INVARIANT SUBMANIFOLDS OF (Κ, μ)-CONTACT MANIFOLD USING CURVATURE TENSOR

SAGARIKA NANDY*

*Department of Mathematics, Deshabandhu Mahavidyalaya,
Chittaranjan, Paschim Bardhaman, West Bengal, India
E-mail: dbmsagarika@gmail.com

Abstract: *In this paper, we investigate various properties of invariant submanifolds within (k,μ)-contact manifold. Specifically, we derive the necessary and sufficient conditions for these invariant submanifolds to be totally geodesic under the constraints $Q(\sigma,W_6), Q(\sigma,W_7), Q(\sigma,W_8), Q(\sigma,W_9), Q(\sigma,W_0^*)$ and $Q(\sigma,W_1^*)$, where W_i 's and W_j^*'s (for i = 6, · · · , 9 and j = 0, 1) represent specific curvature tensors, and σ denotes the second fundamental form.*

Keywords: *(k,μ)-contact manifold; invariant submanifold; totally geodesic.*

***MSC2010**: 53D35, 53D20.*

1 Introduction

In 1995, Blair, Koufogiorgos, and Papantoniou [4] introduced contact metric manifolds such that the (k, μ)-nullity distribution contains the characteristic vector field ξ. This specific type of contact manifold is termed a (k, μ)-contact manifold.

In modern mathematics, contact manifolds and invariant submanifolds of various types of contact manifolds are areas of significant interest to many researchers. M. M. Tripathi et al. [10] introduced the concept of invariant submanifolds within (k, μ)-contact manifolds. Recently, M. S. Siddesha and C. S. Bagewadi [12] investigated conditions under which invariant submanifolds of (k, μ)-contact manifolds are geodesic, focusing on specific properties of the second fundamental form. M. M. Tripathi et al. [9] introduced a novel type of curvature tensor, known as the τ-curvature tensor, noting that several known curvature tensors are particular cases of the τ-curvature tensor. Nagaraja and Somashekhara [7] examined the τ-curvature tensor within the context of (k, μ)-contact manifolds.

In modern mathematics contact manifolds and invariant submanifolds of various types of contact manifolds are interesting parts to many researchers. M. M. Tripathi et al. [10] introduced invariant submanifold of (k, μ)-contact manifold. M. S. Siddesha and C. S. Bagewadi [12] recently studied invariant submanifold of (k, μ)-contact manifold to be geodesic under some conditions of the second fundamental form. M. M. Tripathi and et al. [9] initiated the new type of curvature tensor called τ-curvature tensor. Some known curvature tensors were the particular cases of the τ-curvature tensor. Nagaraja and Somashekhara [7] studied τ-curvature tensor in (k, μ)-contact manifold.

The invariant submanifold of a Kenmotsu manifold was studied by the authors in [15] under the conditions $Q(\sigma, R)$ and $Q(S, \sigma)$. Additionally, the invariant submanifold of a (k, μ)-contact manifold was examined by the authors in [12] with the conditions $Q(\sigma, R)$, $Q(S, \sigma)$, and $Q(\sigma, C)$, including an example. In [5], the authors investigated the necessary condition for an invariant submanifold to be geodesic on $(LCS)_n$-manifolds, satisfying $Q(\sigma, R)$, $Q(S, \sigma)$, and $Q(\sigma, C)$. Here, R, S, and C denote the curvature tensor, Ricci tensor, and concircular curvature tensor, respectively.

Motivated by the aforementioned considerations, we study the invariant submanifold of a (k, μ)-contact manifold that satisfies the conditions $Q(\sigma, W_6), Q(\sigma, W_7), Q(\sigma, W_8), Q(\sigma, W_9), Q(\sigma, W_0^*)$ and $Q(\sigma, W_1^*)$, where W_i 's and W_j^*'s (for i = 6, · · · , 9 and j = 0, 1) are curvature tensors, and σ is the second fundamental form.

