A Study On The Lattice Of Subgroups Of All 2x2 Non-Singular Matrices Over Z_3

N.Pushparani[1], A. Vethamanickam[2]

[1] Research Scholar (Reg.No.17221172092013), PG and Research Department of Mathematics, Rani Anna Government College for Women, Tirunelveli - 627008. Email ID:npushparani1983@gmail.com Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, India – 627012

[2]Former Associate Professor and Research Supervisor, PG and Research Department of Mathematics, Rani Anna Government College for Women, Tirunelveli - 627008. Email ID:dr_vethamanickam@yahoo.co.in

Abstract

In this paper, we display the lattice structure of the lattice of subgroups of the 2x2 non-singular matrices over Z_3.

Keywords: *Matrix group, Subgroups, Lagrange's theorem, Sylow's theorem, Poset, Lattice, Atom.*

1. Introduction

The study of subgroup lattices has a quite long history, starting with Richard Dedekind's [2] work in 1877. After that a number of authors made contributions in the subgroup lattice theory. In 2015, Jebaraj Thiraviam[7] has worked in the lattice of subgroups of 2x2 matrices over Z_p, $p\leq 7$, with determinant value 1. In this paper we continue the same work for all non-singular matrices over Z_p, $p\leq 7$.

Let $G = \left\{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}: a_1, a_2, a_3, a_4 \in Zp,\ a_1a_4 - a_2a_3 \neq 0\right\}$. Then G is a group under the binary operation of matrix multiplication modulo p of order $(p^2-1)(p^2-p)$

2. Preliminaries

In this section, we give the necessary definitions and theorems for the development of the paper.

Definition 2.1

A partially ordered set $(A, \leq)$ consists of a non-empty set A and a binary relation $\leq$ on A such that $\leq$ satisfies reflexive, anti-symmetric and transitive. A Poset$(A, \leq)$ that also satisfies either $a \leq b$ or $b \leq a$ for every $a, b \in A$ is called a chain(totally ordered set).

Definition 2.2

Let $(A, \leq)$ be a Poset. LetS be a non-empty subset of A. An element $u \in A$ is called an upper bound of S if $a \leq u$ for all $a \in S$. The least upper bound of S is called the supremum or join of S. An element $l \in A$ is called a lower bound of S if $l \leq a$for all $a \in S$. The greatest lower bound of S is called the infimum or meet of S.

Definition 2.3

A Poset $(A, \leq)$ is a lattice if every pair of elements of A have infimum and supremum, we denote the infimum and supremum of two elementsa and b $\in A$ by $a\wedge b$ and $a\vee b$ respectively.

Definition 2.4

In the Poset $(A, \leq)$, a covers b or b is covered by a (in notation $a \succ b$ or $b \prec a$) if only if $b < a$ and for no $x \in A, b < x < a$ holds.

Theorem 2.5 (Lagrange's theorem) If G is a finite group and H is a subgroup of G, then the order of H is a divisor of the order of G.

Theorem 2.6. (Sylow's theorem) If p is a prime number and $p^{\alpha}|o(G)$ and $p^{\alpha+1} \nmid o(G)$, then G has a subgroup of order p^{α}, called a p-Sylow subgroup.

Theorem 2.7 The number of p-Sylow subgroups in G, for a given prime p, is of the form 1+mp.

.3. Elements of G order-wise

Let G denote the collection of all 2 x 2 non-singular matrices over Z_3. Then G is a group under the binary operation of matrix multiplication modulo 3 and $o(G) = (3^2-1)(3^2-3)$

$$= 8 \times 6 = 48$$

The order-wise arrangement of elements of G.

