A NEW CLASS OF DEGENERATE HERMITE-FUBINI POLYNOMIALS AND THEIR APPLICATIONS

[1]MD Jawed Miandad , [2]Naresh Menaria

[1,2]Department of Mathematics, Faculty of Science, Pacific Academy of Higher Education and Research University, Udaipur, India.

Abstract.
In this study, a new kind of degenerate Hermite-Fubini numbers and polynomials is presented, and the characteristics of these numbers and polynomials are investigated. The series summation methods are utilised in the process of formulating summation formulae for these polynomials. By using generating functions, we are able to obtain symmetric identities for the degenerate Hermite-Fubini numbers and polynomials. This is accomplished by the use of generating functions.

Keywords: *Hermite polynomials, degenerate Hermite polynomials, degenerate Fubini polynomials, degenerate Hermite-Fubini polynomials.*

2010 Mathematics Subject Classification 2020.: 11B68, 11B83, 33C45, 33C99.

1. Introduction

We can investigate both Hermite numbers and Fubini numbers using the generating function. The Hermite polynomials $H_N(\alpha)$ are utilized in a wide range of applications, such as quantum harmonic osculation, probability theory, gaussian quadrature in mathematical physics, partial differential equations (PDEs), probability theory, numerical analysis, quantum physics, approximation theory, signal and image processing, and more. Fubini polynomials $F_N(\alpha)$ frequently count permutations and partitions in the field of combinatorial mathematics.

The two-variable Hermite Kampe' de Fe'riet polynomials (2VHKdFP) $H_N(\alpha, \beta)$ [1, 4] can be defined as follows:

$$H_N(\alpha, \beta) = N! \sum_{K=0}^{\left[\frac{N}{2}\right]} \frac{\beta^r \alpha^{N-2K}}{r!\,(N-2K)!}. \tag{1.1}$$

It is clear that

$$H_N(2\alpha, -1) = H_N(\alpha, H_N(\alpha, -\frac{1}{2}) = He_N(\alpha), H_N(\alpha, 0) = \alpha^N,$$

$H_N(\alpha)$ and $He_N(\alpha)$ represent the ordinary Hermite polynomials.
The Hermite polynomial $H_N(\alpha, \beta)$ (see ([12, 13]) is defined with the following generating function:

$$e^{\alpha\rho+\beta\rho^2} = \sum_{n=0}^{\infty} H_N(\alpha, \beta) \frac{\rho^N}{N!}. \tag{1.2}$$

Khan [7] recently talked about degenerate Hermite polynomials using the following generating function:

$$(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{y\beta}{\theta}} = \sum_{n=0}^{\infty} H_n(\alpha,\beta;\theta)\frac{\rho^N}{N!}. \tag{1.3}$$

Note that

$$\lim_{\theta\to 0}(1+\theta\rho)^{\frac{\alpha}{\theta}} = e^{\alpha\rho}.$$

It is clear that equation (1.3) simplifies to equation (1.2).
It is evident that (1.3) reduces to (1.2). The $H_N(\alpha,\beta)$ limiting case of $H_N(\alpha,\beta;\theta)$,occurs as the limit of θ approaches 0.
The closed form of degenerate Hermite polynomials $H_N(\alpha,\beta;\theta)$ is below:

$$H_N(\alpha,\beta;\theta) = N!\sum_{r=0}^{\left[\frac{N}{2}\right]} \frac{\theta^{N-r}(\frac{\alpha}{\theta})_{N-2r}(\frac{y}{\theta})_r}{r!\,(N-2r)!}. \tag{1.4}$$

Carlitz introduced the degenerate Bernoulli polynomials for θ∈C, which are defined by the generating function.

$$\frac{\rho}{(1+\theta\rho)^{\frac{1}{\theta}}-1}(1+\theta\rho)^{\frac{\alpha}{\theta}} = \sum_{n=0}^{\infty}\beta_N(\alpha;\theta)\frac{\rho^N}{N!}, (\text{see}[3,8,9,10,11]) \tag{1.5}$$

so that

$$\beta_N(\alpha;\theta) = \sum_{N=0}^{m}\binom{N}{m}\beta_m(\theta)(\frac{\alpha}{\theta})_{N-m}. \tag{1.6}$$

The degenerate Bernoulli numbers are referred to as $\beta_N(\theta) = \beta_N(0;\theta)$ when α is equal to 0.
Based on equation (1.5), we have noticed that

$$\sum_{n=0}^{\infty}\lim_{\theta\to 0}\beta_N(\alpha;\theta)\frac{\rho^N}{N!} = \lim_{\theta\to 0}\frac{\rho}{(1+\theta\rho)^{\frac{1}{\theta}}-1}(1+\theta\rho)^{\frac{\alpha}{\theta}}$$

$$= \frac{\rho}{e^{\rho}-1}e^{\alpha\rho} = \sum_{n=0}^{\infty}B_N(\alpha)\frac{\rho^n}{n!} \tag{1.7}$$

In which $B_N(\alpha)$ are referred as the Bernoulli polynomials (see [1-15]).
Geometric polynomials (also known as Fubini polynomials) are expressed as follows (see [2]):

$$F_N(\alpha) = \sum_{k=0}^{N}\left\{{N \atop k}\right\}k!\,\alpha^k, \tag{1.8}$$

