10$ and

$\mathcal{U}_{A}(0) = 0.22, \mathcal{U}_{A}(\mathfrak{f}) = 0.42, \mathcal{U}_{A}(\mathfrak{w}) = 0.50, \mathcal{U}_{A}(\mathfrak{o}) = 0.65.$

Then A is $\mathcal{NFI}$ of $\acute{G}$ and $NF^{\beta} - T$ of A, where $C = 0.22$ and we take $\beta = 0.20 \in [0, C]$ is given as follows

$(\mathbb{P}_{A})_{\beta}^{T}(0) = 0.62, (\mathbb{P}_{A})_{\beta}^{T}(\mathfrak{f}) = 0.53, (\mathbb{P}_{A})_{\beta}^{T}(\mathfrak{w}) = 0.42, (\mathbb{P}_{A})_{\beta}^{T}(\mathfrak{o}) = 0.32$.

$(\sigma_{A})_{\beta}^{T}(0) = 0.52, (\sigma_{A})_{\beta}^{T}(\mathfrak{f}) = 0.41, (\sigma_{A})_{\beta}^{T}(\mathfrak{w}) = 0.40, (\sigma_{A})_{\beta}^{T}(\mathfrak{o}) = 0.30$ and

$(\mathcal{U}_{A})_{\beta}^{T}(0) = 0.02, (\mathcal{U}_{A})_{\beta}^{T}(\mathfrak{f}) = 0.22, (\mathcal{U}_{A})_{\beta}^{T}(\mathfrak{w}) = 0.30, (\mathcal{U}_{A})_{\beta}^{T}(\mathfrak{o}) = 0.40$

Clearly it is not a $\mathcal{NFCI}$ of $\acute{G}$, because

$(\mathbb{P}_{A})_{\beta}^{T}\left(\mathfrak{w} * (\mathfrak{o} * (\mathfrak{o} * \mathfrak{w}))\right) = 0.42 < 0.62 = min\{(\mathbb{P}_{A})_{\beta}^{T}((\mathfrak{w} * \mathfrak{o}) * 0), (\mathbb{P}_{A})_{\beta}^{T}(0)\}$

$(\sigma_{A})_{\beta}^{T}\left(\mathfrak{w} * (\mathfrak{o} * (\mathfrak{o} * \mathfrak{w}))\right) = 0.40 < 0.52 = min\{(\sigma_{A})_{\beta}^{T}((\mathfrak{w} * \mathfrak{o}) * 0), (\sigma_{A})_{\beta}^{T}(0)\}$ and

$(\mathcal{U}_{A})_{\beta}^{T}\left(\mathfrak{w} * (\mathfrak{o} * (\mathfrak{o} * \mathfrak{w}))\right) = 0.30 > 0.02 = max\{(\mathcal{U}_{A})_{\beta}^{T}((\mathfrak{w} * \mathfrak{o}) * 0), (\mathcal{U}_{A})_{\beta}^{T}(0)\}$.

Corollary 3.6. Every A_{β}^{T} $\mathcal{NFCI}$ of $\acute{G}$ must be a $\mathcal{NFSA}$ of $\acute{G}$.

Corollary 3.7. Let the $NF^{\beta} - T$ A_{β}^{T} of A be a $\mathcal{NFCI}$ of $\acute{G}$ $\forall$ $\beta \in [0, C]$, then we have the following

$\mathfrak{f} \leq \mathfrak{g}$ implies $(\mathbb{P}_{A})_{\beta}^{T}(\mathfrak{f}) \geq (\mathbb{P}_{A})_{\beta}^{T}(\mathfrak{g})$, $(\sigma_{A})_{\beta}^{T}(\mathfrak{f}) \geq (\sigma_{A})_{\beta}^{T}(\mathfrak{g})$ and $(\mathcal{U}_{A})_{\beta}^{T}(\mathfrak{f}) \leq (\mathcal{U}_{A})_{\beta}^{T}(\mathfrak{g})$, $\forall$ $\mathfrak{f}, \mathfrak{g} \in \acute{G}$.

Theorem 3.8. Let A_{β}^{T} be a $\mathcal{NFI}$ of $\acute{G}$, then A_{β}^{T} is a $\mathcal{NFCI}$ of $\acute{G}$ iff

$(\mathbb{P}_{A})_{\beta}^{T}\left(\mathfrak{b} * (\mathfrak{u} * (\mathfrak{u} * \mathfrak{b}))\right) \geq (\mathbb{P}_{A})_{\beta}^{T}(\mathfrak{b} * \mathfrak{u})$, $(\sigma_{A})_{\beta}^{T}\left(\mathfrak{b} * (\mathfrak{u} * (\mathfrak{u} * \mathfrak{b}))\right) \geq (\sigma_{A})_{\beta}^{T}(\mathfrak{b} * \mathfrak{u})$ and

$(\mathcal{U}_{A})_{\beta}^{T}\left(\mathfrak{b} * (\mathfrak{u} * (\mathfrak{u} * \mathfrak{b}))\right) \leq (\mathcal{U}_{A})_{\beta}^{T}(\mathfrak{b} * \mathfrak{u})$ $\forall$ $\mathfrak{b}, \mathfrak{u} \in \acute{G}$.

Proof: Let $Ѧ_\beta^T$ be a $\mathcal{NFI}$ of $Ǥ$. Assume that $Ѧ_\beta^T$ is a $\mathcal{NFCI}$ of $Ǥ$. Put $ɣ = 0$ in $\mathcal{NFCI}$ -2, 3 and 4

We get $(\mathbb{P}_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big) \geq min\{(\mathbb{P}_Ѧ)_\beta^T((ƀ * ɰ) * 0), (\mathbb{P}_Ѧ)_\beta^T(0)\}$

$= min\{(\mathbb{P}_Ѧ)_\beta^T(ƀ * ɰ), (\mathbb{P}_Ѧ)_\beta^T(0)\}$

$= (\mathbb{P}_Ѧ)_\beta^T(ƀ * ɰ)$

$(\sigma_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big) \geq min\{(\sigma_Ѧ)_\beta^T((ƀ * ɰ) * 0), (\sigma_Ѧ)_\beta^T(0)\}$

$= min\{(\sigma_Ѧ)_\beta^T(ƀ * ɰ), (\sigma_Ѧ)_\beta^T(0)\}$

$= (\sigma_Ѧ)_\beta^T(ƀ * ɰ)$ and

$(\text{Ч}_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big) \leq max\{(\text{Ч}_Ѧ)_\beta^T((ƀ * ɰ) * 0), (\text{Ч}_Ѧ)_\beta^T(0)\}$

$= max\{(\text{Ч}_Ѧ)_\beta^T(ƀ * ɰ), (\text{Ч}_Ѧ)_\beta^T(0)\}$

$= (\text{Ч}_Ѧ)_\beta^T(ƀ * ɰ)$.

Therefore, $(\mathbb{P}_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big) \geq (\mathbb{P}_Ѧ)_\beta^T(ƀ * ɰ)$

$(\sigma_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big) \geq (\sigma_Ѧ)_\beta^T(ƀ * ɰ)$ and

$(\text{Ч}_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big) \leq (\text{Ч}_Ѧ)_\beta^T(ƀ * ɰ)$ $\forall$ $ƀ, ɰ \in Ǥ$.

