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INTRODUCING A NOVEL FAMILY OF LAGUERRE-HERMITE-FUBINI POLYNOMIALS WITH THEIR APPLICATIONS

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ABSTRACT
This paper proposes a novel class of Laguerre-Hermite-Fubini polynomials, exploring their various properties and applications. We evolve summation axioms for these polynomials using series summation techniques. Additionally, we will construct symmetric identity axioms for the Laguerre-Hermite-Fubini polynomials through the use of generating functions.

Keywords: *Hermite polynomials, Fubini numbers and polynomials, Laguerre-Hermite-Fubini polynomials, Summation axioms, Symmetric identity axioms.*

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1. Introduction

Throughout the entire contents of this chapter, the theory of Laguerre-Hermite-Fubini polynomials, as well as applications associated with unique notions and corollaries are discussed. The Laguerre polynomials, together with their novel ideas and theories for additional families, will be the primary topic of discussion in this section.
In the first step of our process, we establish the following conclusion using the Laguerre-Hermite-Fubini polynomials by employing series rearrangement techniques. Beyond that, we will apply the use of the generating function, which has been carried out by Pathan and Khan in their recent research [13, 14] in order to construct broad symmetry identity axioms for the generalized Laguerre-Hermite-Fubini polynomials.

It is feasible to define the generating function for the Laguerre polynomials in two variables $Ł_n(\eta,\xi)$ [5] by using the subsequent formula:

$$\frac{1}{(1-\xi\tau)}\exp\left(\frac{-\eta\tau}{1-\xi\tau}\right)=\sum_{n=0}^{\infty} Ł_n(\eta,\xi)\tau^n, (|\xi\tau|<1) \tag{1}$$

which is equivalently expressed in [6], can be expressed by the following:

$$\exp(\xi\tau)C_0(\eta\tau)=\sum_{n=0}^{\infty} Ł_n(\eta,\xi)\frac{\tau^n}{n!}, \tag{2}$$

In the existing equation (2), the operator $C_0(\eta)$ represents the Tricomi function of the zero-th order, thus the Tricomi functions of the n^{th}order, denoted as $C_n(\eta)$, are formally defined as follows:

$$C_n(\eta)=\sum_{n=0}^{\infty}\frac{(-1)^r\eta^r}{r!\,(n+r)!},(n\in N_0) \tag{3}$$

By use the provided generating function:

$$\exp\left(\tau-\frac{\eta}{\tau}\right)=\sum_{n=0}^{\infty}C_n(\eta)\tau^n, \tag{4}$$

for $\tau\neq 0$ and $\forall$ finite η . There appears a connection between the Tricomi functions $C_n(\eta)$ and the Bessel function $J_n(\eta)$ by means of the following relation, which can be expressed in the following manner,

$$C_n(\eta)=\eta^{\frac{n}{2}}J_n\big(2\sqrt{\eta}\big). \tag{5}$$

So, from equations (2) and (3), we derive

$$L_n(\eta,\xi)=n!\sum_{s=0}^{n}\frac{(-1)^s\eta^s\xi^{n-s}}{(s!)^2(n-s)!}=\xi^nL_n\left(\frac{\eta}{\xi}\right). \tag{6}$$

as a result, we have

$$Ł_n(\eta,\xi)=\frac{(-1)^n\eta^n}{n!},Ł_n(0,\xi)=\xi^n,Ł_n(\eta,1)=Ł_n(\eta), \tag{7}$$

In the above-obtained equation (6) L_n denotes the ordinary Laguerre polynomials referenced in [1],
The mathematical formulation of the Hermite Kampé de Fériet polynomials $H_n(\eta,\xi)$ is two-variable and can be understood as follows [2, 4]:

$$H_n(\eta,\xi)=n!\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{\xi^r\eta^{n-2r}}{r!\,(n-2r)!}, \tag{8}$$

and are assisted by the following generating function:

$$e^{\eta\tau+\xi\tau^2}=\sum_{n=0}^{\infty}H_n(\eta,\xi)\frac{\tau^n}{n!}. \tag{9}$$

and if $\xi=-1$ and η is substituted with 2η, in equation (9), the result is the ordinary Hermite polynomials $H_n(\eta)$[2].
The three-variable Laguerre-Hermite polynomials denoted as ${}_LH_n(\eta,\xi,\mu)$ have been introduced by Dattoli et al. in [8, p.241] which are defined as:

$${}_LH_n(\eta,\xi,\mu)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\mu^kL_{n-2k}(\eta,\xi)}{k!\,(n-2k)!}. \tag{10}$$

Now, the Three-variable Laguerre-Hermite polynomials denoted as ${}_LH_n(\eta,\xi,\mu)$ are determined by the generating function that is as follows:

$$\frac{1}{(1-\mu\tau)}\exp\left(\frac{-\eta\tau}{1-\mu\tau}+\frac{\xi\tau^2}{1-\mu\tau^2}\right)=\sum_{n=0}^{\infty}{}_LH_n(\eta,\xi,\mu)\tau^n, \tag{11}$$

above equation equivalent to

$$\exp(\xi\tau+\mu\tau^2)C_0(\eta\tau)=\sum_{n=0}^{\infty}{}_LH_n(\eta,\xi,\mu)\frac{\tau^n}{n!}. \tag{12}$$

It is evident that

$$ {}_L\mathrm{H}_n\left(\eta,\xi,-\frac{1}{2}\right) = {}_L\mathrm{H}_n(\eta,\xi), $$
$$ {}_LH_n(\eta,1,-1) = {}_LH_n(\eta), $$

where ${}_LH_n(\eta,\xi)$ denotes the Two-variable Laguerre-Hermite polynomials [6] and ${}_LH_n(\eta)$ denotes the Laguerre-Hermite polynomials [7] .