.3.1.1. Element of order 1(one element)

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.1.2 The list of elements of order 2(13 elements)

$\alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, $\alpha_5 = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$, $\alpha_6 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

$\alpha_7 = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$, $\alpha_8 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha_9 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\alpha_{10} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, $\alpha_{11} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$, $\alpha_{12} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$,

$\alpha_{13} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$

3.1.3 The list of elements of order 3(8 elements)

$\beta_1 = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$, $\beta_2 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$, $\beta_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\beta_4 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $\beta_5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\beta_6 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,

$\beta_7 = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$, $\beta_8 = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$

3.1.4 The list of elements of order 4(6 elements)

$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $\gamma_3 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\gamma_4 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$, $\gamma_5 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\gamma_6 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$

3.1.5 The list of elements of order 6(8 elements)

$\delta_1 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$, $\delta_2 = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$, $\delta_3 = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$, $\delta_4 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$, $\delta_5 = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$, $\delta_6 = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$,

$\delta_7 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $\delta_8 = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$

3.1.6 The list of elements of order 8(12 elements)

$\eta_1=\begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}$, $\eta_2=\begin{pmatrix}0 & 1\\1 & 2\end{pmatrix}$, $\eta_3=\begin{pmatrix}0 & 2\\2 & 1\end{pmatrix}$, $\eta_4=\begin{pmatrix}0 & 2\\2 & 2\end{pmatrix}$, $\eta_5=\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$, $\eta_6=\begin{pmatrix}1 & 1\\2 & 1\end{pmatrix}$,

$\eta_7=\begin{pmatrix}1 & 2\\1 & 1\end{pmatrix}$, $\eta_8=\begin{pmatrix}1 & 2\\2 & 0\end{pmatrix}$, $\eta_9=\begin{pmatrix}2 & 1\\1 & 0\end{pmatrix}$, $\eta_{10}=\begin{pmatrix}2 & 1\\2 & 2\end{pmatrix}$, $\eta_{11}=\begin{pmatrix}2 & 2\\1 & 2\end{pmatrix}$, $\eta_{12}=\begin{pmatrix}2 & 2\\2 & 0\end{pmatrix}$

4. Subgroups of G of various orders

In the present section we find all the subgroups of G of various orders. Based on Lagrange's theorem, we have to look only among the divisors of 48 for identifying the subgroups G.

4.1 Subgroups of G which have order 2

Consider a subgroup H of G which has order 2. Then all the subgroups of order 2 are

$H_1 = \{e, \alpha_1\}$, $H_2 = \{e, \alpha_2\}$, $H_3 = \{e, \alpha_3\}$, $H_4 = \{e, \alpha_4\}$, $H_5 = \{e, \alpha_5\}$,

$H_6=\{e, \alpha_6\}$, $H_7= \{e, \alpha_7\}$, $H_8= \{e, \alpha_8\}$, $H_9= \{e, \alpha_9\}$, $H_{10}= \{e, \alpha_{10}\}$,

$H_{11}= \{e\ \alpha_{11}\}$, $H_{12}= \{e, \alpha_{12}\}$, $H_{13}= \{e, \alpha_{13}\}$

4.2 Subgroups of G which have order 3

Since $o(G) = 2^4 \times 3$, $3 \mid o(G)$ and $3^2 \nmid o(G)$, by Sylow's theorem, G has a 3- Sylow subgroup which has order 3. Hence the number of 3 – Sylow subgroups of G is of the form 1+3m and we have $1+3m \mid o(G)$.

That is, $1+3m \mid 2^4 \times 3$.Then, $1+3m \mid 2^4$. Therefore, the probable values for m = 0, 1.

Hence, there are atmost four 3- Sylow subgroups corresponding to m = 1.

The subgroups are

$L_1= \{e, \beta_1, \beta_8\}$, $L_2 = \{e, \beta_2, \beta_7\}$, $L_3 = \{e, \beta_3, \beta_4\}$, $L_4= \{e, \beta_5, \beta_6\}$

4.3 Subgroups of G which have order 4

Consider an arbitrary subgroup M of G which has order 4. Then M consists of elements of orders1, 2 or 4. If M consists of an element which has order 4, then M is generated by that element. Then the first three subgroups are cyclic and the remaining 6 are non-cyclic.