Conversely, consider

$(\mathbb{P}_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big) \geq (\mathbb{P}_Ѧ)_\beta^T(ƀ * ɰ) \geq min\{(\mathbb{P}_Ѧ)_\beta^T((ƀ * ɰ) * ɣ), (\mathbb{P}_Ѧ)_\beta^T(ɣ)\}$

$(\sigma_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big) \geq (\sigma_Ѧ)_\beta^T(ƀ * ɰ) \geq min\{(\sigma_Ѧ)_\beta^T((ƀ * ɰ) * ɣ), (\sigma_Ѧ)_\beta^T(ɣ)\}$ and

$(\text{Ч}_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big) \leq (\text{Ч}_Ѧ)_\beta^T(ƀ * ɰ) \leq max\{(\text{Ч}_Ѧ)_\beta^T((ƀ * ɰ) * ɣ), (\text{Ч}_Ѧ)_\beta^T(ɣ)\}$

$\forall$ $ƀ, ɰ, ɣ \in Ǥ$. Thus, $Ѧ_\beta^T$ is a $\mathcal{NFCI}$ of $Ǥ$.

Note: $(ƀ * ɰ) * \big(ƀ * (ɰ * (ɰ * ƀ))\big) = \Big(ƀ * \big(ƀ * (ɰ * (ɰ * ƀ))\big)\Big) * ɰ$ [by P-3]

$= \big(ɰ * (ɰ * ƀ)\big) * ɰ$ [by P-3]

$= (ɰ * ɰ) * (ɰ * ƀ) = 0$

Therefore, $(ƀ * ɰ) \leq \big(ƀ * (ɰ * (ɰ * ƀ))\big)$, $\forall$ $ƀ, ɰ \in Ǥ$.

By Theorem 3.8, we have $(\mathbb{P}_Ѧ)_\beta^T(ƀ * ɰ) \leq (\mathbb{P}_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big)$

$(\sigma_Ѧ)_\beta^T(ƀ * ɰ) \leq (\sigma_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big)$ and

$(\text{Ч}_Ѧ)_\beta^T(ƀ * ɰ) \geq (\text{Ч}_Ѧ)_\beta^T\big(ƀ * (ɰ * (ɰ * ƀ))\big)$ $\forall$ $ƀ, ɰ \in Ǥ$.

Theorem 3.9. A $NF^{\beta}-T\text{Ꞙ}_{\beta}^{T}$ of Ꞙ is a $\mathcal{NFCI}$ of Ǵ iff $(\mathbb{P}_{Ꞙ})_{\beta}^{T}(ѣ * щ) \leq (\mathbb{P}_{Ꞙ})_{\beta}^{T}(ѣ * (щ * (щ * ѣ)))$, $(\sigma_{Ꞙ})_{\beta}^{T}(ѣ * щ) \leq (\sigma_{Ꞙ})_{\beta}^{T}(ѣ * (щ * (щ * ѣ)))$ and $(Ч_{Ꞙ})_{\beta}^{T}(ѣ * щ) \geq (Ч_{Ꞙ})_{\beta}^{T}(ѣ * (щ * (щ * ѣ)))$ $\forall$ $ѣ, щ \in Ǵ$.

4. Neutrosophic Fuzzy Translations of n-Fold H-ideal of BCK-Algebra

Theorem 4.1: Whenever $Ꞙ = (\mathbb{P}_{Ꞙ}, \sigma_{Ꞙ}, Ч_{Ꞙ})$ is a NFnHI of Ǵ, its N$F^{\beta}-T$ given by $Ꞙ_{\beta}^{T} = ((\mathbb{P}_{Ꞙ})_{\beta}^{T}, (\sigma_{Ꞙ})_{\beta}^{T}, (Ч_{Ꞙ})_{\beta}^{T})$ necessarily inherits the NFnHI property $\forall$ β in $[0, C]$.

Proof: Assume $Ꞙ = (\mathbb{P}_{Ꞙ}, \sigma_{Ꞙ}, Ч_{Ꞙ})$ is a NFnHI of Ǵ and β in $[0, C]$.

Accordingly $(\mathbb{P}_{Ꞙ})_{\beta}^{T}(0) = \mathbb{P}_{Ꞙ}(0) + \beta \geq \mathbb{P}_{Ꞙ}(ѣ) + \beta = (\mathbb{P}_{Ꞙ})_{\beta}^{T}(ѣ)$

$(\sigma_{Ꞙ})_{\beta}^{T}(0) = \sigma_{Ꞙ}(0) + \beta \geq \sigma_{Ꞙ}(ѣ) + \beta = (\sigma_{Ꞙ})_{\beta}^{T}(ѣ)$ and

$(Ч_{Ꞙ})_{\beta}^{T}(0) = Ч_{Ꞙ}(0) - \beta \leq Ч_{Ꞙ}(ѣ) - \beta = (Ч_{Ꞙ})_{\beta}^{T}(ѣ)$

Meanwhile, $(\mathbb{P}_{Ꞙ})_{\beta}^{T}(ѣ * ɣ^{n}) = \mathbb{P}_{Ꞙ}(ѣ * ɣ^{n}) + \beta$

$\geq min\{\mathbb{P}_{Ꞙ}(ѣ * (щ * ɣ^{n})), \mathbb{P}_{Ꞙ}(щ)\} + \beta$

$= min\{\mathbb{P}_{Ꞙ}(ѣ * (щ * ɣ^{n})) + \beta, \mathbb{P}_{Ꞙ}(щ) + \beta\}$

$= min\{(\mathbb{P}_{Ꞙ})_{\beta}^{T}(ѣ * (щ * ɣ^{n})), (\mathbb{P}_{Ꞙ})_{\beta}^{T}(щ)\}$

$(\sigma_{Ꞙ})_{\beta}^{T}(ѣ * ɣ^{n}) = \sigma_{Ꞙ}(ѣ * ɣ^{n}) + \beta$

$\geq min\{\sigma_{Ꞙ}(ѣ * (щ * ɣ^{n})), \sigma_{Ꞙ}(щ)\} + \beta$

$= min\{\sigma_{Ꞙ}(ѣ * (щ * ɣ^{n})) + \beta, \sigma_{Ꞙ}(щ) + \beta\}$

$= min\{(\sigma_{Ꞙ})_{\beta}^{T}(ѣ * (щ * ɣ^{n})), (\sigma_{Ꞙ})_{\beta}^{T}(щ)\}$ and

$(Ч_{Ꞙ})_{\beta}^{T}(ѣ * ɣ^{n}) = Ч_{Ꞙ}(ѣ * ɣ^{n}) - \beta$

$\leq max\{Ч_{Ꞙ}(ѣ * (щ * ɣ^{n})), Ч_{Ꞙ}(щ)\} - \beta$

$= max\{Ч_{Ꞙ}(ѣ * (щ * ɣ^{n})) - \beta, Ч_{Ꞙ}(щ) - \beta\}$

$= max\{(Ч_{Ꞙ})_{\beta}^{T}(ѣ * (щ * ɣ^{n})), (Ч_{Ꞙ})_{\beta}^{T}(щ)\}$ $\forall$ $ѣ, щ, ɣ \in Ǵ$.