The following is a definition of Fubini polynomials, which can also be referred to as geometric polynomials [3]:

$$ F_n(\eta) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k!\, \eta^k, \tag{13} $$

in which the stirling number of the second kind is denoted by the symbol $\left\{ {n \atop k} \right\}$ [9]

For $\eta = 1$ in equation (13), we get n^{th} Fubini number, also known as the ordered Bell number or the geometric number

F_n [3, 9, 10, 12, 16] is defined by the following equation.

$$ F_n(1) = F_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k!. \tag{14} $$

From [3], we can express the exponential generating functions of geometric polynomials which can be given by :

$$ \frac{1}{1-\eta(e^\tau - 1)} = \sum_{n=0}^{\infty} F_n(\eta) \frac{\tau^n}{n!}, \tag{15} $$

the notions that refer to the geometric series are as follows [3]:

$$ \left(\eta \frac{d}{d\eta}\right)^m \frac{1}{1-\eta} = \sum_{k=0}^{\infty} k^m \eta^k = \frac{1}{1-\eta} F_m\left(\frac{\eta}{1-\eta}\right), |\eta| < 1. $$

The following, is a brief list of these polynomials and numbers which are included in this segment:

$F_0(\eta) = 1, F_1(\eta) = \eta, F_2(\eta) = \eta + 2\eta^2, F_3(\eta) = \eta + 6\eta^2 + 6\eta^3, F_4(\eta) = \eta + 14\eta^2 + 36\eta^3 + 24\eta^4$, and

$F_0 = 1, F_1 = 1, F_2 = 3, F_3 = 13, F_4 = 75.$

A relation between Geometric and exponential polynomials [3] can be given by:

$$ F_n(\eta) = \int_0^\infty \phi_n(\eta) e^{-\lambda} d\lambda. \tag{16} $$

The Laguerre-based Hermite-Bernoulli polynomials with three variables have been introduced by Khan et al. [11], which are defined using the following generating function:

$$ \left(\frac{\tau^m}{e^\tau - \sum_{h=0}^{m-1} \frac{\tau^h}{h!}} \right)^\alpha e^{\xi\tau + \mu\tau^2} C_0(\eta\tau) = \sum_{n=0}^{\infty} {}_LH^{B_n^{[\alpha,m-1]}}(\eta,\xi,\mu) \frac{\tau^n}{n!}. \tag{17} $$

By setting $\alpha = 1$ in equation (13), the result reduces to a known result that was cited by Pathan and Khan in [13], futhermore setting $\alpha = 1$, it yields the result of Dattoli et al. [4].

The structure of this manuscript is as follows: In the second segment, we examine generating functions for Laguerre-Hermite-Fubini numbers additionally this segment explores the properties and applications associated with these polynomials and numbers.

In the third segment, we establish summation axioms for Laguerre-Hermite-Fubini numbers and polynomials.

And, the fourth segment, is devoted to the construction of symmetric identities axioms for Laguerre- Hermite-Fubini numbers and polynomials using generating functions.

2. Laguerre-Hermite-Fubini numbers and polynomials

In this segment, we define three-variable Laguerre-Hermite-Fubini polynomials and derive a fundamental features that yield a novel formula for ${}_LH^{F_n}(\eta,\xi,\mu;\zeta)$.

We present a novel approach to 4-variable Laguerre-based Hermite-Fubini polynomials which can be obtained using the following generating function:

$$\frac{e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)}{1-\zeta(e^\tau-1)}=\sum_{n=0}^{\infty}{}_LH^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}. \tag{18}$$

Clearly from definition we get from the equation (18), we have

$${}_LH^{F_n}(0,0,0;\zeta)=F_n(\zeta),{}_L H^{F_n}(0,0,0;1)=F_n.$$

By substituting $\eta=\mu=0$ in equation (18) we derive 2-variable Fubini polynomials which is precisely specified by Kargin [12].

$$\frac{e^{\xi\tau}}{1-\zeta(e^\tau-1)}=\sum_{n=0}^{\infty}F_n(\eta;\mu)\frac{\tau^n}{n!}. \tag{19}$$

Theorem 2.1. The axiom for Laguerre-based Hermite-Fubini polynomials is valid for $\forall\, n\geq 0$

$${}_LH^{F_n}(\eta,\xi,\mu;\zeta)=\sum_{m=0}^{n}\binom{n}{r}F_{n-m}(\mu){}_LH_m(\eta,\xi). \tag{20}$$

Proof. From definition of equation (18), we Obtain

$$\sum_{n=0}^{\infty}{}_LH^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}=\frac{e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)}{1-\zeta(e^\tau-1)}$$
$$=\sum_{n=0}^{\infty}F_n(\mu)\frac{\tau^n}{n!}\sum_{m=0}^{\infty}{}_LH_m(\eta,\xi)\frac{\tau^m}{m!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{r}F_{n-m}(\mu){}_LH_m(\eta,\xi)\right)\frac{\tau^n}{n!}.$$

thus, by equating the coefficients of $\frac{\tau^n}{n!}$ from both sides, we get the desired result mentioned in the equation (20).