$M_1= \{e, \alpha_1, \alpha_2, \alpha_9\}$, $M_2= \{e, \alpha_3, \alpha_8, \alpha_9\}$, $M_3= \{e, \alpha_4, \alpha_9, \alpha_{11}\}$,$M_4= \{e, \alpha_5, \alpha_9, \alpha_{10}\}$, $M_5= \{e, \alpha_6, \alpha_9, \alpha_{13}\}$,$M_6= \{e, \alpha_7, \alpha_9, \alpha_{12}\}$,$M_7= \{e, \alpha_9, \gamma_1, \gamma_2\}$, $M_8= \{e, \alpha_9, \gamma_3, \gamma_6\}$, $M_9= \{e, \alpha_9, \gamma_4, \gamma_5\}$.

4.4 Subgroups of G which have order 6

Consider a subgroup N of G which has order 6. Since $o(N) = 2 \times 3$, by Sylow's theorem, N has only one subgroup of order 3. Further, if N contains an element which has order 6, then N is generated by that element. Then the subgroups of order 6 are

$N_1 = \{e, \alpha_1, \alpha_5, \alpha_{13}, \beta_1, \beta_8\}$, $N_2 = \{e, \alpha_2, \alpha_4, \alpha_{12}, \beta_2, \beta_7\}$, $N_3 = \{e, \alpha_3, \alpha_4, \alpha_5, \beta_3, \beta_4\}$,

$N_4= \{e, \alpha_8, \alpha_{12}, \alpha_{13}, \beta_5, \beta_6\}$, $N_5= \{e, \alpha_9, \beta_1, \beta_8, \delta_2, \delta_3\}$, $N_6= \{e, \alpha_9, \beta_2, \beta_7\ \delta_1, \delta_4\}$,

$N_7 = \{e, \alpha_9, \beta_3, \beta_4, \delta_5, \delta_6\}$, $N_8 = \{e, \alpha_9, \beta_5, \beta_6, \delta_7, \delta_8\}$.

Here each of the last four subgroups of order 6 has two elements of order 6 and we find that all the 8 elements of order 6 have been taken care of and we note that every subgroup of order 6 contains exactly one subgroup of order 3. Thus, there is no other possibility for any other subgroups of order 6.

4.5 Subgroups of G which have order 8

Consider an arbitrary subgroup K of order 8 in G. Then K consists of elements of order 1, 2, 4 or 8. If K has an element of order 8, then K is generated by that element. Then the subgroups are

$K_1 = \{e, \alpha_9, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$, $K_2 = \{e, \alpha_9\, \gamma_1, \gamma_2, \eta_6, \eta_7, \eta_{10}, \eta_{11}\}$

$K_3 = e, \alpha_9, \gamma_3, \gamma_6, \eta_1, \eta_4, \eta_8, \eta_9\}$, $K_4 = \{e, \alpha_9, \gamma_4, \gamma_5, \eta_2, \eta_3, \eta_5, \eta_{12}\}$,

Here each of the last three subgroups of order 8 has four elements of order 8 and we find that all the 12 elements of order 8 have been taken care of and we note that only one subgroup of order 8 contains six elements of order 4. Thus, there is no other possibility for any other subgroups of order 8.

4.6 Subgroups of G which have order 12

Let P be an arbitrary subgroupof G of order 12. Since, $o(P) = 2^2 \text{ x } 3$, by Sylow's theorem, P has a 2-Sylow subgroup which has order 4. The number of 2-Sylow subgroups is of the form 1+2m and we have 1+2m | 3. Then the probable values for m= 0,1.

Also, by Sylow's theorem P hasonly one subgroup of order 3.

Two cases arise:

i. Only one subgroup of order 3 and three subgroups of order 4.
ii. Only one subgroup of order 3 and one subgroup of order 4.