Where $\left\{{N \atop k}\right\}$ is the Stirling number of the second kind (see [5]).
For $\alpha = 1$ in (1.8), we get N^{th} Fubini number (ordered Bell number or geometric number) F_N [2, 5, 6, 15] is defined by

$$F_N(1) = F_N = \sum_{k=0}^{N}\left\{{N \atop k}\right\}k!. \tag{1.9}$$

The exponential generating functions of geometric polynomials can be written as (see [2]):

$$\frac{1}{1-\alpha(e^{\rho}-1)} = \sum_{n=0}^{\infty}F_N(\alpha)\frac{\rho^N}{N!}, \tag{1.10}$$

and related to the geometric series (see [2]):

$$\left(\alpha\frac{d}{d\alpha}\right)^m\frac{1}{1-\alpha} = \sum_{k=0}^{\infty}k^m\alpha^k = \frac{1}{1-\alpha}F_m(\frac{\alpha}{1-\alpha}), |\alpha| < 1.$$

Here's a brief list of these polynomials and values:

$$F_0(\alpha)=1, F_1(\alpha)=\alpha, F_2(\alpha)=\alpha+2\alpha^2, F_3(\alpha)=\alpha+6\alpha^2+6\alpha^3, F_4(\alpha)=\alpha+14\alpha^2+36\alpha^3+24\alpha^4,$$

and

$$F_0=1, F_1=1, F_2=3, F_3=13, F_4=75.$$

Geometric and exponential polynomials are related by the relation (see [2]):

$$F_N(\alpha)=\int_0^\infty \phi_N(\alpha)e^{-\theta}d\theta. \tag{1.11}$$

In (2016), Khan [7] introduced two variable degenerate Hermite-poly-Bernoulli polynomials is defined by means of the following generating function:

$$\frac{Li_k(1-e^{-\rho})}{(1+\theta\rho)^{\frac{1}{\theta}}-1}(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}=\sum_{n=0}^{\infty}{}_H\beta_N^{(k)}(\alpha,\beta;\theta)\frac{\rho^N}{N!}, \tag{1.12}$$

so that

$${}_H\beta_N^{(k)}(\alpha,\beta;\theta)=\sum_{m=0}^{N}\binom{N}{m}\beta_{N-m}^{(k)}(\theta)H_m(\alpha,\beta;\theta).$$

The focus of this paper, we consider generating functions for degenerate Hermite-Fubini numbers and polynomials and give some properties of these numbers and polynomials. We derive for degenerate Hermite-Fubini numbers and polynomials, and we develop symmetric identities for these numbers and polynomials using generating functions.

2. Degenerate Hermite-Fubini numbers and polynomials

In this section, we introduce three-variable degenerate Hermite-Fubini polynomials and derive fundamental properties, leading to a new formula for ${}_HF_{N,\theta}(\alpha,\beta;\gamma)$ as follows:

$$\frac{1}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}=\sum_{n=0}^{\infty}{}_HF_{N,\theta}(\alpha,\beta;\gamma)\frac{N}{N!}. \tag{2.1}$$

Under the conditions $\alpha=\beta=0.\ \gamma=1$ in (2.1), we have

$${}_HF_{N,\theta}(0,0;\gamma)=F_{N,\theta}(\gamma), {}_HF_{N,\theta}(0,0;1)=F_{N,\theta}.$$

Not that $\lim_{\theta\to 0}{}_HF_{N,\theta}(\alpha,\beta;\gamma)={}_HF_N(\alpha,\beta;\delta)$.

We are able to obtain two-variable Fubini polynomials, which are described by Kim et al. [9], by stating that β is equal to zero in equation (2.1).

$$\frac{1}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}(1+\theta\rho)^{\frac{\alpha}{\theta}}=\sum_{N=0}^{\infty}F_{N,\theta}(\alpha;\gamma)\frac{\rho^N}{N!}. \tag{2.2}$$

Theorem 2.1. For $N\geq 0$, we have

$${}_HF_{N,\theta}(\alpha,\beta;\gamma)=\sum_{m=0}^{N}\binom{N}{r}F_{N-m,\theta}(z)H_m(\alpha,\beta;\theta). \tag{2.3}$$

Proof. Utilizing definition (2.1), we have

$$\sum_{n=0}^{\infty}{}_HF_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!}=\frac{1}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}$$

$$= \sum_{N=0}^{\infty} F_{N,\theta}(\gamma)\frac{\rho^N}{N!}\sum_{m=0}^{\infty} H_m(\alpha,\beta;\theta)\frac{\rho^m}{m!}$$

$$= \sum_{N=0}^{\infty}\left(\sum_{m=0}^{n}\binom{N}{r}F_{N-m,\theta}(z)H_m(\alpha,\beta;\theta)\right)\frac{\rho^N}{N!}.$$

Equating the coefficients of $\frac{\rho^N}{N!}$ yields (2.3).

Theorem 2.2. For $N \geq 0$, we have

$$_{H}F_{N,\theta}(\alpha,\beta;\gamma) = \sum_{r=0}^{N}\binom{N}{r}H_{N-r}(\alpha,\beta;\theta)\sum_{k=0}^{r}\gamma^k k!\,S_2(r,k). \quad (2.4)$$

Proof. From definition (2.1), we have

$$\sum_{N=0}^{\infty} {}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!} = \frac{1}{1-\gamma((1+\theta t)^{\frac{1}{\theta}}-1)}(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}$$

$$= (1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}\sum_{k=0}^{\infty}\gamma^k((1+\theta v)^{\frac{1}{\theta}}-1)^k$$

$$= (1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}\sum_{k=0}^{\infty}\gamma^k\sum_{N=k}^{\infty}S_2(r,k)\frac{\rho^r}{r!}$$

$$= \sum_{N=0}^{\infty}H_N(\alpha,\beta;\theta)\frac{\rho^N}{N!}\left(\sum_{r=0}^{\infty}\sum_{k=0}^{r}\gamma^k k!\,S_2(r,k)\frac{\rho^r}{r!}\right)$$

$$\text{L.H.S} = \sum_{N=0}^{\infty}\left(\sum_{r=0}^{N}\binom{N}{r}H_{N-r}(\alpha,\beta;\theta)\sum_{k=0}^{r}\gamma^k k!\,S_2(r,k)\right)\frac{\rho^N}{N!}.$$

By matching the coefficients of $\frac{\rho^N}{N!}$ on both sides, we obtain (2.4).