As a direct consequence, the N$F^{\beta}-T$ of Ꞙ is given by $Ꞙ_{\beta}^{T} = ((\mathbb{P}_{Ꞙ})_{\beta}^{T}, (\sigma_{Ꞙ})_{\beta}^{T}, (Ч_{Ꞙ})_{\beta}^{T})$ is indeed a NFnHI of Ǵ.

Theorem 4.2: Let $Ꞙ = (\mathbb{P}_{Ꞙ}, \sigma_{Ꞙ}, Ч_{Ꞙ})$ be a Neutrosophic Fuzzy subset of Ǵ such that its N$F^{\beta}-T$ $Ꞙ_{\beta}^{T} = ((\mathbb{P}_{Ꞙ})_{\beta}^{T}, (\sigma_{Ꞙ})_{\beta}^{T}, (Ч_{Ꞙ})_{\beta}^{T})$, is a NFnHI of Ǵ for some β in $[0, C]$. Then $Ꞙ = (\mathbb{P}_{Ꞙ}, \sigma_{Ꞙ}, Ч_{Ꞙ})$ is necessarily a NFnHI of Ǵ.

Proof: Assume that $Ꞙ_{\beta}^{T} = ((\mathbb{P}_{Ꞙ})_{\beta}^{T}, (\sigma_{Ꞙ})_{\beta}^{T}, (Ч_{Ꞙ})_{\beta}^{T})$ possesses the property of being a NFnHI of Ǵ for some β in $[0, C]$.

Observe that $\mathbb{P}_{A}(0)+\beta=(\mathbb{P}_{A})_{\beta}^{T}(0)\geq(\mathbb{P}_{A})_{\beta}^{T}(ъ)=\mathbb{P}_{A}(ъ)+\beta$

$\sigma_{A}(0)+\beta=(\sigma_{A})_{\beta}^{T}(0)\geq(\sigma_{A})_{\beta}^{T}(ъ)=\sigma_{A}(ъ)+\beta$ and

$\psi_{A}(0)-\beta=(\psi_{A})_{\beta}^{T}(0)\leq(\psi_{A})_{\beta}^{T}(ъ)=\psi_{A}(ъ)-\beta$

This yields $\mathbb{P}_{A}(0)\geq\mathbb{P}_{A}(ъ)$

$\sigma_{A}(0)\geq\sigma_{A}(ъ)$ and

$\psi_{A}(0)\leq\psi_{A}(ъ)$.

Presently, we observe

$\mathbb{P}_{A}(ъ*\gamma^{n})+\beta=(\mathbb{P}_{A})_{\beta}^{T}(ъ*\gamma^{n})\geq min\{(\mathbb{P}_{A})_{\beta}^{T}(ъ*(щ*\gamma^{n})),(\mathbb{P}_{A})_{\beta}^{T}(щ)\}$

$=min\{\mathbb{P}_{A}(ъ*(щ*\gamma^{n}))+\beta,\mathbb{P}_{A}(щ)+\beta\}$

$=min\{\mathbb{P}_{A}(ъ*(щ*\gamma^{n})),\mathbb{P}_{A}(щ)\}+\beta$.

$\sigma_{A}(ъ*\gamma^{n})+\beta=(\sigma_{A})_{\beta}^{T}(ъ*\gamma^{n})\geq min\{(\sigma_{A})_{\beta}^{T}(ъ*(щ*\gamma^{n})),(\sigma_{A})_{\beta}^{T}(щ)\}$

$=min\{\sigma_{A}(ъ*(щ*\gamma^{n}))+\beta,\sigma_{A}(щ)+\beta\}$

$=min\{\sigma_{A}(ъ*(щ*\gamma^{n})),\sigma_{A}(щ)\}+\beta$ and

$\psi_{A}(ъ*\gamma^{n})-\beta=(\psi_{A})_{\beta}^{T}(ъ*\gamma^{n})\leq max\{(\psi_{A})_{\beta}^{T}(ъ*(щ*\gamma^{n})),(\psi_{A})_{\beta}^{T}(щ)\}$

$=max\{\psi_{A}(ъ*(щ*\gamma^{n}))-\beta,\psi_{A}(щ)-\beta\}$

$=max\{\psi_{A}(ъ*(щ*\gamma^{n})),\psi_{A}(щ)\}-\beta$.

This leads to $\mathbb{P}_{A}(ъ*\gamma^{n})\geq min\{\mathbb{P}_{A}(ъ*(щ*\gamma^{n})),\mathbb{P}_{A}(щ)\}$

$\sigma_{A}(ъ*\gamma^{n})\geq min\{\sigma_{A}(ъ*(щ*\gamma^{n})),\sigma_{A}(щ)\}$

$\psi_{A}(ъ*\gamma^{n})\leq max\{\psi_{A}(ъ*(щ*\gamma^{n})),\psi_{A}(щ)\}$ $\forall$ $ъ,щ,\gamma\in G$.

Therefore, we can deduce that $A=(\mathbb{P}_{A},\sigma_{A},\psi_{A})$ is a NFnHI of G.

Theorem 4.3: If the N$F^{\beta}-T$ $A_{\beta}^{T}=((\mathbb{P}_{A})_{\beta}^{T},(\sigma_{A})_{\beta}^{T},(\psi_{A})_{\beta}^{T})$ induced by A is a NFnHI of G $\forall\,\beta$ in $[0,C]$ then it must be a $\mathcal{NFI}$ of G.

Proof: Assume the N$F^{\beta}-T$ defined as $A_{\beta}^{T}=((\mathbb{P}_{A})_{\beta}^{T},(\sigma_{A})_{\beta}^{T},(\psi_{A})_{\beta}^{T})$ of A forms a NFnHI of G.

Sequently, we obtain $(\mathbb{P}_{A})_{\beta}^{T}(ъ*\gamma^{n})\geq min\{(\mathbb{P}_{A})_{\beta}^{T}(ъ*(щ*\gamma^{n})),(\mathbb{P}_{A})_{\beta}^{T}(щ)\}$

$(\sigma_{A})_{\beta}^{T}(ъ*\gamma^{n})\geq min\{(\sigma_{A})_{\beta}^{T}(ъ*(щ*\gamma^{n})),(\sigma_{A})_{\beta}^{T}(щ)\}$ and

$(\psi_{A})_{\beta}^{T}(ъ*\gamma^{n})\leq max\{(\psi_{A})_{\beta}^{T}(ъ*(щ*\gamma^{n})),(\psi_{A})_{\beta}^{T}(щ)\}$ $\forall$ $ъ,щ,\gamma\in G$.