This paper is structured as follows. Section 2 addresses the fundamental results of the (k, μ)-contact manifold, including the curvature tensors W_i 's for (i = 6, · · · , 9), W_j^*'s for (j = 0, 1). Section 3 presents the basic results of invariant submanifolds within the (k, μ)-contact manifold. Sections 4 through 9 discuss the results for invariant submanifolds satisfying the conditions $Q(\sigma, W_6), Q(\sigma, W_7), Q(\sigma, W_8), Q(\sigma, W_9), Q(\sigma, W_0^*)$ and $Q(\sigma, W_1^*)$, respectively. Here, W_i 's are curvature tensors for i = 6, · · · , 9, W_j^*'s are curvature tensors for j = 0, 1, and σ denotes the second fundamental form.

2. Some basic and essential results of (k, μ)-contact manifolds

An almost contact manifold is a (2m+1)-dimensional differentiable manifold M equipped with the structure (ϕ, ξ, η, g) that satisfies the following conditions [3],

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \ g(X, \xi) = \eta(X), \ \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ g(\phi X, Y) = -g(X, \phi Y), \tag{2.3}$$

$$g(X, \phi X) = 0, \tag{2.4}$$

for all X, Y on M, ϕ is a (1,1)-tensor field, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M.

An almost contact metric structure (ϕ, ξ, η, g) is defined as a contact metric structure if it satisfies the following condition

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.5}$$

for all vector fields X, Y on M.

In a contact metric manifold M^{2m+1} with structure (ϕ, ξ, η, g), a (1, 1)-tensor field h is defined by $h = \frac{1}{2}\pounds_\xi \phi$ where £ denotes the Lie-differentiation. This tensor field h is symmetric and satisfies the following properties:

$$h\phi = -\phi h, \ Tr.h = Tr.\phi h \text{ and } h\xi = 0.$$

Given that ∇ denotes the Riemannian connection of g, the following equation holds:

$$\nabla_X \xi = -\phi X - \phi h X. \tag{2.6}$$

The (k, μ)-nullity distribution for a contact metric manifold $M^{2m+1}(\phi, \xi, \eta, g)$ is defined as:

$$p \to N_p(k, \mu) = \{Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY]\}$$

where $X, Y \in T_pM$, R is the curvature tensor and k, μ are real numbers. If the (k, μ)-nullity distribution includes the characteristic vector field ξ, then:

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \quad (2.7)$$

A contact metric manifold that satisfies the condition given in equation (2.7) is referred to as a (k, μ)-contact manifold [4]. In a (k, μ)-contact metric manifolds the following relations hold [4, 8]:

$$h^2 = (k-1)\phi^2, k \leq 1, \quad (2.8)$$

$$(\nabla_X \phi)Y = g(X + hX, Y) - \eta(Y)(X + hX), \quad (2.9)$$

$$S(X, \xi) = 2mk\eta(X) \quad (2.10)$$

for all vector fields X and Y on M, where ∇ denotes the covariant differentiation operator and S represents the Ricci tensor of type (0,2) of the manifold M.

The authors [2] studied the T curvature tensor of generalized Sasakian space form and deduced that the curvature tensors W_6, W_7, W_8, W_9, W_0^* and W_1^* are specific forms of the T curvature tensor. Utilizing these curvature tensors in a (k, μ)-contact manifold of dimension (2n + 1) the following results can be derived.

$$W_6(X, Y)\xi = k[g(X, Y)\xi - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \quad (2.11)$$

$$W_7(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] - \frac{1}{2n}[2nk\eta(Y)X - \eta(Y)QX], \quad (2.12)$$

$$W_8(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] + \frac{1}{2n}[2nk\eta(Y)X - S(X, Y)\xi], \quad (2.13)$$

$$W_9(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] + \frac{1}{2n}[S(X, Y)\xi - \eta(Y)QX], \quad (2.14)$$

$$W_0^*(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] + \frac{1}{2n}[2nk\eta(Y)X - \eta(X)QY], \quad (2.15)$$

$$W_1^*(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] - \frac{1}{2n}[2nk\eta(Y)X - \eta(X)QY], \quad (2.16)$$