Theorem 2.2. The axiom for Laguerre- Hermite-Fubini polynomials verifies for $\forall\, n\geq 0$

$${}_LH_n(\eta,\xi,\mu)={}_L H^{F_n}(\eta,\xi,\mu;\zeta)-\zeta{}_LH^{F_n}(\eta,\xi,\mu;\zeta)+\zeta{}_LH^{F_n}(\eta,\xi,\mu;\zeta). \tag{21}$$

Proof. Let's start with the definition of equation (18) so, we write

$$e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)=\frac{1-\zeta(e^\tau-1)}{1-\zeta(e^\tau-1)}e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)$$
$$=\frac{e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)}{1-\zeta(e^\tau-1)}-\frac{\zeta(e^\tau-1)}{1-\mu(e^\tau-1)}e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)$$

Then, according to the concept of Laguerre-based Hermite polynomials ${}_LH_n(\eta,\xi,\mu)$ and eqaution (18), we have

$$\sum_{n=0}^{\infty} {}_LH_n(\eta,\xi,\mu)\frac{\tau^n}{n!} = \sum_{n=0}^{\infty} [\,{}_LH^{F_n}(\eta,\xi,\mu;\zeta) - \zeta_L H^{F_n}(\eta,\xi,\mu;\zeta) + \zeta_L H^{F_n}(\eta,\xi,\mu;\zeta)]\frac{\tau^n}{n!}.$$

Thus, finally,by equating the coefficients of $\frac{\tau^n}{n!}$, we obtain the desired equation (21)

Theorem 2.3. The axiom.

$$\sum_{k=0}^{n} \binom{n}{k} {}_LH_n{}^{F_{n-k}}(\eta_1,\xi_1,\mu_1;\zeta_1)_L H^{F_k}(\eta_2,\xi_2,\mu_2;\zeta_2)$$

$$= \frac{\zeta_{2L}H^{F_n}(\eta_1,\xi_1+\xi_2,\mu_1+\mu_2;\zeta_1) - \zeta_{1L}H^{F_n}(\eta_2,\xi_1+\xi_2,\mu_1+\mu_2;\zeta_2)}{\zeta_2-\zeta_1}. \quad (22)$$

is valid $\forall\, n \geq 0$ and $\zeta_1 \neq \zeta_2$

Proof. The products of two similar expressions from equation (18) can be expressed as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_LH^{F_n}(\eta_1,\xi_1,\mu_1;\zeta_1)\frac{\tau^n}{n!}\,{}_LH^{F_k}(\eta_2,\xi_2,\mu_2;\zeta_2)\frac{\tau^k}{k!} = \frac{e^{\xi_1\tau+\mu_1\tau^2}C_0(\eta_1\tau)}{1-\zeta_1(e^\tau-1)}\frac{e^{\xi_2\tau+\mu_2\tau^2}C_0(\eta_2\tau)}{1-\zeta_2(e^\tau-1)}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\,{}_LH^{F_{n-k}}(\eta_1,\xi_1,\mu_1;\zeta_1)_L H^{F_k}(\eta_2,\xi_2,\mu_2;\zeta_2)\right)\frac{\tau^n}{n!}$$

$$= \frac{\zeta_2}{\zeta_2-\zeta_1}\frac{e^{(\xi_1+\xi_2)\tau+(\mu_1+\mu_2)\tau^2}C_0(\eta_1\tau)}{1-\zeta_1(e^\tau-1)} - \frac{\zeta_1}{\zeta_2-\zeta_1}\frac{e^{(\xi_1+\xi_2)\tau+(\mu_1+\mu_2)\tau^2}C_0(\eta_2\tau)}{1-\zeta_2(e^\tau-1)}$$

$$= \left(\frac{\zeta_{2L}H^{F_n}(\eta_1,\xi_1+\xi_2,\mu_1+\mu_2;\zeta_1) - \zeta_{1L}H^{F_n}(\eta_2,\xi_1+\xi_2,\mu_1+\mu_2;\zeta_2)}{\zeta_2-\zeta_1}\right)\frac{\tau^n}{n!}.$$

thus, by equating the coefficients of $\frac{\tau^n}{n!}$ from both sides, we obtain the desired result of the equation (22).