Given any $ъ\in G$, $ъ*0^{n}=ъ$, thus with the setting of $\gamma=0$, we attain

$(\mathbb{P}_{A})_{\beta}^{T}(ъ)\geq min\{(\mathbb{P}_{A})_{\beta}^{T}(ъ*(щ*0^{n})),(\mathbb{P}_{A})_{\beta}^{T}(щ)\}$

$=min\{(\mathbb{P}_{A})_{\beta}^{T}(ъ*щ),(\mathbb{P}_{A})_{\beta}^{T}(щ)\}$

$(\sigma_{Ⱥ})^{T}_{\beta}(ƀ) \geq min\{(\sigma_{Ⱥ})^{T}_{\beta}(ƀ * (ɰ * 0^{n})), (\sigma_{Ⱥ})^{T}_{\beta}(ɰ)\}$

$= min\{(\sigma_{Ⱥ})^{T}_{\beta}(ƀ * ɰ), (\sigma_{Ⱥ})^{T}_{\beta}(ɰ)\}$ and

$(Ч_{Ⱥ})^{T}_{\beta}(ƀ) \leq max\{(Ч_{Ⱥ})^{T}_{\beta}(ƀ * (ɰ * 0^{n})), (Ч_{Ⱥ})^{T}_{\beta}(ɰ)\}$

$= max\{(Ч_{Ⱥ})^{T}_{\beta}(ƀ * ɰ), (Ч_{Ⱥ})^{T}_{\beta}(ɰ)\}$

Accordingly, $Ⱥ^{T}_{\beta} = ((\mathbb{P}_{Ⱥ})^{T}_{\beta}, (\sigma_{Ⱥ})^{T}_{\beta}, (Ч_{Ⱥ})^{T}_{\beta})$ is a $\mathcal{NFI}$ of $Ǵ$.

Proposition 4.4: For any β between 0 and C, If the N $F^{\beta} - T$ of $Ⱥ$ given by $A^{T}_{\beta} = ((\mathbb{P}_{Ⱥ})^{T}_{\beta}, (\sigma_{Ⱥ})^{T}_{\beta}, (Ч_{Ⱥ})^{T}_{\beta})$ is a NFnHI of $Ǵ$, then it necessarily follows that it is also a $\mathcal{NFSA}$ of $Ǵ$.

Theorem 4.5: Let $Ⱥ = (\mathbb{P}_{Ⱥ}, \sigma_{Ⱥ}, Ч_{Ⱥ})$ be a $\mathcal{NFS}$ such that the N$F^{\beta} - T$ $Ⱥ^{T}_{\beta} = ((\mathbb{P}_{Ⱥ})^{T}_{\beta}, (\sigma_{Ⱥ})^{T}_{\beta}, (Ч_{Ⱥ})^{T}_{\beta})$ of $Ⱥ$ is a NFnHI of $Ǵ$ $\forall$ β in $[0, C]$ then the sets $\mathfrak{B} = \{ƀ | ƀ \in Ǵ\ and\ (\mathbb{P}_{Ⱥ})^{T}_{\beta}(ƀ) = (\mathbb{P}_{Ⱥ})^{T}_{\beta}(0)\}$, $\mathfrak{C} = \{ƀ | ƀ \in Ǵ\ and\ (\sigma_{Ⱥ})^{T}_{\beta}(ƀ) = (\sigma_{Ⱥ})^{T}_{\beta}(0)\}$ and $\mathfrak{D} = \{ƀ | ƀ \in Ǵ\ and\ (Ч_{Ⱥ})^{T}_{\beta}(ƀ) = (Ч_{Ⱥ})^{T}_{\beta}(0)\}$ are nHI of $Ǵ$.

Proof: Assume that $Ⱥ^{T}_{\beta} = ((\mathbb{P}_{Ⱥ})^{T}_{\beta}, (\sigma_{Ⱥ})^{T}_{\beta}, (Ч_{Ⱥ})^{T}_{\beta})$ constitutes a NFnHI of $Ǵ$.

Then $(\mathbb{P}_{Ⱥ})^{T}_{\beta}, (\sigma_{Ⱥ})^{T}_{\beta}$ and $(Ч_{Ⱥ})^{T}_{\beta}$ become FnHI's of $Ǵ$.

It is evident that $0 \in \mathfrak{B}, 0 \in \mathfrak{C}$ and $0 \in \mathfrak{D}$.

Thus $\mathfrak{B} \neq \emptyset, \mathfrak{C} \neq \emptyset$ and $\mathfrak{D} \neq \emptyset$.

Assume any $n \in Ǵ$

$ƀ * (ɰ * ɣ^{n}) \in \mathfrak{B}$ and $ɰ \in \mathfrak{B} \Rightarrow (\mathbb{P}_{Ⱥ})^{T}_{\beta}(ƀ * (ɰ * ɣ^{n})) = (\mathbb{P}_{Ⱥ})^{T}_{\beta}(0) = (\mathbb{P}_{Ⱥ})^{T}_{\beta}(ɰ)$

We now turn to $(\mathbb{P}_{Ⱥ})^{T}_{\beta}(ƀ * ɣ^{n}) \geq min\{(\mathbb{P}_{Ⱥ})^{T}_{\beta}(ƀ * (ɰ * ɣ^{n})), (\mathbb{P}_{Ⱥ})^{T}_{\beta}(ɰ)\}$

$= min\{(\mathbb{P}_{Ⱥ})^{T}_{\beta}(0), (\mathbb{P}_{Ⱥ})^{T}_{\beta}(0)\}$

$= (\mathbb{P}_{Ⱥ})^{T}_{\beta}(0)$

Which entails $(\mathbb{P}_{Ⱥ})^{T}_{\beta}(ƀ * ɣ^{n}) \geq (\mathbb{P}_{Ⱥ})^{T}_{\beta}(0)$

This shows that $\mathbb{P}_{Ⱥ}(ƀ * ɣ^{n}) + \beta \geq \mathbb{P}_{Ⱥ}(0) + \beta$ (or) $\mathbb{P}_{Ⱥ}(ƀ * ɣ^{n}) \geq \mathbb{P}_{Ⱥ}(0)$.

In order that $ƀ * ɣ^{n} \in \mathfrak{B}$, $\forall$ $ƀ, ɰ, ɣ \in Ǵ$.

Thus $\mathfrak{B}$ is nHI of $Ǵ$.

Using a similar approach, we can prove that $\mathfrak{C}$ is nHI of $Ǵ$.

And $ƀ * (ɰ * ɣ^{n}) \in \mathfrak{D}$ and $ɰ \in \mathfrak{D} \Rightarrow (Ч_{Ⱥ})^{T}_{\beta}(ƀ * (ɰ * ɣ^{n})) = (Ч_{Ⱥ})^{T}_{\beta}(0) = (Ч_{Ⱥ})^{T}_{\beta}(ɰ)$

Moving on, consider $(Ч_{Ⱥ})^{T}_{\beta}(ƀ * ɣ^{n}) \leq max\{(Ч_{Ⱥ})^{T}_{\beta}(ƀ * (ɰ * ɣ^{n})), (Ч_{Ⱥ})^{T}_{\beta}(ɰ)\}$

$= max\{(Ч_{Ⱥ})^T_\beta(0), (Ч_{Ⱥ})^T_\beta(0)\}$

$= (Ч_{Ⱥ})^T_\beta(0)$

Which results in $(Ч_{Ⱥ})^T_\beta(ƀ * ɣ^n) \leq (Ч_{Ⱥ})^T_\beta(0)$

This means that $Ч_{Ⱥ}(ƀ * ɣ^n) + \beta \leq Ч_{Ⱥ}(0) + \beta$ (or) $Ч_{Ⱥ}(ƀ * ɣ^n) \leq Ч_{Ⱥ}(0)$.