Lemma 2.1. [14] Let (M, ϕ, ξ, η, g) be a contact Riemannian manifold, and suppose that ξ belongs to the (k, μ)-nullity distribution. Then, $k \leq 1$. If $k < 1$, then the manifold admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$, and $D(-\lambda)$ defined by the eigen spaces of h, where $\lambda = \sqrt{(1-k)}$. Further, if $X \in D(\lambda)$, then $hX = \lambda X$ and $\phi h = -\phi h$ imply $h(\phi X) = -\lambda(\phi X)$, and hence $\phi X \in D(-\lambda)$.

3. Invariant submanifolds of (k, μ)-contact manifolds

Definition 3.1. [1] A submanifold N of a manifold M is called an invariant submanifold if the structure vector field ξ is tangent to at every point of N and ϕX is tangent to N for any vector field X tangent to N at every point on N, that is, $\phi(TN) \subset TN$ at every point on N.

Definition 3.2. [1] In the context of differential geometry, consider N as a submanifold of a manifold M. N is described as totally geodesic when its second fundamental form σ vanishes identically across the manifold.

In the context of N being an invariant submanifold within a (k, μ)-contact manifold M, certain specific relations are satisfied on N [10].

$$\widetilde{\nabla}_X \xi = -\phi X - \phi hX, \quad (3.1)$$

$$\sigma(X, \xi) = 0, \quad (3.2)$$

$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX],$ (3.3)
$S(X,\xi) = 2nk\eta(X),$ (3.4)
$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX),$ (3.5)
$\sigma(X, \phi Y) = \phi\sigma(X,Y),$ (3.6)

for any vector fields X, Y defined on a manifold N, where R denotes the curvature tensor and $\widetilde{\nabla}$ represents the covariant differentiation operator associated with the induced metric g, the tensor S refers to the Ricci tensor, expressed in the form of a (0,2) tensor on the submanifold N, and σ signifies the second fundamental form.

Theorem 2.4. [10] An invariant submanifold N of a (k,μ)-contact manifold M is a (k,μ)-contact manifold.

The following result holds on (k,μ)-contact manifold M. The above theorem states that an invariant submanifold N of a (k,μ)-contact manifold M is also a (k,μ)-contact manifold. Therefore, the following result is also true for an invariant submanifold N of a (k,μ)-contact manifold M

$QX = [2(n-1) - n\mu] + [2(n-1) + \mu]hX + [2(1-n) + n(2k+\mu)]\eta(X)\xi,$ (3.7)

where Q is the Ricci operator and $g(QX,Y) = S(X,Y)$.

4. INVARIANT SUBMANIFOLD OF $(\mathbf{k},\boldsymbol{\mu})$-CONTACT MANIFOLD SATISFYING $\mathbf{Q}(\boldsymbol{\sigma},\mathbf{W_6}) = \mathbf{0}$

This section establishes the necessary and sufficient condition for the invariant submanifolds of (k,μ)-contact manifold satisfying $Q(\sigma, W_6) = 0$, to be totally geodesic.

Theorem 4.1. Let N be an invariant submanifold of a (k,μ)-contact manifold M satisfying $Q(\sigma, W_6) = 0$. Then N is totally geodesic if and only if $2nk \pm \lambda\mu \neq 0$.