Case(i): At a time, combining a subgroup of order 3 with three subgroups of order 4, we find the following subgroups of order 12.

$P_1 = \{e, \alpha_1, \alpha_2, \alpha_5, \alpha_6, \alpha_9, \alpha_{10}, \alpha_{13}, \beta_1, \beta_8, \delta_2, \delta_3\}$,
$P_2 = \{e, \alpha_1, \alpha_2, \alpha_4, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{12}, \beta_2, \beta_7, \delta_1, \delta_4\}$,
$P_3 = \{e, \alpha_3, \alpha_4, \alpha_5, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \beta_3, \beta_4, \delta_5, \delta_6\}$,
$P_4 = \{e, \alpha_3, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{12}, \alpha_{13}, \beta_5, \beta_6, \delta_7, \delta_8\}$.

Here every subgroup of order 12 have two elements of order 6 and we get exactly 8 elements of order 6 and 4 subgroups which have order 3. Then there is no other possibility for any other subgroups.

Case(ii): Multiplying a subgroup of order 4 by a subgroup of order 3 produces another subgroup of order 3, which is not in the original group and hence this case does not at all.

4.7 Subgroups of G which have order 16

Since, $o(G)=48=2^4x3$ and $2^4 \mid o(G)$ but $2^{4+1} \nmid o(G)$, then G has a 2-Sylow subgroup which has order 2^4.Thenumber of 2 –Sylow subgroups is of the form 1+2m and we have 1+2m | o(G).

Hence 1+2m | 2^4x3.Then 1+2m | 3. There are two probable values for m, namely, 0and1.

Hence there are atmost three 2-Sylow subgroups corresponding to m=1 which has order 16.

But G does not have an element of order 16, so that the subgroups of order 16 must have elements of orders 1,2,4 or 8. Then the subgroups are

$Q_1=\{e, \alpha_1, \alpha_2, \alpha_3, \alpha_8, \alpha_9, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \eta_6, \eta_7, \eta_{10}, \eta_{11}\}$,

$Q_2= \{e, \alpha_4, \alpha_6, \alpha_9, \alpha_{11}, \alpha_{13}, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \eta_1, \eta_4, \eta_8, \eta_9\}$,

$Q_3= \{e, \alpha_5, \alpha_7, \alpha_9, \alpha_{10}, \alpha_{12}, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \eta_2, \eta_3, \eta_5, \eta_{12}\}$.

4.8 Subgroups of G which have order 24

Consider an arbitrary subgroup R of G of order 24 and $o(R)=24=2^3x3$. By Sylow's theorem, R has a 3-Sylow subgroup which has order 3. The number of 3-Sylow subgroup of order 3 is 1+3m and we have $1+3m \mid 2^3x3$. Then $1+3m \mid 2^3$. The probable values for m are 0,1. Hence, the number of subgroups of R of order 3 is either 1 or 4.

Also, R has 2-Sylow subgroups which have order 8. The number of 2- Sylow subgroups of order 8 is 1+2m and we have $1+2m \mid 3$. The probable values for mare 0,1. Hence the number of subgroups of R of order 8 is either 1 or 3.

Four cases arise:

i. Only one subgroup of order 3 and one subgroup of order 8.
ii. Four subgroups of order 3 and three subgroups of order 8.
iii. One subgroup of order 3 and three subgroups of order 8.
iv. Four subgroups of order 3 and one subgroup of order 8.

Case(i): Let S be the one subgroup of order 3 and T be another one subgroup which has order 8 in R. But S and T are normal subgroups in R. Therefore R=ST should be abelian, but it is found to be false by verifying all cases of S and T. Hence, this case does not arise at all.

Case(ii): Will not exist, since combing four subgroups of order 3with three subgroups of order 8 we will have more than 24 elements.