Theorem 2.4. The axiom for Laguerre-Hermite-Fubini polynomials is valid $\forall\, n \geq 0$,:

$$\zeta_L H^{F_n}(\eta,\xi+1,\mu;\zeta) = (1+\zeta)_L H^{F_n}(\eta,\xi,\mu;\zeta) - {}_LH_n(\eta,\xi,\mu). \quad (23)$$

Proof. We start with the equation (18) so, we write the expressions as

$$\sum_{n=0}^{\infty} [\,{}_LH^{F_n}(\eta,\xi+1,\mu;\zeta) - {}_LH^{F_n}(\eta,\xi+1,\mu;\zeta)]\frac{\tau^n}{n!} = \frac{e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)}{1-\zeta(e^\tau-1)}(e^\tau-1)$$

$$= \frac{1}{\zeta}\left[\frac{e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)}{1-\zeta(e^\tau-1)} - e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)\right]$$

$$= \frac{1}{\zeta}\sum_{n=0}^{\infty} [\,{}_LH^{F_n}(\eta,\xi,\mu;w) - {}_LH_n(\eta,\xi,\mu)]\frac{\tau^n}{n!}.$$

thus, by comparing the coefficients of $\frac{\tau^n}{n!}$ from both the sides, we obtain the desired result mentioned in equation (23)

Remark 2.3. On Substituting $\eta = \xi = \mu = 0$ and $\xi = -1$ in Theorem 2.4, we obtain

$$\zeta_L H^{F_n}(0,1,0;\zeta) = (1+\zeta)_L H^{F_n}(0,0,0;\zeta), \tag{24}$$

and

$$\zeta_L H^{F_n}(0,0,0;\zeta) = (1+\zeta)_L H^{F_n}(0,-1,0;\zeta) - (-1)^n. \tag{25}$$

Theorem 2.5. The **axiom** for Laguerre- Hermite-Fubini polynomials

$${}_L H^{F_n}(\eta, p\xi, q\mu;\zeta) = n! \sum_{k=0}^{n} \sum_{j=0}^{\left[\frac{k}{2}\right]} {}_L H_n{}^{F_{n-k}}(\eta,\xi,\mu;\zeta)((p-1)\eta)^k((q-1)\xi)^j \frac{1}{(n-k-2j)!\,j!}. \tag{26}$$

is valid for $\forall\, n \geq 0$, and $p, q \in R$.

Proof. By rewriting the generating function we get from equation (18), we have

$$\sum_{n=0}^{\infty} {}_L H_n{}^{F_n}(\eta, q\xi, q\mu;\zeta)\frac{\tau^n}{n!} = \frac{1}{1-\zeta(e^\tau-1)} e^{\xi\tau+\mu\tau^2} e^{(p-1)\xi\tau} e^{(q-1)\mu\tau^2} C_0(\eta\tau)$$

$$= \left(\sum_{n=0}^{\infty} {}_L H_n{}^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}\right)\left(\sum_{k=0}^{\infty} ((p-1)\eta)^k \frac{\tau^k}{k!}\right)\left(\sum_{j=0}^{\infty} ((q-1)\xi)^j \frac{\tau^{2j}}{j!}\right)$$

$$= \left(\sum_{n=0}^{\infty} {}_L H_n{}^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} ((p-1)\eta)^k((q-1)\xi)^j \frac{\tau^{k+2j}}{n!\,k!\,j!}\right)$$

Now, by substituting k with $k-2j$ in the above equation, so, we take

$$Left\ Hand\ Side = \left(\sum_{n=0}^{\infty} {}_L H_n{}^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}\right)\left(\sum_{k=2j}^{\infty} ((p-1)\eta)^{k-2j}((q-1)\xi)^j \frac{\tau^k}{(k-2j)!\,j!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=2j}^{\infty} {}_L H_n{}^{F_n}(\eta,\xi,\mu;\zeta)((p-1)\eta)^{k-2j}((q-1)\xi)^j \frac{\tau^{n+k}}{(k-2j)!\,j!\,n!}$$

Again substituting n with $n-k$ in the above equation, we get

$$Left\ Hand\ Side = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{\left[\frac{k}{2}\right]} {}_L H_n{}^{F_{n-k}}(\eta,\xi,\mu;\zeta)((p-1)\eta)^{k-2j}((q-1)\xi)^j \frac{\tau^n}{(n-k-2j)!\,j!\,k!}.$$

thus, by comapiring the coefficients of τ^n on both sides, we obtain the result as per equation (26).

Theorem 2.6. The axiom of Laguerre-Hermite-Fubini polynomials

$${}_L H^{F_n}(\eta,\xi,\mu;\zeta) = \sum_{l=0}^{n} \binom{n}{l}_L H_{n-l}(\eta,\xi) \sum_{k=0}^{l} \zeta^k k!\, S_2(l,k). \tag{27}$$

is valid $\forall\, n \geq 0$.

Proof. We use equation (18), so we have

$$\sum_{n=0}^{\infty} {}_L H_n{}^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!} = \frac{e^{\xi\tau+\mu\tau^2}}{1-\zeta(e^\tau-1)} C_0(\eta\tau)$$

$$= e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)\sum_{k=0}^{\infty}\zeta^k(e^\tau-1)^k = e^{\eta\tau+\xi\tau^2}\sum_{k=0}^{\infty}\zeta^k\sum_{l=k}^{\infty}k!\,S_2(l,k)\frac{\tau^l}{l!}$$

$$=\sum_{n=0}^{\infty}{}_LH_n(\eta,\xi,\mu)\frac{\tau^n}{n!}\sum_{l=0}^{\infty}\zeta^k\sum_{k=0}^{l}k!\,S_2(l,k)\frac{\tau^l}{l!}.$$

Now, substituting n with $n-l$ in the above equation, we get

$$\sum_{n=0}^{\infty}{}_LH^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}{}_LH_{n-l}(\eta,\xi,\mu)\sum_{k=0}^{l}\zeta^k k!\,S_2(l,k)\right)\frac{\tau^n}{n!}.$$

thus, by equating the coefficients of $\frac{\tau^n}{n!}$ on both sides, we obtain the desired result as mentioned in equation (27).