Theorem 2.3. For $N \geq 0$, the following formula for degenerate Hermite-Fubini polynomials valid:

$$\frac{1}{1-\gamma}\sum_{m=0}^{N}\binom{N}{m}F_{m,\theta}\left(\frac{\gamma}{1-\gamma}\right)H_{N-m,\theta}(\alpha,\beta) = \sum_{r=0}^{N}\binom{N}{r}\sum_{k=0}^{\infty}\gamma^k(k)_{r,\theta}H_{N-r,\theta}(\alpha,\beta). \quad (2.5)$$

Proof. We start with the definition (2.1)

$$\sum_{N=0}^{\infty} {}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!} = \frac{1}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}$$

Let

$$\frac{1}{1-\gamma}\left(\frac{1}{1-\frac{\gamma}{1-\gamma}(1+\theta\rho)^{\frac{1}{\theta}}-1}\right)=\frac{1}{1-\gamma(1+\theta\rho)^{\frac{1}{\theta}}}=\sum_{k=0}^{\infty}\gamma^{k}(1+\theta\rho)^{\frac{k}{\theta}} \tag{2.6}$$

$$=\sum_{r=0}^{\infty}\left(\sum_{k=0}^{\infty}\gamma^{k}(k)_{r,\theta}\right)\frac{\rho^{r}}{r!}$$

$$\sum_{N=0}^{\infty}{}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^{N}}{N!}=\sum_{r=0}^{\infty}\left(\sum_{k=0}^{\infty}\gamma^{k}(k)_{r,\theta}\right)\frac{\rho^{r}}{r!}\left(\sum_{n=0}^{\infty}H_{N,\theta}(\alpha,\beta)\frac{\rho^{N}}{N!}\right)$$

$$=\sum_{N=0}^{\infty}\left(\sum_{r=0}^{n}\binom{N}{r}\sum_{k=0}^{\infty}\gamma^{k}(k)_{r,\theta}H_{N-r,\theta}(\alpha,\beta)\right)\frac{\rho^{N}}{N!}. \tag{2.7}$$

Now, we recognize that, according to (2.6), we obtain

$$\frac{1}{1-\gamma}\left(\frac{1}{1-\frac{\gamma}{1-\gamma}(1+\theta\rho)^{\frac{1}{\theta}}-1}\right)=\frac{1}{1-\gamma}\sum_{n=0}^{\infty}F_{N,\theta}\left(\frac{\gamma}{1-\gamma}\right)\frac{\rho^{N}}{N!}$$

Hence, we achived

$$\sum_{N=0}^{\infty}{}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^{N}}{N!}=\frac{1}{1-\gamma}\sum_{m=0}^{\infty}F_{m,\theta}\left(\frac{\gamma}{1-\gamma}\right)\frac{\rho^{m}}{m!}\left(\sum_{n=0}^{\infty}H_{N,\theta}(\alpha,\beta)\frac{\rho^{N}}{N!}\right)$$

$$=\frac{1}{1-\gamma}\sum_{N=0}^{\infty}\left(\sum_{m=0}^{N}\binom{N}{m}F_{m,\theta}\left(\frac{\gamma}{1-\gamma}\right)H_{N-m,\theta}(\alpha,\beta)\right)\frac{\rho^{N}}{N!}. \tag{2.8}$$

Comparing the coefficients of $\frac{\rho^{N}}{N!}$ in equation (2.7) and (2.8), we get (2.5).

Theorem 2.4. For $N\geq 0$, the following formula for degenerate Hermite-Fubini polynomials holds true:

$$H_{N,\theta}(\alpha,\beta)={}_{H}F_{N,\theta}(\alpha,\beta;\gamma)-z_{H}F_{N,\theta}(\alpha+1,\beta;\gamma)+z_{H}F_{N,\theta}(\alpha,\beta;\gamma). \tag{2.9}$$

Proof. We begin with the definition (2.1) and write

$$(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^{2})^{\frac{\beta}{\theta}}=\frac{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^{2})^{\frac{\beta}{\theta}}$$

$$=\frac{(1+\theta\rho(1+\theta\rho^{2})^{\frac{\beta}{\theta}}}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}-\frac{\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^{2})^{\frac{\beta}{\theta}}.$$

Then using the definition of Kamp e′ de F e′riet generalization of the degenerate Hermite polynomials $H_{N,\theta}(\alpha,\beta)$ (1.3) and (2.1), we have

$$\sum_{N=0}^{\infty}H_{N,\theta}(\alpha,\beta)\frac{\rho^{N}}{N!}=\sum_{N=0}^{\infty}\left[{}_{H}F_{N,\theta}(\alpha,\beta;\gamma)-z_{H}F_{N,\theta}(\alpha+1,\beta;\gamma)+z_{H}F_{N,\theta}(\alpha,\beta;\gamma)\right]\frac{\rho^{N}}{N!}.$$

Finally, comparing the coefficients of $\frac{\rho^N}{N!}$, we get (2.9).