Insofar as $ƀ * ɣ^n \in \mathfrak{D}$, $\forall$ $ƀ, щ, ɣ \in Ǵ$.

Thus we can conclude that $\mathfrak{D}$ is nHI of $Ǵ$.

Proposition 4.6: Suppose $Ⱥ = (\mathbb{P}_{Ⱥ}, \mathcal{O}_{Ⱥ}, Ч_{Ⱥ})$ is a NFnHI of $Ǵ$. Then it is straightforward to see that $\mathbb{P}_{Ⱥ}(ƀ * ɣ^n) \geq \mathbb{P}_{Ⱥ}(ƀ * (0 * ɣ^n))$, $\mathcal{O}_{Ⱥ}(ƀ * ɣ^n) \geq \mathcal{O}_{Ⱥ}(ƀ * (0 * ɣ^n))$ and $Ч_{Ⱥ}(ƀ * ɣ^n) \leq Ч_{Ⱥ}(ƀ * (0 * ɣ^n))$ $\forall$ $ƀ, ɣ \in Ǵ$.

Lemma 4.7: Let the N$F^\beta - T$ $Ⱥ^T_\beta = ((\mathbb{P}_{Ⱥ})^T_\beta, (\mathcal{O}_{Ⱥ})^T_\beta, (Ч_{Ⱥ})^T_\beta)$ of $Ⱥ$ be a NFnHI of $Ǵ$ $\forall$ β in $[0, C]$. Then for any $ƀ, ɟ \in Ǵ$, we have: $(\mathbb{P}_{Ⱥ})^T_\beta(ƀ) \geq (\mathbb{P}_{Ⱥ})^T_\beta(ɟ)$, $(\mathcal{O}_{Ⱥ})^T_\beta(ƀ) \geq (\mathcal{O}_{Ⱥ})^T_\beta(ɟ)$ and $(Ч_{Ⱥ})^T_\beta(ƀ) \leq (Ч_{Ⱥ})^T_\beta(ɟ)$, whenever $ƀ \leq ɟ$.

Proof: Suppose $ƀ, ɟ \in Ǵ$ satisfy $ƀ \leq ɟ$, then $ƀ * ɟ = 0$.

Consider $(\mathbb{P}_{Ⱥ})^T_\beta(ƀ) = (\mathbb{P}_{Ⱥ})^T_\beta(ƀ * 0) \geq min\{(\mathbb{P}_{Ⱥ})^T_\beta(ƀ * (ɟ * 0^n)), (\mathbb{P}_{Ⱥ})^T_\beta(ɟ)\}$

$= min\{(\mathbb{P}_{Ⱥ})^T_\beta(ƀ * ɟ), (\mathbb{P}_{Ⱥ})^T_\beta(ɟ)\}$

$= min\{(\mathbb{P}_{Ⱥ})^T_\beta(0), (\mathbb{P}_{Ⱥ})^T_\beta(ɟ)\}$

$= (\mathbb{P}_{Ⱥ})^T_\beta(ɟ)$

$\Rightarrow (\mathbb{P}_{Ⱥ})^T_\beta(ƀ) \geq (\mathbb{P}_{Ⱥ})^T_\beta(ɟ)$

Analogously, $(\mathcal{O}_{Ⱥ})^T_\beta(ƀ) = (\mathcal{O}_{Ⱥ})^T_\beta(ƀ * 0) \geq min\{(\mathcal{O}_{Ⱥ})^T_\beta(ƀ * (ɟ * 0^n)), (\mathcal{O}_{Ⱥ})^T_\beta(ɟ)\}$

$= min\{(\mathcal{O}_{Ⱥ})^T_\beta(ƀ * ɟ), (\mathcal{O}_{Ⱥ})^T_\beta(ɟ)\}$

$= min\{(\mathcal{O}_{Ⱥ})^T_\beta(0), (\mathcal{O}_{Ⱥ})^T_\beta(ɟ)\}$

$= (\mathcal{O}_{Ⱥ})^T_\beta(ɟ)$

$\Rightarrow (\mathcal{O}_{Ⱥ})^T_\beta(ƀ) \geq (\mathcal{O}_{Ⱥ})^T_\beta(ɟ)$

And $(Ч_{Ⱥ})^T_\beta(ƀ) = (Ч_{Ⱥ})^T_\beta(ƀ * 0) \leq max\{(Ч_{Ⱥ})^T_\beta(ƀ * (ɟ * 0^n)), (Ч_{Ⱥ})^T_\beta(ɟ)\}$

$= max\{(Ч_{Ⱥ})^T_\beta(ƀ * ɟ), (Ч_{Ⱥ})^T_\beta(ɟ)\}$

$= max\{(Ч_{Ⱥ})^T_\beta(0), (Ч_{Ⱥ})^T_\beta(ɟ)\}$

$= (Ч_{Ⱥ})^T_\beta(ɟ)$

$\Rightarrow (Ч_{Ⱥ})^T_\beta(ƀ) \leq (Ч_{Ⱥ})^T_\beta(ɟ)$.

Therefore, $(\mathbb{P}_{Ⱥ})^T_\beta, (\mathcal{O}_{Ⱥ})^T_\beta$ are of order-reversing, while $(Ч_{Ⱥ})^T_\beta$ is order-preserving.

5. Neutrosophic Fuzzy Translations of n-Fold Implicative Ideal of BCK-Algebra

Theorem 5.1. If ₳ is a $NFnII$ of Ǵ, then the N $F^{\beta}-T₳^{T}_{\beta}=\left((\mathbb{P}_{₳})^{T}_{\beta},(\sigma_{₳})^{T}_{\beta},(ч_{₳})^{T}_{\beta}\right)$ of ₳ is a $NFnII$ of Ǵ $\forall$ $\beta\in[0,C]$.

Proof: Let ₳ be a $NFnII$ of Ǵ and $\beta\in[0,C]$

Then $(\mathbb{P}_{₳})^{T}_{\beta}(0)=\mathbb{P}_{₳}(0)+\beta\geq\mathbb{P}_{₳}(ƀ)+\beta=(\mathbb{P}_{₳})^{T}_{\beta}(ƀ)$

$(\sigma_{₳})^{T}_{\beta}(0)=\sigma_{₳}(0)+\beta\geq\sigma_{₳}(ƀ)+\beta=(\sigma_{₳})^{T}_{\beta}(ƀ)$

and $(ч_{₳})^{T}_{\beta}(0)=ч_{₳}(0)-\beta\leq ч_{₳}(ƀ)-\beta=(ч_{₳})^{T}_{\beta}(ƀ)$

Now, $(\mathbb{P}_{₳})^{T}_{\beta}(ƀ)=\mathbb{P}_{₳}(ƀ)+\beta$

$$\geq min\left\{\mathbb{P}_{₳}\left((ƀ*(ɰ*ƀ^{n}))*ɣ\right),\mathbb{P}_{₳}(ɣ)\right\}+\beta$$

$$=min\left\{\mathbb{P}_{₳}\left((ƀ*(ɰ*ƀ^{n}))*ɣ\right)+\beta,\mathbb{P}_{₳}(ɣ)+\beta\right\}$$

$$=min\left\{(\mathbb{P}_{₳})^{T}_{\beta}\left((ƀ*(ɰ*ƀ^{n}))*ɣ\right),(\mathbb{P}_{₳})^{T}_{\beta}(ɣ)\right\}$$