Theorem 2.7. The axiom for Laguerre-Hermite-Fubini polynomials:

$${}_LH^{F_n}(\eta,\xi+r,\mu;\zeta)=\sum_{l=0}^{n}\binom{n}{l}{}_LH_{n-l}(\eta,\xi,\mu)\sum_{k=0}^{l}\zeta^k k!\,S_2(l+r,k+r).\quad l \tag{28}$$

is valid for $\forall\, n\geq 0$

Proof. Using the equation (18) and replacing ξ by $\xi+r$ in this equation, we get

$$\sum_{n=0}^{\infty}{}_LH^{F_n}(\eta,\xi+r,\mu;\zeta)\frac{\tau^n}{n!}=\frac{e^{(\xi+r)\tau+\mu\tau^2}}{1-\zeta(e^t-1)}C_0(\eta\tau)$$

$$=e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)e^{r\tau}\sum_{k=0}^{\infty}\zeta^k(e^\tau-1)^k=e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)e^{r\tau}\sum_{k=0}^{\infty}\zeta^k\sum_{l=k}^{\infty}k!\,S_2(l,k)\frac{\tau^l}{l!}$$

$$=\sum_{n=0}^{\infty}{}_LH_n(\eta,\xi,\mu)\frac{\tau^n}{n!}\sum_{l=0}^{\infty}\zeta^k\sum_{k=0}^{l}k!\,S_2(l+r,k+r)\frac{\tau^l}{l!}.$$

Now, substituting n with $n-l$ in the above equation, we get

$$\sum_{n=0}^{\infty}{}_LH^{F_n}(\eta,\xi+r,\mu;\zeta)\frac{\tau^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}{}_LH_{n-l}(\eta,\xi,\mu)\sum_{k=0}^{l}\zeta^k k!\,S_2(l+r,k+r)\right)\frac{\tau^n}{n!}.$$

thus, by comparing the coefficients of $\frac{\tau^n}{n!}$ on both sides, we get result as per the equation (28).

3. Summation Axioms for Laguerre-Hermite-Fubini polynomials

we establish the axioms involving the Laguerre-Hermite-Fubini polynomials ${}_LH^{F_n}(\eta,\xi,\mu;\zeta)$ by using series rearrangement techniques. Additionally, we take into consideration the specific case of these polynomials.

Theorem 3.1. The summation axiom for Laguerre- Hermite-Fubini polynomials ${}_HF_n(\eta,v,\mu)$ is valid for

$${}_LH^{F_{q+l}}(\eta,v,\mu;\zeta)\;=\sum_{n,p=0}^{q,l}\binom{q}{n}\binom{l}{p}(v-\xi)^{n+p}{}_LH^{F_{q+l-n-p}}(\eta,\xi,\mu;\zeta). \tag{29}$$

Proof. Substituting τ with $\tau+u$ in the equation (18) and then applying the formula [15,p.52(2)]:

$$\sum_{N=0}^{\infty} f(N)\frac{(\eta+\xi)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{\eta^n}{n!}\frac{\xi^m}{m!}, \tag{30}$$

Now, the generating function for the Laguerre-Hermite-Fubini polynomials ${}_LH^{F_n}(\eta,\xi,\mu;\zeta)$ is obtained by the following equation:

$$\frac{1}{1-\zeta(e^{\tau+u}-1)}e^{\mu(\tau+u)^2}C_0(\eta\tau) = e^{-\xi(\tau+u)}\sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(\eta,\xi,\mu;\zeta)\frac{\tau^q}{q!}\frac{u^l}{l!}. \tag{31}$$

substituting ξ with v in the previous equation and compairing the resulting expression to the previous equation, we find

$$exp\big((v-\xi)(\tau+u)\big)\sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(\eta,\xi,\mu;\zeta)\frac{\tau^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(\eta,v,\mu;\zeta)\frac{\tau^q}{q!}\frac{u^l}{l!}. \tag{32}$$

Expanding the exponential function in equation (32),yields

$$\sum_{N=0}^{\infty}\frac{[(v-\xi)(\tau+u)]^N}{N!}\sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(\eta,\xi,\mu;\zeta)\frac{\tau^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(\eta,v,\mu;\zeta)\frac{\tau^q}{q!}\frac{u^l}{l!}, \tag{33}$$

which by applying formula from equation (30) to the first summation on the left hand side, it yields

$$\sum_{n,p=0}^{\infty}\frac{(v-\xi)^{n+p}\tau^n u^p}{n!\,p!}\sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(\eta,\xi,\mu;\zeta)\frac{\tau^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(\eta,v,\mu;\zeta)\frac{\tau^q}{q!}\frac{u^l}{l!}. \tag{34}$$

Next, by substituting q with $q-n$, and l with $l-p$ and applying the lemma ([15, p.100(1)]), we obtain:

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A(n,k) = \sum_{k=0}^{\infty}\sum_{n=0}^{k} A(n,k-n), \tag{35}$$

On the Left Hand Side of the equation (34), we get

$$\sum_{q,l=0}^{\infty}\sum_{n,p=0}^{q,l}\frac{(v-\xi)^{n+p}}{n!\,p!}\,{}_LH^{F_{q+l-n-p}}(\eta,\xi,\mu;\zeta)\frac{\tau^q}{(q-n)!}\frac{u^l}{(l-p)!}$$
$$= \sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(\eta,v,\mu;\zeta)\frac{\tau^q}{q!}\frac{u^l}{l!}. \tag{36}$$

thus, by compairing the coefficients of the corresponding powers of τ and u in the above equation, thus the assertion of Theorem 3.1 is derived.