Proof. Assume N is an invariant submanifold of a (k,μ)-contact manifold M satisfying $Q(\sigma, W_6) = 0$. Therefore

$0 = Q(\sigma, W_6) = Q(\sigma, W_6)(X,Y,Z;U,V) = ((U \wedge_\sigma V).W_6)(X,Y)Z = -W_6\big((U \wedge_\sigma V)X, Y\big)Z -$
$W_6(X, (U \wedge_\sigma V)Y)Z - W_6(X,Y)(U \wedge_\sigma V)Z$ (4.1)

Where $U \wedge_\sigma V$ is defined by

$(U \wedge_\sigma V)P = \sigma(V,P)U - \sigma(U,P)V$ (4.2)

Using (4.2) in (4.1) we get

$-\sigma(V,X)W_6(U,Y)Z + \sigma(U,X)W_6(V,Y)Z - \sigma(V,Y)W_6(X,U)Z + \sigma(U,Y)W_6(X,V)Z - \sigma(V,Z)W_6(X,Y)U +$
$\sigma(U,Z)W_6(X,Y)V = 0.$ (4.3)

Putting $Z = V = \xi$ in (4.3) and using (3.2), we get

$\sigma(U,X)W_6(\xi,Y)\xi + \sigma(U,Y)W_6(X,\xi)\xi = 0$ (4.4)

Using (2.11) in (4.4) we obtain

$\sigma(U,X)[k\{\eta(Y)\xi - Y\} - \mu hY] + \sigma(U,Y)\mu hX = 0$ (4.5)

Taking inner product with W, we get

$\sigma(U,X)[k\{\eta(Y)\eta(W) - g(Y,W)\} - \mu g(hY,W)] + \sigma(U,Y)\mu g(hX,W) = 0$ (4.6)

Contracting Y and W we get

$\sigma(U,X)2k(1 - 2n - 1) + \mu\sigma(U,hX) = 0$ (4.7)

Therefore, we get

$\sigma(U,X)[2nk \pm \lambda\mu] = 0$ (4.8)

Hence $\sigma(U, X) = 0$, provided $2nk \pm \lambda\mu \neq 0$. Therefore, the manifold is totally geodesic. The converse part of the theorem is trivial. Thus, the theorem is proved.

Corollary 4.1. Let N be an invariant submanifold of a $N(K)$-contact manifold M of odd dimension. Then $Q(\sigma, W_6) = 0$ if and only if N is totally geodesic, provided $2nk \neq 0$.

5. INVARIANT SUBMANIFOLD OF $(\mathbf{k}, \boldsymbol{\mu})$-CONTACT MANIFOLD SATISFYING $\mathbf{Q}(\boldsymbol{\sigma}, \mathbf{W_7}) = \mathbf{0}$

This section contains the necessary and sufficient condition for the invariant submanifolds of (k, μ)-contact manifold satisfying $Q(\sigma, W_7) = 0$ to be totally geodesic.

Theorem 5.1. Let N be an invariant submanifold of a (k, μ)-contact manifold M satisfying $Q(\sigma, W_7) = 0$. Then N is totally geodesic if and only if $[-4n^2k - (2n+1)\{2(n-1) - n\mu\} \pm \lambda\{(2n-1)\mu - 2(n-1)\}] \neq 0$.

Proof. Let N be an invariant submanifold of a (k, μ)-contact manifold M satisfying $Q(\sigma, W_7) = 0$. Therefore

$$0 = Q(\sigma, W_7) = Q(\sigma, W_7)(X, Y, Z; U, V) = ((U \wedge_\sigma V).W_7)(X, Y)Z = -W_7\big((U \wedge_\sigma V)X, Y\big)Z - W_7(X, (U \wedge_\sigma V)Y)Z - W_7(X, Y)(U \wedge_\sigma V)Z \tag{5.1}$$

Using (4.2) in (5.1) we get

$$-\sigma(V, X)W_7(U, Y)Z + \sigma(U, X)W_7(V, Y)Z - \sigma(V, Y)W_7(X, U)Z + \sigma(U, Y)W_7(X, V)Z - \sigma(V, Z)W_7(X, Y)U + \sigma(U, Z)W_7(X, Y)V = 0 \tag{5.2}$$