Case(iii): Let $\mathscr{B}$ be a collection of three number of subgroups of order 8. Consider a subgroup P which has order 3. Clearly P is a normal subgroup of R, because P is the only one subgroup in R of order 3. Hence, $rpr^{-1} \in P$ for every $r \in R$, $p \in P$. At a time, consider a subgroup of order 3, combining it with three subgroups of order 8, we find that this condition is not true. Hence, this case does not arise at all.

Case(iv): At a time, combining a subgroup of order 8 with four subgroups of order 3, we get exactly one subgroup of order 24, by examination.

$R_1= \{e, \alpha_9, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8\}$.

5. The structure of L(G)

Using all the above subgroups of G, we draw the Hasse diagram of L(G) as in Figure1.

Lattice Structure of L(G)

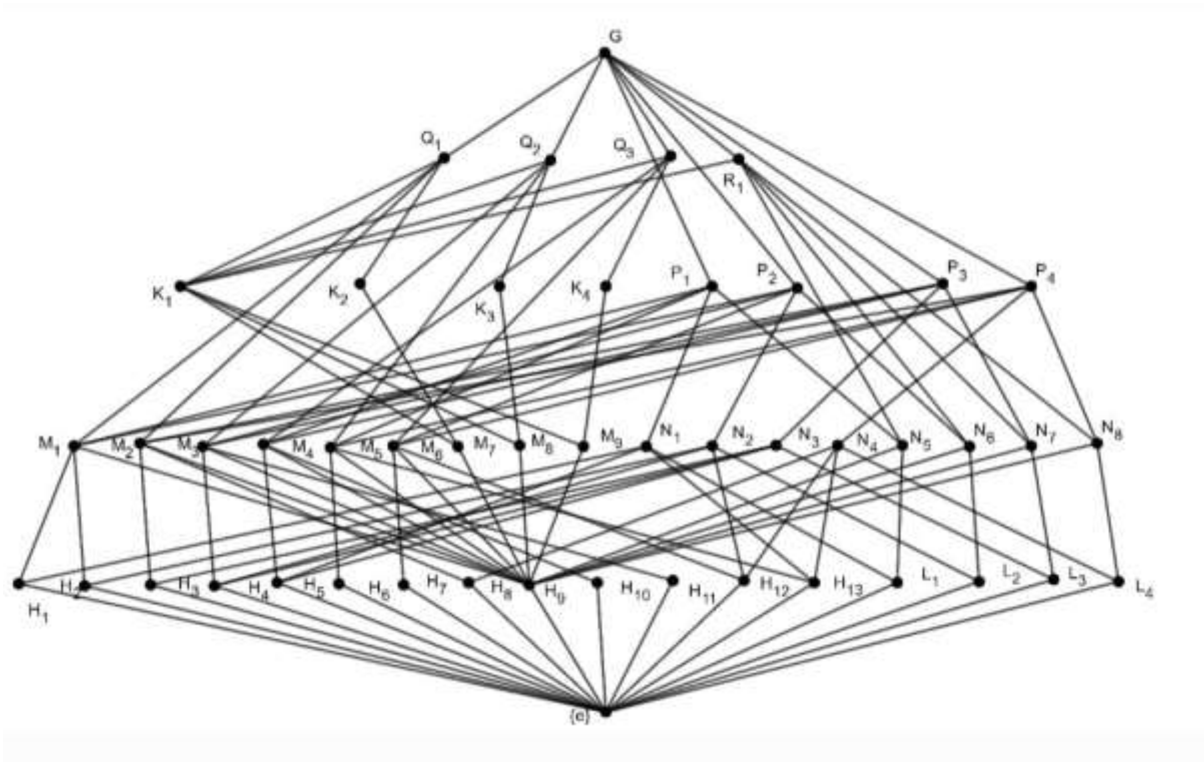

Figure 1

6. Conclusion:

We have displayed in this paper the lattice structure formed by all subgroups of the 2x2non-singular matrices over Z_3. There is a scope for studying some lattice - theoretic properties of the lattice such as super solvability, 0-distributivity and complementedness in L(G).

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