Remark 3.1. By setting $l=0$ in the proof of Theorem (29), we are able to draw the following implication.

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Corollary 3.1. The summation axiom for Hermite-Fubini polynomials ${}_HF_n(\eta,\xi;\mu)$ is true:

$${}_LH^{F_q}(\eta,v,\mu;\zeta) = \sum_{n=0}^{q}\binom{q}{n}(v-\xi)^n\,{}_LH^{F_{q-n}}(\eta,\xi,\mu;\zeta). \tag{37}$$

Remark 3.2. By substituting v with $v+\xi$ in the equation (37), we obtain

$$ {}_LH^{F_q}(\eta, v+\xi, \mu; \zeta) = \sum_{n=0}^{q} \binom{q}{n} v^n {}_LH^{F_{q-n}}(\eta, \xi, \mu; \zeta). \tag{38} $$

Theorem 3.2. The axiom for Laguerre-Hermite-Fubini polynomials ${}_HF_n(\eta, \xi; \mu)$ is valid for:

$$ {}_LH^{F_n}(\eta, u, v; \zeta) {}_LH^{F_m}(\eta', u', v'; \zeta') = \sum_{r,k=0}^{n,m} \binom{n}{r}\binom{m}{k} H_r(u-\xi, v-\mu) {}_LH^{F_{n-r}}(\eta, \xi, \mu; \zeta) \times H_k(u'-\xi', v'-\mu') {}_LH^{F_{m-k}}(\eta', \xi', \mu'; \zeta'). \tag{39} $$

Proof. Let us say, the product of the Laguerre-Hermite-Fubini polynomials, which can be expressed using the generating function in (18) as follows:

$$ \frac{1}{1-\zeta'(e^\tau - 1)} e^{\xi\tau+\mu\tau^2} C_0(\eta\tau) \frac{1}{1-\zeta'(e^T-1)} e^{\xi' T + \mu' T^2} C_0(\eta' T) = \sum_{n=0}^{\infty} {}_LH^{F_n}(\eta, \xi, \mu; \zeta) \frac{\tau^n}{n!} \sum_{m=0}^{\infty} {}_LH^{F_m}(\eta', \xi', \mu'; \zeta') \frac{T^m}{m!}. \tag{40} $$

By substituting ξ with u, μ with v, ξ' with u' and μ' with v' in the above equation (40) and equating the resulting expression to itself,

$$ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_LH^{F_n}(\eta, u, v; \zeta) {}_LH^{F_m}(\eta', u', v'; \zeta') \frac{\tau^n}{n!}\frac{T^m}{m!} = exp((u-\xi)\tau + (v-\mu)\tau^2) exp((u'-\xi')T + (v'-\mu')T^2) $$

$$ \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_LH^{F_n}(\eta, \xi, \mu; \zeta) {}_LH^{F_m}(\eta', \xi', \mu'; \zeta') \frac{\tau^n}{n!}\frac{T^m}{m!}, $$

with the application of the generating function (35) in the exponential on the Right Hand Side, becomes

$$ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_LH^{F_n}(\eta, u, v; \zeta) {}_LH^{F_m}(\eta', u', v'; \zeta') \frac{\tau^n}{n!}\frac{T^m}{m!} $$

$$ = \sum_{n,r=0}^{\infty} H_r(u-\xi, v-\mu) {}_LH^{F_n}(\eta, \xi, \mu; \zeta) \frac{\tau^{n+r}}{n!\,r!} \sum_{m,k=0}^{\infty} H_k(u'-\xi', v'-\mu') {}_LH^{F_m}(\eta', \xi', \mu'; \zeta') \frac{T^{m+k}}{m!\,k!}. \tag{41} $$

Finally, by substituting n with $n-r$ and m with $m-k$ and using the equation (35) on the Right Hand Side of the above equation and then by equating the coefficients of corresponding powers of τ and T, the assertion (39) of theorem 3.2 is derived.

Remark 3.3. By substituting u with ξ and u' with ξ' in assertion (39) of Theorem 3.2, we deduce the the following consequence of the theorem .