Theorem 2.5. For $N \geq 0$ and $\gamma_1 \neq \gamma_2$, the following formula for degenerate Hermite-Fubini polynomials holds true:

$$\sum_{k=0}^{N} \binom{N}{k} {}_HF_{N-k,\theta}(\alpha_1, \beta_1; \gamma_1) {}_HF_{k,\theta}(\alpha_2, \beta_2; \gamma_2)$$

$$= \frac{\gamma_2 {}_HF_{N,\theta}(\alpha_1+\alpha_2, \beta_1+\beta_2; \gamma_2) - z_1 {}_HF_{N,\theta}(\alpha_1+\alpha_2, \beta_1+\beta_2; \gamma_1)}{\gamma_2 - \gamma_1}. \tag{2.10}$$

Proof. The products of (2.1) can be written as

$$\sum_{N=0}^{\infty}\sum_{k=0}^{\infty} {}_HF_N(\alpha_1, \beta_1; \gamma_1)\frac{t^N}{N!} {}_HF_k(\alpha_2, \beta_2; \gamma_2)\frac{\rho^k}{k!} = \frac{(1+\theta\rho)^{\frac{\alpha_1}{\theta}}(1+\theta\rho^2)^{\frac{\beta_1}{\theta}}}{1-\gamma_1((1+\theta\rho)^{\frac{1}{\theta}}-1)} \frac{(1+\theta\rho)^{\frac{\alpha_2}{\theta}}(1+\theta\rho^2)^{\frac{\beta_2}{\theta}}}{1-\gamma_2((1+\theta\rho)^{\frac{1}{\theta}}-1)}$$

$$\sum_{N=0}^{\infty}\left(\sum_{k=0}^{N} \binom{N}{k} {}_HF_{N-k}(\alpha_1, \beta_1; \gamma_1) {}_HF_k(\alpha_2, \beta_2; \gamma_2)\right)\frac{\rho^N}{N!}$$

$$= \frac{\gamma_2}{\gamma_2-\gamma_1}\frac{(1+\theta\rho)^{\frac{\alpha_1+\alpha_2}{\theta}}(1+\theta\rho^2)^{\frac{\beta_1+\beta_2}{\theta}}}{1-\gamma_1((1+\theta\rho)^{\frac{1}{\theta}}-1)} - \frac{\gamma_1}{\gamma_2-\gamma_1}\frac{(1+\theta\rho)^{\frac{\alpha_1+\alpha_2}{\theta}}(1+\theta\rho^2)^{\frac{\beta_1+\beta_2}{\theta}}}{1-\gamma_2((1+\theta\rho)^{\frac{1}{\theta}}-1)}$$

$$= \left(\frac{\gamma_2 {}_HF_N(\alpha_1+\alpha_2, \beta_1+\beta_2; \gamma_2) - \gamma_1 {}_HF_N(\alpha_1+\alpha_2, \beta_1+\beta_2; \gamma_1)}{\gamma_2-\gamma_1}\right)\frac{\rho^N}{N!}.$$

By equating the coefficients of $\frac{\rho^N}{N!}$ on both sides, we get (2.10).

Theorem 2.6. For $N \geq 0$, the following formula for degenerate Hermite-Fubini polynomials holds true:

$$z_H F_{N,\theta}(\alpha+1, \beta; \gamma) = (1+z)_H F_{N,\theta}(\alpha, \beta; \gamma) - H_{N,\theta}(\alpha, \beta). \tag{2.11}$$

Proof. From (2.1), we have

$$\sum_{N=0}^{\infty}\left[{}_HF_{N,\theta}(\alpha+1, \beta; \gamma) - {}_HF_{N,\theta}(\alpha, \beta; \gamma)\right]\frac{\rho^N}{N!} = \frac{(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}((1+\theta\rho)^{\frac{1}{\theta}}-1)$$

$$= \frac{1}{\gamma}\left[\frac{(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)} - (1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}\right]$$

$$= \frac{1}{\gamma}\sum_{n=0}^{\infty}\left[{}_HF_N(\alpha, \beta; \gamma) - H_N(\alpha, \beta)\right]\frac{\rho^N}{N!}.$$

Comparing the coefficients of $\frac{\rho^N}{N!}$ on both sides, we obtain (2.11).

Remark 2.3. On setting $\alpha = \beta = 0$ and $\alpha = -1$ in Theorem 2.6, we find

$$z\,{}_HF_{N,\theta}(1,0;\gamma) = (1+z)\,{}_HF_{N,\theta}(0,0;\gamma), \tag{2.12}$$

and

$$z\,{}_HF_{N\theta}(0,0;\gamma) = (1+z)\,{}_HF_{N,\theta}(-1,0;\gamma) - (-\theta)^N\left(\frac{1}{\theta}\right)_N. \tag{2.13}$$

Theorem 2.7. For $N \geq 0,\ p,q \in \mathbb{R}$, the following formula for degenerate Hermite-Fubini polynomials holds true:

$$ {}_HF_{N,\theta}(p\alpha, q\beta;\gamma)$$

$$= N!\sum_{k=0}^{N}\sum_{j=0}^{\left[\frac{k}{2}\right]} \theta^{k-j}\,{}_HF_{N-k,\theta}(\alpha,\beta;\gamma)\left(\frac{(p-1)\alpha}{\theta}\right)_{k-2j}\left(\frac{(q-1)\beta}{\theta}\right)_j \frac{1}{(N-k-2j)!\,j!}. \tag{2.14}$$