$(\sigma_{₳})^{T}_{\beta}(ƀ)=\sigma_{₳}(ƀ)+\beta$

$$\geq min\left\{\sigma_{₳}\left((ƀ*(ɰ*ƀ^{n}))*ɣ\right),\sigma_{₳}(ɣ)\right\}+\beta$$

$$=min\left\{\sigma_{₳}\left((ƀ*(ɰ*ƀ^{n}))*ɣ\right)+\beta,\sigma_{₳}(ɣ)+\beta\right\}$$

$$=min\left\{(\sigma_{₳})^{T}_{\beta}\left((ƀ*(ɰ*ƀ^{n}))*ɣ\right),(\sigma_{₳})^{T}_{\beta}(ɣ)\right\}$$

And $(ч_{₳})^{T}_{\beta}(ƀ)=ч_{₳}(ƀ)-\beta$

$$\leq max\left\{ч_{₳}\left((ƀ*(ɰ*ƀ^{n}))*ɣ\right),ч_{₳}(ɣ)\right\}-\beta$$

$$=max\left\{ч_{₳}\left((ƀ*(ɰ*ƀ^{n}))*ɣ\right)-\beta,ч_{₳}(ɣ)-\beta\right\}$$

$$=max\left\{(ч_{₳})^{T}_{\beta}\left((ƀ*(ɰ*ƀ^{n}))*ɣ\right),(ч_{₳})^{T}_{\beta}(ɣ)\right\}\ \forall\ ƀ,ɰ,ɣ\in Ǵ.$$

Hence N $F^{\beta}-T₳^{T}_{\beta}=\left((\mathbb{P}_{₳})^{T}_{\beta},(\sigma_{₳})^{T}_{\beta},(ч_{₳})^{T}_{\beta}\right)$ of ₳ is a $NFnII$ of Ǵ.

Theorem 5.2. If the N $F^{\beta}-T₳^{T}_{\beta}=\left((\mathbb{P}_{₳})^{T}_{\beta},(\sigma_{₳})^{T}_{\beta},(ч_{₳})^{T}_{\beta}\right)$ of ₳ is a $NFnII$ of Ǵ $\forall$ $\beta\in[0,C]$ then it must be a $\mathcal{NFJ}$ of Ǵ.

Proof: Let the N $F^{\beta}-T₳^{T}_{\beta}=\left((\mathbb{P}_{₳})^{T}_{\beta},(\sigma_{₳})^{T}_{\beta},(ч_{₳})^{T}_{\beta}\right)$ of ₳ is a $NFnII$ of Ǵ, then we have

$(\mathbb{P}_{₳})^{T}_{\beta}(ƀ)\geq min\left\{(\mathbb{P}_{₳})^{T}_{\beta}\left((ƀ*(ɰ*ƀ^{n}))*ɣ\right),(\mathbb{P}_{₳})^{T}_{\beta}(ɣ)\right\},$

$(\sigma_{₳})^{T}_{\beta}(ƀ)\geq min\left\{(\sigma_{₳})^{T}_{\beta}\left((ƀ*(ɰ*ƀ^{n}))*ɣ\right),(\sigma_{₳})^{T}_{\beta}(ɣ)\right\}$ and

$(\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \leq max\left\{(\text{ч}_{Ⱥ})_{\beta}^{T}\left((\text{ѣ} * (\text{щ} * \text{ѣ}^{n})) * \text{ɣ}\right), (\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ɣ})\right\} \; \forall \; \text{ѣ}, \text{щ}, \text{ɣ} \in Ǵ.$

Since for any $\text{ѣ} \in Ǵ$, $\text{ѣ} * (0 * \text{ѣ}) = \text{ѣ}$,

Therefore by setting $\text{щ} = 0$ and $\text{ɣ} = \text{щ}$, we get

$$(\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \geq min\left\{(\mathbb{P}_{Ⱥ})_{\beta}^{T}\left((\text{ѣ} * (0 * \text{ѣ}^{n})) * \text{щ}\right), (\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{щ})\right\}$$

$$= min\{(\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ѣ} * \text{щ}), (\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{щ})\}$$

$$(\sigma_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \geq min\left\{(\sigma_{Ⱥ})_{\beta}^{T}\left((\text{ѣ} * (0 * \text{ѣ}^{n})) * \text{щ}\right), (\sigma_{Ⱥ})_{\beta}^{T}(\text{щ})\right\}$$

$$= min\{(\sigma_{Ⱥ})_{\beta}^{T}(\text{ѣ} * \text{щ}), (\sigma_{Ⱥ})_{\beta}^{T}(\text{щ})\}$$

And $(\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \leq max\left\{(\text{ч}_{Ⱥ})_{\beta}^{T}\left((\text{ѣ} * (0 * \text{ѣ}^{n})) * \text{щ}\right), (\text{ч}_{Ⱥ})_{\beta}^{T}(\text{щ})\right\}$

$$= max\{(\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ѣ} * \text{щ}), (\text{ч}_{Ⱥ})_{\beta}^{T}(\text{щ})\}$$

Therefore, $Ⱥ_{\beta}^{T} = \left((\mathbb{P}_{Ⱥ})_{\beta}^{T}, (\sigma_{Ⱥ})_{\beta}^{T}, (\text{ч}_{Ⱥ})_{\beta}^{T}\right)$ is a $\mathcal{NFI}$ of $Ǵ$.

Theorem 5.3. Let $Ⱥ_{\beta}^{T}$ be a $\mathcal{NFI}$ of $Ǵ$.Then N $F^{\beta} - TȺ_{\beta}^{T} = \left((\mathbb{P}_{Ⱥ})_{\beta}^{T}, (\sigma_{Ⱥ})_{\beta}^{T}, (\text{ч}_{Ⱥ})_{\beta}^{T}\right)$ of $Ⱥ$ is a $NFnII$ of $Ǵ$ $\forall$ $\beta \in [0, C] \Leftrightarrow$ it satisfies the conditions

$(\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \geq (\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ѣ} * (\text{щ} * \text{ѣ}^{n})), (\sigma_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \geq (\sigma_{Ⱥ})_{\beta}^{T}(\text{ѣ} * (\text{щ} * \text{ѣ}^{n}))$ and

$(\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \leq (\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ѣ} * (\text{щ} * \text{ѣ}^{n}))$ $\forall$ $\text{ѣ}, \text{щ} \in Ǵ$.

Proof: Suppose $Ⱥ_{\beta}^{T} = \left((\mathbb{P}_{Ⱥ})_{\beta}^{T}, (\sigma_{Ⱥ})_{\beta}^{T}, (\text{ч}_{Ⱥ})_{\beta}^{T}\right)$ is a $NFnII$ of $Ǵ$.