Corollary 3.2. The summation axiom for Hermite-Fubini polynomials ${}_HF_n(\eta, \xi; \mu)$ is true:

$$ {}_LH^{F_n}(\eta, \xi, v; \zeta) {}_LH^{F_m}(\eta', \xi', v'; \zeta') = \sum_{r,k=0}^{n,m} \binom{n}{r}\binom{m}{k} (v-\mu)^r {}_LH^{F_{n-r}}(\eta, \xi, \mu; \zeta) \times (v'-\mu')^k {}_LH^{F_{m-k}}(\eta', \xi', \mu'; \zeta') \quad (42) $$

Theorem 3.3. The summation axiom for Laguerre- Hermite-Fubini polynomials ${}_LH^{F_n}(\eta, \xi, \mu; \zeta)$ is valid:

$$ {}_LH^{F_n}(\eta,\xi+u,\mu+v;\zeta)=\sum_{s=0}^{n}\binom{n}{s}\ {}_LH^{F_{n-s}}(\eta,u,v;\zeta)H_s(u,v). \quad (43)$$

Proof. Let's say, by substituting ξ with $\xi+u$ and μ with $\mu+v$ in the equation (18), we use (9) and rewrite the generating function as:

$$\frac{1}{1-\zeta(e^\tau-1)}exp((\xi+u)\tau+(\mu+v)\tau^2)C_0(\eta\tau)=\sum_{n=0}^{\infty}\ {}_LH^{F_n}(\eta,u,v;\zeta)\frac{\tau^n}{n!}\sum_{s=0}^{\infty}H_s(u,v)\frac{\tau^s}{s!}$$

$$=\sum_{n=0}^{\infty}\ {}_LH^{F_n}(\eta,\xi+u,\mu+v;\zeta)\frac{\tau^n}{n!}.$$

Now, By substituting n with $n-s$ on the Left Hand Side and equating the coefficients of τ^n on both sides, we obtain the result of equation (43).

Theorem 3.4. The axiom of summation for Laguerre-Hermite-Fubini polynomials ${}_LH^{F_n}(\eta,\xi,\mu;\zeta)$ is valid for:

$$ {}_LH^{F_n}(\eta,\xi,\mu;\zeta)=\sum_{r=0}^{n}\binom{n}{r}\ {}_LF_{n-r}(\eta,\xi-u;\zeta)H_r(u,\mu). \quad (44)$$

Proof. By utilizing the generating function in equation (2), we can express the equation (18) as

$$\frac{1}{1-\zeta(e^\tau-1)}e^{(\xi-u)\tau}e^{u\tau+\mu\tau^2}C_0(\eta\tau)=\sum_{n=0}^{\infty}\ {}_LF_n(\eta,\xi-u;\zeta)\frac{\tau^n}{n!}\sum_{r=0}^{\infty}H_r(u,\mu)\frac{\tau^r}{r!}. \quad (45)$$

By substituting n with $n-r$ in the above equation, we get

$$\sum_{n=0}^{\infty}\ {}_LH^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\ {}_LF_{n-r}(\eta,\xi-u;\zeta)H_r(u,\mu)\frac{\tau^n}{(n-r)!\,r!}.$$

By correlating the coefficients of corresponding powers of τ of both sides, we obtain the equation (45)

Theorem 3.5. The subsequent axiom for Laguerre-Hermite-Fubini polynomials ${}_LH^{F_n}(\eta,\xi,\mu;\zeta)$ is valid for

$$ {}_LH^{F_n}(\eta,\xi+1,\mu;\zeta)=\sum_{r=0}^{n}\binom{n}{r}\ {}_LH^{F_{n-r}}(\eta,\xi,\mu;\zeta). \quad (46)$$

Proof. By utilizing the generating function (18), we may derive the following result:

$$\sum_{n=0}^{\infty}\ {}_LH^{F_n}(\eta,\xi+1,\mu;\zeta)\frac{\tau^n}{n!}-\sum_{n=0}^{\infty}\ {}_LH^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}$$

$$=\left(\frac{1}{1-\zeta(e^\tau-1)}\right)(e^\tau-1)e^{\xi\tau+\mu\tau^2}C_0(\eta\tau)$$

$$=\sum_{n=0}^{\infty}\ {}_LH^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}\left(\sum_{r=0}^{\infty}\frac{\tau^r}{r!}-1\right)$$

$$=\sum_{n=0}^{\infty}\ {}_LH^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}\sum_{r=0}^{\infty}\frac{\tau^r}{r!}-\sum_{n=0}^{\infty}\ {}_LH^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\ {}_LH^{F_{n-r}}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}-\sum_{n=0}^{\infty}\ {}_LH^{F_n}(\eta,\xi,\mu;\zeta)\frac{\tau^n}{n!}.$$

Therefore, by correlating the coefficients of the appropriate powers of τ equal on both sides, we derive equation (46).

4. Identity Axioms

In this segment, we establish axioms based on identity for the generalized Laguerre-Hermite-Fubini polynomials ${}_LH^{F_n}(\eta,\xi,\mu;\zeta)$ by utilizing the generating functions (18) and (19). Previous works shows the introduction of symmetric identities by Pathan and Khan in [13,14] .