Proof. By formulating the generating function (2.1), we have

$$\sum_{N=0}^{\infty} {}_HF_N(p\alpha, q\beta;\gamma)\frac{\rho^N}{N!} = \frac{1}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}(1+\theta\rho)^{\frac{(p-1)\alpha}{\theta}}(1+\theta\rho^2)^{\frac{(q-1)\beta}{\theta}}$$

$$= \left(\sum_{n=0}^{\infty} {}_HF_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!}\right)\left(\sum_{k=0}^{\infty}\left(\frac{(p-1)\alpha}{\theta}\right)_k \theta^k\frac{\rho^k}{k!}\right)\left(\sum_{j=0}^{\infty}\left(\frac{(q-1)\beta}{\theta}\right)_j \theta^j\frac{\rho^{2j}}{j!}\right)$$

$$= \left(\sum_{N=0}^{\infty} {}_HF_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!}\right)\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{(p-1)\alpha}{\theta}\right)_k\left(\frac{(q-1)\beta}{\theta}\right)_j \theta^{k+j}\frac{t^{k+2j}}{k!\,j!}\right)$$

Substituting k by $k-2j$ in the above equation, we obtained

$$\text{L.H.S.} = \left(\sum_{N=0}^{\infty} {}_HF_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!}\right)\left(\sum_{k=2j}^{\infty}\theta^{k-j}\left(\frac{(p-1)\alpha}{\theta}\right)_{k-2j}\left(\frac{(q-1)\beta}{\theta}\right)_j\frac{\rho^k}{(k-2j)!\,j!}\right).$$

Again by substitutin N by $N-k$ in above equation, we obtain

$$\text{L.H.S.} = \sum_{N=0}^{\infty}\sum_{k=0}^{N}\sum_{j=0}^{[\frac{k}{2}]}\theta^{k-j}\,{}_HF_{N-k,\theta}(\alpha,\beta;\gamma)\left(\frac{(p-1)\alpha}{\theta}\right)_{k-2j}\left(\frac{(q-1)\beta}{\theta}\right)_j\frac{\rho^n}{(n-k-2j)!\,j!\,k!}.$$

Finally, by equating the coefficients of ρ^N on both sides, we obtain the result (2.14).

Theorem 2.8. For $N \geq 0$, the following formula for degenerate Hermite-Fubini polynomials is established:

$$ {}_HF_{N,\theta}(\alpha+r,\beta;\gamma) = \sum_{l=0}^{n}\binom{N}{l}H_{N-l,\theta}(\alpha,\beta)\sum_{k=0}^{l}\gamma^k k!\,S_{2,\theta}(l+r,k+r). \tag{2.15}$$

Proof. Changing α by $\alpha+r$ in (2.1), we get

$$\sum_{n=0}^{\infty} {}_HF_{N,\theta}(\alpha+r,\beta;\gamma)\frac{t^N}{N!} = \frac{(1+\theta\rho)^{\frac{\alpha+r}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}$$

$$= (1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}(1+\theta\rho)^{\frac{r}{\theta}}\sum_{k=0}^{\infty}\gamma^k((1+\theta\rho)^{\frac{1}{\theta}}-1)^k$$

$$= (1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}(1+\theta\rho)^{\frac{r}{\theta}}\sum_{k=0}^{\infty}\gamma^k\sum_{l=k}^{\infty}k!\,S_2(l,k)\frac{\rho^l}{l!}$$

$$= \sum_{N=0}^{\infty}H_{N,\theta}(\alpha,\beta)\frac{\rho^N}{N!}\sum_{l=0}^{\infty}\gamma^k\sum_{k=0}^{l}k!\,S_{2,\theta}(l+r,k+r)\frac{\rho^l}{l!}.$$

Again changing N by $N-l$ in above equation, we have

$$\sum_{n=0}^{\infty}{}_HF_{N,\theta}(\alpha+r,\beta;\gamma)\frac{\rho^N}{N!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{N}\binom{N}{l}H_{N-l,\theta}(\alpha,\beta)\sum_{k=0}^{l}z^k k!\,S_{2,\theta}(l+r,k+r)\right)\frac{\rho^N}{N!}.$$

Analyzing the coefficients of $\frac{\rho^N}{N!}$ in both sides, we get (2.15).

3. Summation Formulae for degenerate Hermite-Fubini polynomials

First, we establish the following result concerning the degenerate Hermite-Fubini polynomials ${}_HF_{n,\theta}(\alpha,\beta;\gamma)$ by using series rearrangement techniques and considered its specific case:

Theorem 3.1. The following summation formula for degenerate Hermite-Fubini polynomials ${}_HF_n(\alpha,\beta;\gamma)$ is valid:

$${}_HF_{N,\theta}(u,v;z)\,{}_HF_{m,\theta}(U,V;Z) = \sum_{r,k=0}^{N,m}\binom{N}{r}\binom{m}{k}H_{r,\theta}(u-\alpha,v-\beta)\,{}_HF_{N-r,\theta}(\alpha,\beta;\gamma)$$

$$\times H_{k,\theta}(U-X,V-Y)\,{}_HF_{m-k,\theta}(X,Y;Z). \quad (3.1)$$

Proof. Look into the product of the degenerate Hermite-Fubini polynomials, we can express it as the generating function (2.1) in this form:

$$\frac{1}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}\frac{1}{1-\gamma((1+\theta T)^{\frac{1}{\theta}}-1)}(1+\theta T)^{\frac{\alpha}{\theta}}(1+\theta T^2)^{\frac{\beta}{\theta}}$$

$$= \sum_{N=0}^{\infty}{}_HF_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!}\sum_{m=0}^{\infty}{}_HF_{m,\theta}(X,Y;Z)\frac{T^m}{m!}. \quad (3.2)$$