$(\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \geq min\left\{(\mathbb{P}_{Ⱥ})_{\beta}^{T}\left((\text{ѣ} * (\text{щ} * \text{ѣ}^{n})) * \text{ɣ}\right), (\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ɣ})\right\}$, ------(i)

$(\sigma_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \geq min\left\{(\sigma_{Ⱥ})_{\beta}^{T}\left((\text{ѣ} * (\text{щ} * \text{ѣ}^{n})) * \text{ɣ}\right), (\sigma_{Ⱥ})_{\beta}^{T}(\text{ɣ})\right\}$ and---------(ii)

$(\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \leq max\left\{(\text{ч}_{Ⱥ})_{\beta}^{T}\left((\text{ѣ} * (\text{щ} * \text{ѣ}^{n})) * \text{ɣ}\right), (\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ɣ})\right\}$ ------(iii)

Put $\text{ɣ} = 0$ in (i), (ii), (iii)

We get $(\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \geq (\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ѣ} * (\text{щ} * \text{ѣ}^{n}))$

$(\sigma_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \geq (\sigma_{Ⱥ})_{\beta}^{T}(\text{ѣ} * (\text{щ} * \text{ѣ}^{n}))$

And $(\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \leq (\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ѣ} * (\text{щ} * \text{ѣ}^{n}))$ $\forall$ $\text{ѣ}, \text{щ} \in Ǵ$.

Conversely suppose that, $\forall$ $\text{ѣ}, \text{щ}, \text{ɣ} \in Ǵ, (\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \geq (\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ѣ} * (\text{щ} * \text{ѣ}^{n}))$,

$(\sigma_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \geq (\sigma_{Ⱥ})_{\beta}^{T}(\text{ѣ} * (\text{щ} * \text{ѣ}^{n}))$ and $(\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \leq (\text{ч}_{Ⱥ})_{\beta}^{T}(\text{ѣ} * (\text{щ} * \text{ѣ}^{n}))$.

Since $Ⱥ_{\beta}^{T} = \left((\mathbb{P}_{Ⱥ})_{\beta}^{T}, (\sigma_{Ⱥ})_{\beta}^{T}, (\text{ч}_{Ⱥ})_{\beta}^{T}\right)$ is $\mathcal{NFI}$ of $Ǵ$

$(\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ѣ}) \geq (\mathbb{P}_{Ⱥ})_{\beta}^{T}(\text{ѣ} * (\text{щ} * \text{ѣ}^{n}))$

$\geq min\{(\mathbb{P}_{A})^{T}_{\beta}\left((ƀ * (ɰ * ƀ^{n})) * ɣ\right), (\mathbb{P}_{A})^{T}_{\beta}(ɣ)\}$,

$(\sigma_{A})^{T}_{\beta}(ƀ) \geq (\sigma_{A})^{T}_{\beta}(ƀ * (ɰ * ƀ^{n}))$

$\geq min\{(\sigma_{A})^{T}_{\beta}\left((ƀ * (ɰ * ƀ^{n})) * ɣ\right), (\sigma_{A})^{T}_{\beta}(ɣ)\}$ and

$(ч_{A})^{T}_{\beta}(ƀ) \leq (ч_{A})^{T}_{\beta}(ƀ * (ɰ * ƀ^{n}))$

$\leq max\{(ч_{A})^{T}_{\beta}\left((ƀ * (ɰ * ƀ^{n})) * ɣ\right), (ч_{A})^{T}_{\beta}(ɣ)\} \forall ƀ, ɰ, ɣ \in G$.

Therefore $A^{T}_{\beta} = \left((\mathbb{P}_{A})^{T}_{\beta}, (\sigma_{A})^{T}_{\beta}, (ч_{A})^{T}_{\beta}\right)$ is a $NFnII$ of G

Clearly N$F^{\beta} - TA^{T}_{\beta} = \left((\mathbb{P}_{A})^{T}_{\beta}, (\sigma_{A})^{T}_{\beta}, (ч_{A})^{T}_{\beta}\right)$ of A is a $NFnII$ of G.

Lemma 5.4. Let the N$F^{\beta} - TA^{T}_{\beta} = \left((\mathbb{P}_{A})^{T}_{\beta}, (\sigma_{A})^{T}_{\beta}, (ч_{A})^{T}_{\beta}\right)$ of G be a $NFnII$ of G $\forall$ $\beta \in [0, C]$, then we have the following

$ƀ \leq ɰ \Rightarrow (\mathbb{P}_{A})^{T}_{\beta}(ƀ) \geq (\mathbb{P}_{A})^{T}_{\beta}(ɰ)$,

$(\sigma_{A})^{T}_{\beta}(ƀ) \geq (\sigma_{A})^{T}_{\beta}(ɰ)$ and

$(ч_{A})^{T}_{\beta}(ƀ) \leq (ч_{A})^{T}_{\beta}(ɰ)$ $\forall$ $ƀ, ɰ \in G$.

Proof: Let $ƀ, ɰ \in G$ such that $ƀ \leq ɰ$ implies $ƀ * ɰ = 0$.

Put $ɰ = 0$ and $ɣ = ɰ$ in $NFnII$ -2, 3 and 4.

Consider $(\mathbb{P}_{A})^{T}_{\beta}(ƀ) \geq min\left\{(\mathbb{P}_{A})^{T}_{\beta}\left((ƀ * (ɰ * ƀ^{n})) * ɣ\right), (\mathbb{P}_{A})^{T}_{\beta}(ɣ)\right\}$

$$\geq min\left\{(\mathbb{P}_{A})^{T}_{\beta}\left((ƀ * (0 * ƀ^{n})) * ɰ\right), (\mathbb{P}_{A})^{T}_{\beta}(ɰ)\right\}$$

$$= min\{(\mathbb{P}_{A})^{T}_{\beta}(ƀ * ɰ), (\mathbb{P}_{A})^{T}_{\beta}(ɰ)\}$$

$$= min\{(\mathbb{P}_{A})^{T}_{\beta}(0), (\mathbb{P}_{A})^{T}_{\beta}(ɰ)\}$$

$$= (\mathbb{P}_{A})^{T}_{\beta}(ɰ)$$

Similarly, $(\sigma_{A})^{T}_{\beta}(ƀ) \geq min\left\{(\sigma_{A})^{T}_{\beta}\left((ƀ * (ɰ * ƀ^{n})) * ɣ\right), (\sigma_{A})^{T}_{\beta}(ɣ)\right\}$

$$\geq min\left\{(\sigma_{A})^{T}_{\beta}\left((ƀ * (0 * ƀ^{n})) * ɰ\right), (\sigma_{A})^{T}_{\beta}(ɰ)\right\}$$

$$= min\{(\sigma_{A})^{T}_{\beta}(ƀ * ɰ), (\sigma_{A})^{T}_{\beta}(ɰ)\}$$

$$= min\{(\sigma_{A})^{T}_{\beta}(0), (\sigma_{A})^{T}_{\beta}(ɰ)\}$$

$$= (\sigma_{A})^{T}_{\beta}(ɰ).$$

and $(ч_{A})^{T}_{\beta}(ƀ) \leq max\left\{(ч_{A})^{T}_{\beta}\left((ƀ * (ɰ * ƀ^{n})) * ɣ\right), (ч_{A})^{T}_{\beta}(ɣ)\right\}$

$$\leq max\left\{(ч_{A})^{T}_{\beta}\left((ƀ * (0 * ƀ^{n})) * ɰ\right), (ч_{A})^{T}_{\beta}(ɰ)\right\}$$

$$= max\{(ч_{A})^{T}_{\beta}(ƀ * ɰ), (ч_{A})^{T}_{\beta}(ɰ)\}$$

$= max\{(\text{Ч}_{\text{Ѧ}})^T_\beta(0), (\text{Ч}_{\text{Ѧ}})^T_\beta(\text{щ})\}$

$= (\text{Ч}_{\text{Ѧ}})^T_\beta(\text{щ})$.