Putting $Z = V = \xi$ in (5.2) and using (3.2), we get

$$\sigma(U, X)W_7(\xi, Y)\xi + \sigma(U, Y)W_7(X, \xi)\xi = 0 \tag{5.3}$$

Using (2.12) in (5.3) we obtain

$$\sigma(U, X)[k\{\eta(Y)\xi - Y\} - \mu hY] + \sigma(U, Y)\left[k\{X - \eta(X)\xi\} + \mu hX - kX + \frac{1}{2n}QX\right] = 0 \tag{5.4}$$

Taking inner product with W, we get

$$\sigma(U, X)[k\{\eta(Y)\eta(W) - g(Y, W)\} - \mu g(hY, W)] + \sigma(U, Y)\left[k\{g(X, W) - \eta(X)\eta(W)\} + \mu g(hX, W) - kg(X, W) - \frac{1}{2n}g(QX, W)\right] = 0 \tag{5.5}$$

Contracting Y and W and using (3.7) we get

$$\sigma(U, X)[-4n^2k - (2n+1)\{2(n-1) - n\mu\} \pm \lambda\{(2n-1)\mu - 2(n-1)\}] \neq 0 \tag{5.6}$$

Hence $\sigma(U, X) = 0$, provided

$$[-4n^2k - (2n+1)\{2(n-1) - n\mu\} \pm \lambda\{(2n-1)\mu - 2(n-1)\}] \neq 0$$

Therefore, the manifold is totally geodesic. The converse part of the theorem is trivial. Therefore, the theorem is proved.

Corollary 5.1. Let N be an invariant submanifold of a Sasakian manifold M of dimension odd satisfying $Q(\sigma, W_7) = 0$ if and only if N is totally geodesic.

Corollary 5.2. Let N be an invariant submanifold of a $N(K)$-contact manifold M of dimension odd satisfying $Q(\sigma, W_7) = 0$ if and only if N is totally geodesic, provided $2n^2k + (n-1)(2n-1) \pm \lambda(n-1) \neq 0$.

6. INVARIANT SUBMANIFOLDS OF $(\mathbf{k}, \boldsymbol{\mu})$-CONTACT MANIFOLD SATISFYING $\mathbf{Q}(\boldsymbol{\sigma}, \mathbf{W_8}) = \mathbf{0}$

This section establishes the necessary and sufficient condition for the invariant submanifolds of (k,μ)-contact manifold satisfying $Q(\sigma, W_8)=0$ to be totally geodesic.

Theorem 6.1. Let N be an invariant submanifold of a (k,μ)-contact manifold M satisfying $Q(\sigma, W_8)=0$ if and only if N is totally geodesic provided $2k(1-2n)\pm\lambda\mu\neq 0$.

Proof. Let N be an invariant submanifold of a (k,μ)-contact manifold M satisfying $Q(\sigma, W_8)=0$. Therefore

$$0=Q(\sigma,W_8)=Q(\sigma,W_8)(X,Y,Z;U,V)=((U\wedge_\sigma V).W_8)(X,Y)Z=-W_8\big((U\wedge_\sigma V)X,Y\big)Z-W_8(X,(U\wedge_\sigma V)Y)Z-W_8(X,Y)(U\wedge_\sigma V)Z \quad (6.1)$$

Using (4.2) in (6.1) we get

$$-\sigma(V,X)W_8(U,Y)Z+\sigma(U,X)W_8(V,Y)Z-\sigma(V,Y)W_8(X,U)Z+\sigma(U,Y)W_8(X,V)Z-\sigma(V,Z)W_8(X,Y)U+\sigma(U,Z)W_8(X,Y)V=0 \quad (6.2)$$

Putting $Z=V=\xi$ in (6.2) and using (3.2), we get

$$\sigma(U,X)W_8(\xi,Y)\xi+\sigma(U,Y)W_8(X,\xi)\xi=0 \quad (6.3)$$

Using (2.13) in (6.3) we obtain

$$\sigma(U,X)[k\{\eta(Y)\xi-Y\}-\mu hY]+\sigma(U,Y)[2k\{X-\eta(X)\xi\}+\mu hX]=0 \quad (6.4)$$