Substituting α by u, y by v, X by U and Y by V in (3.2) and equating the result to itself,

$$\sum_{N=0}^{\infty}\sum_{m=0}^{\infty}{}_HF_{N,\theta}(u,v;z)\,{}_HF_{m,\theta}(U,V;Z)\frac{\rho^N}{N!}\frac{T^m}{m!}$$

$$= (1+\theta\rho)^{\frac{u-\alpha}{\theta}}(1+\theta\rho^2)^{\frac{v-\beta}{\theta}}(1+\theta T)^{\frac{U-X}{\theta}}(1+\theta T^2)^{\frac{V-Y}{\theta}}$$

$$\times\sum_{N=0}^{\infty}\sum_{m=0}^{\infty}{}_HF_{N,\theta}(\alpha,\beta;\gamma)\,{}_HF_{m,\theta}(X,Y;Z)\frac{\rho^N}{N!}\frac{T^m}{m!},$$

which, by applying the generating function [14] to the right-hand side, becomes

$$\sum_{N=0}^{\infty}\sum_{m=0}^{\infty}{}_HF_{N,\theta}(u,v;z)\,{}_HF_{m,\theta}(U,V;Z)\frac{\rho^N}{N!}\frac{T^m}{m!}$$

$$= \sum_{N,r=0}^{\infty} H_{r,\theta}(u-\alpha, v-\beta)\,{}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^{N+r}}{N!\,r!}\sum_{m,k=0}^{\infty} H_{k,\theta}(U-X, V-Y)\,{}_{H}F_{m,\theta}(X,Y;Z)\frac{T^{m+k}}{m!\,k!}. \tag{3.3}$$

Finally, substituting N by $N-r$ and m by $m-k$ and applying the lemma [14] in the right-hand side of the above equation and then equating the coefficients of corresponding powers of ρ and T, we obtained assertion (3.1) of Theorem 3.1.

Theorem 3.2. The following summation formula for degenerate Hermite-Fubini polynomials ${}_{H}F_{n}(\alpha,\beta;\gamma)$ holds true:

$${}_{H}F_{N,\theta}(\alpha+w,\beta+u;\gamma) = \sum_{s=0}^{n}\binom{n}{s}\,{}_{H}F_{N-s,\theta}(\alpha,\beta;\gamma)H_{s,\theta}(w,u). \tag{3.4}$$

Proof. We substitute α by $\alpha+w$ and β by $\beta+u$ in (2.1), use (1.3) and reconstruct the generating function as:

$$\frac{1}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}(1+\theta\rho)^{\frac{\alpha+w}{\theta}}(1+\theta\rho^2)^{\frac{\beta+u}{\theta}} = \sum_{N=0}^{\infty}\,{}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!}\sum_{s=0}^{\infty}H_{s,\theta}(w,u)\frac{\rho^s}{s!}$$

$$= \sum_{N=0}^{\infty}\,{}_{H}F_{N,\theta}(\alpha+w,\beta+u;\gamma)\frac{\rho^N}{N!}.$$

Now again replace N by $N-s$ in l.h.s. and equating the coefficients of ρ^N on both sides, we obtained the result (3.4).

Theorem 3.3. The following summation formula for degenerate Hermite-Fubini polynomials ${}_{H}F_{N,\theta}(\alpha,\beta;\gamma)$ true:

$${}_{H}F_{N,\theta}(\alpha,\beta;\gamma) = \sum_{r=0}^{n}\binom{N}{r}F_{N-r,\theta}(\alpha-w;\gamma)H_{r,\theta}(w,\gamma). \tag{3.5}$$

Proof. By utilizing the generating function (1.3), we can write equation (2.1) as

$$\frac{1}{1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1)}(1+\theta\rho)^{\frac{\alpha-w}{\theta}}(1+\theta\rho)^{\frac{w}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}} = \sum_{n=0}^{\infty}F_{N,\theta}(\alpha-w;\gamma)\frac{\rho^N}{N!}\sum_{r=0}^{\infty}H_{r,\theta}(w,\beta)\frac{\rho^r}{r!}.$$

On interchange N by $N-r$ in above equation, we obtained

$$\sum_{N=0}^{\infty}\,{}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!} = \sum_{N=0}^{\infty}\sum_{r=0}^{N}F_{N-r,\theta}(\alpha-w;\gamma)H_{r,\theta}(w,\beta)\frac{\rho^N}{(N-r)!\,r!}.$$

comparing the coefficients of the like powers of ρ on both sides, we get (3.5).