Theorem 4.1. The summation identity axiom holds for $\eta,\xi,\mu \in R$ and $\forall\ n \geq 0$

$$\sum_{r=0}^{n}\binom{n}{r}b^r a^{n-r}{}_LH^{F_{n-r}}(b\eta,b\xi,b^2\mu;\zeta){}_LH^{F_n}(a\eta,a\xi,a^2\mu;\zeta)$$
$$=\sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}{}_LH^{F_{n-r}}(a\eta,a\xi,a^2\mu;\zeta){}_LH^{F_n}(b\eta,b\xi,b^2\mu;\zeta). \quad (47)$$

Proof. Let us use the expression

$$A(\tau)=\frac{1}{(1-\zeta(e^{a\tau}-1))(1-\zeta(e^{b\tau}-1))}e^{ab\xi\tau+a^2b^2\mu\tau^2}C_0(ab\eta\tau).$$

The expression for $A(\tau)$ demonstrates symmetry with respect to both a and b. We can expand $A(\tau)$ in two different series to derive:

$$A(\tau)=\sum_{n=0}^{\infty}{}_LH^{F_n}(b\eta,b\xi,b^2\mu;\zeta)\frac{(a\tau)^n}{n!}\sum_{r=0}^{\infty}{}_LH^{F_n}(a\eta,a\xi,a^2\mu;\zeta)\frac{(b\tau)^r}{r!}$$
$$A(\tau)=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}b^r a^{n-r}{}_LH^{F_{n-r}}(b\eta,b\xi,b^2\mu;\zeta){}_LH^{F_n}(a\eta,a\xi,a^2\mu;\zeta)\right)\frac{\tau^n}{n!}. \quad (48)$$

In a similar manner, we can demonstrate that

$$A(\tau)=\sum_{n=0}^{\infty}{}_LH^{F_n}(a\eta,a\xi,a^2\mu;\zeta)\frac{(b\tau)^n}{n!}\sum_{r=0}^{\infty}{}_LH^{F_n}(b\eta,b\xi,b^2\mu;\zeta)\frac{(a\tau)^r}{r!}$$
$$A(\tau)=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}{}_LH^{F_{n-r}}(a\eta,a\xi,a^2\mu;\zeta){}_LH^{F_n}(b\eta,b\xi,b^2\mu;\zeta)\right)\frac{\tau^n}{n!}. \quad (49)$$

Therefore, by comparing the coefficients of $\frac{\tau^n}{n!}$ on the right hand sides of the final two equations, we obtain the required conclusion (47).

Theorem 4.2. For any pair of integers a and b and for all integers and $n \geq 0$, the following identity axiom is valid:

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}a^{n-k}b^k{}_LH^{F_{n-k}}\left(b\xi+\frac{b}{a}i+j,b^2\mu,b\eta;\zeta\right)F_k(au,\zeta)$$
$$\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{a-1}\sum_{i=0}^{b-1}b^{n-k}a^k{}_LH^{F_{n-k}}\left(a\xi+\frac{a}{b}i+j,a^2\mu,a\eta;\zeta\right)F_k(bu,\zeta). \quad (50)$$

Proof. Let us write the expression of $B(\tau)$

$$B(\tau)=\frac{e^{ab(\xi+u)\tau+a^2b^2\mu\tau^2}(e^{ab\tau}-1)^2C_0(ab\eta\tau)}{(1-\zeta(e^{a\tau}-1))(1-\zeta(e^{b\tau}-1))(e^{a\tau}-1)(e^{b\tau}-1)}$$
$$=\frac{e^{ab\xi\tau+a^2b^2\mu\tau^2}C_0(\eta\tau)}{1-\zeta(e^{a\tau}-1)}\frac{e^{ab\tau}-1}{e^{b\tau}-1}\frac{e^{abu\tau}}{1-\zeta(e^{b\tau}-1)}\frac{e^{ab\tau}-1}{e^{a\tau}-1}$$
$$B(\tau)=\frac{e^{ab\xi\tau+a^2b^2\mu\tau^2}C_0(ab\eta\tau)}{1-\zeta(e^{a\tau}-1)}\sum_{i=0}^{a-1}e^{b\tau i}\frac{e^{abu\tau}}{1-\zeta(e^{b\tau}-1)}\sum_{j=0}^{b-1}e^{a\tau j}$$
$$=\frac{e^{a^2b^2\mu\tau^2}C_0(ab\eta\tau)}{1-\zeta(e^{a\tau}-1)}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}e^{\left(b\xi+\frac{b}{a}i+j\right)a\tau}\sum_{k=0}^{\infty}F_k(au,\zeta)\frac{(b\tau)^k}{k!}$$
$$B(\tau)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}a^{n-k}b^k{}_LH^{F_{n-k}}\left(b\xi+\frac{b}{a}i+j,b^2\mu,b\eta;\zeta\right)F_k(au,\zeta)\right)\frac{\tau^n}{n!}. \quad (51)$$

However , we also have

$$B(\tau)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{a-1}\sum_{i=0}^{b-1}b^{n-k}a^k{}_LH^{F_n}\left(a\xi+\frac{a}{b}i+j,a^2\mu,a\eta;\zeta\right)F_k(bu,\zeta)\right)\frac{\tau^n}{n!}. \quad (52)$$

thus, by comparing the coefficients of τ^n from the right hand sides of the last two equations (51) and (52), we obtain the desired result.

5. Concluding Remarks

In this study, we convey a special class of Laguerre-Hermite-Fubini polynomials, investigated their diverse characteristics, and examine their potential applications. To derive summing axioms for these polynomials, we employed series summation techniques. Through the utilization of generating functions, we have effectively established symmetric identity axioms for the Laguerre-Hermite-Fubini polynomials.

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