Taking inner product with W, we get

$$\sigma(U,X)[k\{\eta(Y)\eta(W)-g(Y,W)\}-\mu g(hY,W)]+\sigma(U,Y)[2k\{g(X,W)-\eta(X)\eta(W)\}+\mu g(hX,W)]=0 \quad (6.5)$$

Contracting Y and W and using (3.7) we get

$$\sigma(U,X)[2k(1-2n)\pm\lambda\mu]=0 \quad (6.6)$$

Hence $\sigma(U,X)=0$, provided

$$[2k(1-2n)\pm\lambda\mu]\neq 0$$

Therefore, the manifold is totally geodesic. The converse part of the theorem is trivial. Therefore, the theorem is proved.

Corollary 6.1. Let N be an invariant submanifold of a Sasakian manifold N of dimension odd satisfying $Q(\sigma, W_8)=0$ if and only if N is totally geodesic.

Corollary 6.2. Let N be an invariant submanifold of a N(K)-contact manifold M of dimension odd satisfying $Q(\sigma, W_8)=0$ if and only if N is totally geodesic, provided $2k(1-2n)\pm\lambda\mu\neq 0$.

7. INVARIANT SUBMANIFOLDS OF (k,μ)-CONTACT MANIFOLD SATISFYING $Q(\sigma, W_9)=0$

This section establishes the necessary and sufficient condition for the invariant submanifolds of (k,μ)-contact manifold satisfying $Q(\sigma, W_9)=0$ to be totally geodesic.

Theorem 7.1. Let N be an invariant submanifold of a (k,μ)-contact manifold M satisfying $Q(\sigma, W_9)=0$. Then N is totally geodesic if and only if $2nk(1-2n)-\{2(n-1)-n\mu\}\pm\lambda[2n\mu-\{2(n-1)+\mu\}]\neq 0$.

Proof. Let N be an invariant submanifold of a (k,μ)-contact manifold M satisfying $Q(\sigma, W_9)=0$. Therefore

$$0=Q(\sigma,W_9)=Q(\sigma,W_9)(X,Y,Z;U,V)=((U\wedge_\sigma V).W_9)(X,Y)Z=-W_9\big((U\wedge_\sigma V)X,Y\big)Z-W_9(X,(U\wedge_\sigma V)Y)Z-W_9(X,Y)(U\wedge_\sigma V)Z \quad (7.1)$$

Using (4.2) in (7.1) we get

$-\sigma(V,X)W_9(U,Y)Z+\sigma(U,X)W_9(V,Y)Z-\sigma(V,Y)W_9(X,U)Z+\sigma(U,Y)W_9(X,V)Z-\sigma(V,Z)W_9(X,Y)U+\sigma(U,Z)W_9(X,Y)V=0$ (7.2)

Putting $Z=V=\xi$ in (7.2) and using (3.2), we get

$\sigma(U,X)W_9(\xi,Y)\xi+\sigma(U,Y)W_9(X,\xi)\xi=0$ (7.3)

Using (2.14) in (7.3) we obtain

$\sigma(U,X)[k\{\eta(Y)\xi-Y\}-\mu hY]+\sigma(U,Y)\left[k\{X-\eta(X)\xi\}+\mu hX+\frac{1}{2n}\{2nk\eta(X)\xi-QX\}\right]=0$ (7.4)

Taking inner product with W, we get

$\sigma(U,X)[k\{\eta(Y)\eta(W)-g(Y,W)\}-\mu g(hY,W)]+\sigma(U,Y)\left[k\{g(X,W)-\eta(X)\eta(W)\}+\mu g(hX,W)+\frac{1}{2n}\{2nk\eta(X)\eta(W)-g(QX,W)\}\right]=0$ (7.5)