Theorem 3.4. The below summation expression for degenerate Hermite-Fubini polynomials ${}_{H}F_{N,\theta}(\alpha,\beta;\gamma)$ remain true:

$${}_{H}F_{N,\theta}(\alpha+1,\beta;\gamma) = \sum_{r=0}^{N}\binom{N}{r}\,{}_{H}F_{N-r,\theta}(\alpha,\beta;\gamma)\left(\frac{1}{\theta}\right)_r\theta^r. \tag{3.6}$$

Proof. From generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_{H}F_{N,\theta}(\alpha+1,\beta;\gamma)\frac{\rho^N}{N!} - \sum_{n=0}^{\infty} {}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!}$$

$$= \left(\frac{1}{1-1-\gamma((1+\theta\rho)^{\frac{1}{\theta}}-1}\right)((1+\theta\rho)^{\frac{1}{\theta}}-1)(1+\theta\rho)^{\frac{\alpha}{\theta}}(1+\theta\rho^2)^{\frac{\beta}{\theta}}$$

$$= \sum_{N=0}^{\infty} {}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!}\left(\sum_{r=0}^{\infty}\frac{\left(\frac{1}{\theta}\right)_r \theta^r\rho^r}{r!} - 1\right)$$

$$= \sum_{N=0}^{\infty} {}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{\theta}\right)_r \theta^r\rho^r}{r!} - \sum_{N=0}^{\infty} {}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!}$$

$$= \sum_{N=0}^{\infty}\sum_{r=0}^{N}\binom{N}{r} {}_{H}F_{N-r,\theta}(\alpha,\beta;\gamma)\left(\frac{1}{\theta}\right)_r \theta^r\frac{\rho^N}{N!} - \sum_{N=0}^{\infty} {}_{H}F_{N,\theta}(\alpha,\beta;\gamma)\frac{\rho^N}{N!}.$$

Finally,by equating the coefficients of the corresponding powers of ρ on both sides, we get (3.6).

4. Symmetric identities for degenerate Hermite-Fubini polynomials

In this part, we prove broad identities of symmetry for the degenerate Hermite-Fubini polynomials ${}_{H}F_{N,\theta}(\alpha,\beta;\gamma)$ by the use of the generating function (2.1) and (2.2).

Theorem 4.1. Let $\alpha,\beta,\gamma \in \mathbb{R}$ and $N \geq 0$, then the subsequent identify is accurate:

$$\sum_{r=0}^{N}\binom{N}{r} b^r a^{n-r} {}_{H}F_{N-r,\theta}(b\alpha,b^2\beta;\gamma){}_{H}F_{r,\theta}(a\alpha,a^2\beta;\gamma)$$

$$= \sum_{r=0}^{N}\binom{N}{r} a^r b^{N-r} {}_{H}F_{N-r,\theta}(a\alpha,a^2\beta;\gamma){}_{H}F_{r,\theta}(b\alpha,b^2\beta;\gamma). \quad (4.1)$$

Proof. Get start with

$$A(\rho) = \frac{(1+\theta\rho)^{\frac{ab\alpha}{\theta}}(1+\theta\rho^2)^{\frac{a^2b^2\beta}{\theta}}}{(1-\gamma((1+\theta\rho)^{\frac{a}{\theta}}-1))((1-\gamma((1+\theta\rho)^{\frac{b}{\theta}}-1)}.$$

Then the expression for $A(\rho)$ is symmetric in a and b and we can expand $A(t)$ into series in two ways to obtain:

$$A(\rho) = \sum_{N=0}^{\infty} {}_{H}F_{N,\theta}(b\alpha,b^2\beta;\gamma)\frac{(a\rho)^N}{N!}\sum_{r=0}^{\infty} {}_{H}F_{r,\theta}(a\alpha,a^2\beta;\gamma)\frac{(b\rho)^r}{r!}$$

$$A(\rho) = \sum_{N=0}^{\infty}\left(\sum_{r=0}^{N}\binom{N}{r} b^r a^{N-r} {}_{H}F_{N-r,\theta}(b\alpha,b^2\beta;\gamma){}_{H}F_{r,\theta}(a\alpha,a^2\beta;\gamma)\right)\frac{\rho^N}{N!}. \quad (4.2)$$

In the same way, we are able to show that

$$A(\rho) = \sum_{N=0}^{\infty} {}_{H}F_{N,\theta}(a\alpha, a^2\beta; \gamma)\frac{(b\rho)^n}{n!}\sum_{r=0}^{\infty} {}_{H}F_{r,\theta}(b\alpha, b^2\beta; \gamma)\frac{(a\rho)^r}{r!}$$

$$A(\rho) = \sum_{N=0}^{\infty}\left(\sum_{r=0}^{N}\binom{N}{r}a^r b^{N-r}{}_{H}F_{N-r,\theta}(a\alpha, a^2\beta; \gamma)_{H}F_{r,\theta}(b\alpha, b^2\beta; \gamma)\right)\frac{\rho}{N!}. \quad (4.3)$$

By analysing the coefficients of $\frac{\rho^N}{N!}$ on the final two equations right hand sides, we reach our desired result (4.1).

Theorem 4.2. The following identity is true for every pair of integers a, b and $n \geq 0$:

$$\sum_{k=0}^{N}\binom{N}{r}a^{N-k}b^k{}_{H}F_{N-k,\theta}(b\alpha, b^2\beta; \gamma)\sum_{i=0}^{k}\binom{k}{i}\sigma_i\left(\frac{\theta}{b}, a-1\right)F_{k-i,\theta}(au; \gamma)$$

$$\sum_{k=0}^{N}\binom{N}{k}b^{n-k}a^k{}_{H}F_{N-k,\theta}(a\alpha, a^2\beta; \gamma)\sum_{i=0}^{k}\binom{k}{i}\sigma_i\left(\frac{\theta}{a}, b-1\right)F_{k-i,\theta}(bu; \gamma). (4.4)$$