Hence the result.

Theorem 5.5. Let $\text{Ѧ}^T_\beta = ((\mathbb{P}_{\text{Ѧ}})^T_\beta, (\sigma_{\text{Ѧ}})^T_\beta, (\text{Ч}_{\text{Ѧ}})^T_\beta)$ be $\mathcal{NFI}$ of Ǵ. The following items hold the same meaning:

i) Ѧ^T_β is $NFnII$

ii) $(\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{ѣ}) \geq (\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n)), (\sigma_{\text{Ѧ}})^T_\beta(\text{ѣ}) \geq (\sigma_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n))$ and

$(\text{Ч}_{\text{Ѧ}})^T_\beta(\text{ѣ}) \leq (\text{Ч}_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n))$, $\forall\ \text{ѣ}, \text{щ} \in \text{Ǵ}$.

iii) $(\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{ѣ}) = (\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ})), (\sigma_{\text{Ѧ}})^T_\beta(\text{ѣ}) = (\sigma_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}))$ and

$(\text{Ч}_{\text{Ѧ}})^T_\beta(\text{ѣ}) = (\text{Ч}_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}))\ \forall\ \text{ѣ}, \text{щ} \in \text{Ǵ}$.

Proof: $\underline{(i) \Rightarrow (ii)}$

Let Ѧ^T_β be a $NFnII$ of Ǵ.

Put $\text{Ɣ} = 0$ in $NFnII$ -2, 3 and 4, we get

$(\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{ѣ}) \geq (\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n))$

$(\sigma_{\text{Ѧ}})^T_\beta(\text{ѣ}) \geq (\sigma_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n))$

$(\text{Ч}_{\text{Ѧ}})^T_\beta(\text{ѣ}) \leq (\text{Ч}_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n))$, $\forall\ \text{ѣ}, \text{щ} \in \text{Ǵ}$.

Thus, Condition (ii) is upheld.

$\underline{(ii) \Rightarrow (iii)}$

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Observe that in $\text{Ǵ}, \text{ѣ}^n * (\text{ѣ} * \text{щ}) \leq \text{ѣ}$ by (ii)

We have $(\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n)) \geq (\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{ѣ})$

$(\sigma_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n)) \geq (\sigma_{\text{Ѧ}})^T_\beta(\text{ѣ})$ and

$(\text{Ч}_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n)) \leq (\text{Ч}_{\text{Ѧ}})^T_\beta(\text{ѣ})$

It follows from (ii) that $(\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{ѣ}) = (\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n))$,

$(\sigma_{\text{Ѧ}})^T_\beta(\text{ѣ}) = (\sigma_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n))$ and

$(\text{Ч}_{\text{Ѧ}})^T_\beta(\text{ѣ}) = (\text{Ч}_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n))\ \forall\ \text{ѣ}, \text{щ} \in \text{Ǵ}$.

Thus, Condition (iii) is upheld.

$\underline{(iii) \Rightarrow (i)}$

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Since $\text{Ѧ}^T_\beta = ((\mathbb{P}_{\text{Ѧ}})^T_\beta, (\sigma_{\text{Ѧ}})^T_\beta, (\text{Ч}_{\text{Ѧ}})^T_\beta)$ be $\mathcal{NFI}$ of Ǵ,

We have $(\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{ѣ} * (\text{щ} * \text{ѣ}^n)) \geq min\{(\mathbb{P}_{\text{Ѧ}})^T_\beta((\text{ѣ} * (\text{щ} * \text{ѣ}^n)) * \text{Ɣ}), (\mathbb{P}_{\text{Ѧ}})^T_\beta(\text{Ɣ})\}$,

$(\sigma_{A})^{T}_{\beta}(ъ * (щ * ъ^{n})) \geq min\{(\sigma_{A})^{T}_{\beta}((ъ * (щ * ъ^{n})) * γ), (\sigma_{A})^{T}_{\beta}(γ)\}$ and

$(Ч_{A})^{T}_{\beta}(ъ * (щ * ъ^{n})) \leq max\{(Ч_{A})^{T}_{\beta}((ъ * (щ * ъ^{n})) * γ), (Ч_{A})^{T}_{\beta}(γ)\}$, $\forall$ ъ, щ, γ ∈ Ǵ.

Combining (iii) we obtain

$(\mathbb{P}_{A})^{T}_{\beta}(ъ) \geq min\{(\mathbb{P}_{A})^{T}_{\beta}((ъ * (щ * ъ)) * γ), (\mathbb{P}_{A})^{T}_{\beta}(γ)\}$,

$(\sigma_{A})^{T}_{\beta}(ъ) \geq min\{(\sigma_{A})^{T}_{\beta}((ъ * (щ * ъ)) * γ), (\sigma_{A})^{T}_{\beta}(γ)\}$ and

$(Ч_{A})^{T}_{\beta}(ъ) \leq max\{(Ч_{A})^{T}_{\beta}((ъ * (щ * ъ)) * γ), (Ч_{A})^{T}_{\beta}(γ)\}$, $\forall$ ъ, щ, γ ∈ Ǵ.

Obviously A^{T}_{β} satisfies $(\mathbb{P}_{A})^{T}_{\beta}(0) \geq (\mathbb{P}_{A})^{T}_{\beta}(ъ)$, $(\sigma_{A})^{T}_{\beta}(0) \geq (\sigma_{A})^{T}_{\beta}(ъ)$ and $(Ч_{A})^{T}_{\beta}(0) \leq (Ч_{A})^{T}_{\beta}(ъ)$ $\forall$ ъ ∈ Ǵ.

Therefore $A^{T}_{\beta} = ((\mathbb{P}_{A})^{T}_{\beta}, (\sigma_{A})^{T}_{\beta}, (Ч_{A})^{T}_{\beta})$ is $NFnII$ of Ǵ.

6. Conclusion

This research has successfully explored the application of Neutrosophic fuzzy commutative ideal in BCK-algebra, new lights on various related properties with these translations. This study has introduced Neutrosophic fuzzy translations (NFT) as a novel concept. Furthermore, neutrosophic fuzzy translations of n-fold H-ideal and neutrosophic fuzzy translations of n-fold implicative ideal in BCK-algebras, and also demonstrated the significance and versatility of NFT in the context of BCK-algebras. The findings of this study are expected to inspire further research in the field of neutrosophic fuzzy set theory and its applications in algebraic structures, ultimately contributing to the advancement of mathematical knowledge and its practical applications.

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