Proof. Let $B(\rho) = \frac{(1+\theta\rho)^{\frac{ab(\alpha+u)}{\theta}}(1+\theta\rho^2)^{\frac{a^2b^2\beta}{\theta}}((1+\theta\rho)^{\frac{ab}{\theta}}-1)}{(1-\gamma((1+\theta\rho)^{\frac{a}{\theta}}-1))(1-\gamma((1+\theta\rho)^{\frac{b}{\theta}}-1)((1+\theta\rho)^{\frac{a}{\theta}}-1)((1+\theta\rho)^{\frac{b}{\theta}}-1)}$

$$= \frac{(1+\theta\rho)^{\frac{ab\alpha}{\theta}}(1+\theta\rho^2)^{\frac{a^2b^2\beta}{\theta}}}{(1-\gamma((1+\theta\rho)^{\frac{a}{\theta}}-1))}\frac{((1+\theta\rho)^{\frac{ab}{\theta}}-1)}{(1+\theta\rho)^{\frac{b}{\theta}}-1)}\frac{(1+\theta\rho)^{\frac{abu}{\theta}}}{(1-\gamma((1+\theta\rho)^{\frac{b}{\theta}}-1))}$$

$$B(\rho) = \frac{(1+\theta\rho)^{\frac{ab\alpha}{\theta}}(1+\theta t\rho^2)^{\frac{a^2b^2\beta}{\theta}}}{(1-\gamma((1+\theta\rho)^{\frac{a}{\theta}}-1))}\left(\sum_{i=0}^{\infty}\sigma_i\left(\frac{\theta}{b}, a-1\right)\frac{(b\rho)^i}{i!}\right)\left(\sum_{k=0}^{\infty}F_{k,\theta}(au; \gamma)\frac{(b\rho)^k}{k!}\right)$$

$$\left(\sum_{N=0}^{\infty}{}_{H}F_{N,\theta}(b\alpha, b^2\beta; \gamma)\frac{(a\rho)^N}{N!}\right)\left(\sum_{k=0}^{\infty}\sum_{i=0}^{k}\binom{k}{i}b^k\sigma_i\left(\frac{\theta}{b}, a-1\right)F_{k-i,\theta}(au; \gamma)\frac{\rho^k}{k!}\right)$$

$$= \sum_{N=0}^{\infty}\left(\sum_{k=0}^{N}\binom{N}{k}a^{N-k}b^k{}_{H}F_{N-k,\theta}(b\alpha, b^2\beta; \gamma)\sum_{i=0}^{k}\binom{N}{i}\sigma_i\left(\frac{\theta}{b}, a-1\right)F_{k-i,\theta}(au; \gamma)\right)\frac{\rho^N}{N!}. \quad (4.5)$$

In contrast, We've got

$$B(\rho) = \sum_{N=0}^{\infty}\left(\sum_{k=0}^{N}\binom{N}{k}b^{N-k}a^k{}_{H}F_{N-k,\theta}(a\alpha, a^2\beta; \gamma)\sum_{i=0}^{k}\binom{N}{i}\sigma_i\left(\frac{\theta}{a}, b-1\right)F_{k-i,\theta}(bu; \gamma)\right)\frac{\rho^N}{N!}. \quad (4.6)$$

We achieve the intended outcome by comparing the coefficients of $\frac{\rho^N}{N!}$ on the right-hand sides of the final two equations.

Conclusion

We took the Hermite and Fubini polynomial from the Apple type polynomial and found a relationship between them. We then found the degenerate Hermite-Fubini number and polynomial, as well as its properties. Using the series summation method, we were able to prove our results for a new type of polynomial, and we also found identities for the Hermite-Fubini number and polynomial.

References

[1] Bell, E. T, Exponential polynomials, Ann. of Math., 35(1934), 258-277.

[2] Boyadzhiev, K. N, A series transformation formula and related polynomials, Int. J. Math. Math. Sci., 23(2005), 3849-3866.

[3] Carlitz, L, Degenerate Stirling Bernoulli and Eulerian numbers, Util.Math., 15(1979), 51-88.

[4] Dattoli, G, Lorenzutta, S and Cesarano, C, Finite sums and generalized forms of Bernoulli polynomials, Rendiconti di Mathematica, 19(1999), 385-391.

[5] Graham, R. L, Knuth, D. E, Patashnik, O, Concrete Mathematics, ddison-Wesley Publ. Co., New York, 1994.

[6] Gross, O. A, Preferential arrangements, Amer. Math. Monthly, 69(1962), 4-8.

[7] Khan, W. A, A note on degenerate Hermite-poly-Bernoulli numbers and polynomials, J. Classical Anal., 8(1)(2016), 65-76.

[8] Haroon, H, Khan, W. A, Degenerate Bernoulli numbers and polynomials associated with degenerate Hermite polynomials, Commun. Korean Math. Soc., (accepted), (2018), In Press.

[9] Kim, T, Kim, D. S, Jang, G. W, A note on degenerate Fubini polynomials, Proc. Jangjeon Math. Soc., 20(4)(2017), 521-531.

[10] Kim, D. S, Dolgy, T and Komatsu, D. V, Barnes type degenerate Bernoulli polynomials, Adv. Dtud. Contemp. Math., 25(1)(2015), 121-146.

[11] Kim, T, Barnes type multiple degenerate Bernoulli and Euler polynomials, Appl. Math. Comput., 258(2015), 556-564.

[12] Pathan, M. A and Khan, W. A, Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials, Mediterr. J. Math., 12(2015), 679-695.

[13]Pathan, M. A and Khan, W. A: A new class of generalized polynomials associated with Hermite and Euler polynomials, Mediterr. J. Math., 13(2016), 913-928.

[14]Srivastava, H. M and Manocha, H. L, A treatise on generating functions, Ellis Horwood Limited. Co., New York, 1984.

[15]Tanny, S. M, On some numbers related to Bell numbers, Canad. Math. Bull., 17(1974), 733-738.