Contracting Y and W and using (3.7) we get

$\sigma(U,X)[2nk(1-2n)-\{2(n-1)-n\mu\}\pm\lambda[2n\mu-\{2(n-1)+\mu\}]=0$ (7.6)

Hence $\sigma(U,X)=0$, provided

$$2nk(1-2n)-\{2(n-1)-n\mu\}\pm\lambda[2n\mu-\{2(n-1)+\mu\}]\neq 0$$

Therefore, the manifold is totally geodesic. The converse part of the theorem is trivial. Therefore, the theorem is proved.

Corollary 7.1. Let N be an invariant submanifold of a Sasakian manifold M of dimension odd satisfying $Q(\sigma,W_9)=0$ if and only if N is totally geodesic.

Corollary 7.2. Let N be an invariant submanifold of a N(K)-contact manifold M of dimension odd satisfying $Q(\sigma,W_9)=0$ if and only if N is totally geodesic, provided $2nk(1-2n)-2(n-1)(1\pm\lambda)\neq 0$.

8. INVARIANT SUBMANIFOLDS OF $(\mathbf{k},\boldsymbol{\mu})$-CONTACT MANIFOLD SATISFYING $\mathbf{Q}(\boldsymbol{\sigma},\mathbf{W}_0^*)=\mathbf{0}$

This section contains the necessary and sufficient condition for the invariant submanifolds of (k,μ)-contact manifold satisfying $Q(\sigma,W_0^*)=0$ to be totally geodesic.

Theorem 8.1. Let N be an invariant submanifold of a (k,μ)-contact manifold M satisfying $Q(\sigma,W_0^*)=0$ if and only if N is totally geodesic provided $2nk(3-2n)-2(n-1)+n\mu\pm 2n\lambda\mu\neq 0$.

Proof. Let N be an invariant submanifold of a (k,μ)-contact manifold M satisfying $Q(\sigma,W_0^*)=0$. Therefore

$0=Q(\sigma,W_0^*)=Q(\sigma,W_0^*)(X,Y,Z;U,V)=((U\wedge_\sigma V).W_0^*)(X,Y)Z=-W_0^*((U\wedge_\sigma V)X,Y)Z-W_0^*(X,(U\wedge_\sigma V)Y)Z-W_0^*(X,Y)(U\wedge_\sigma V)Z$ (8.1)

Using (4.2) in (8.1) we get

$-\sigma(V,X)W_0^*(U,Y)Z+\sigma(U,X)W_0^*(V,Y)Z-\sigma(V,Y)W_0^*(X,U)Z+\sigma(U,Y)W_0^*(X,V)Z-\sigma(V,Z)W_0^*(X,Y)U+\sigma(U,Z)W_0^*(X,Y)V=0$ (8.2)

Putting $Z=V=\xi$ in (8.2) and using (3.2), we get

$\sigma(U,X)W_0^*(\xi,Y)\xi+\sigma(U,Y)W_0^*(X,\xi)\xi=0$ (8.3)

Using (2.15) in (8.3) we obtain

$\sigma(U,X)[k\{\eta(Y)\xi - Y\} - \mu hY + \frac{1}{2n}\{2nk\eta(Y)\xi - QY\}] + \sigma(U,Y)\Big[k\{X - \eta(X)\xi\} + \mu hX + \frac{1}{2n}\{2nkX - 2nk\eta(X)\xi\}\Big] = 0$ (8.4)

Taking inner product with W, we get

$\sigma(U,X)[k\{\eta(Y)\eta(W) - g(Y,W)\} - \mu g(hY,W) + \frac{1}{2n}\{2nk\eta(Y)\eta(W) - g(QY,W)\}] + \sigma(U,Y)\Big[k\{g(X,W) - \eta(X)\eta(W)\} + \mu g(hX,W) + \frac{1}{2n}\{2nkg(X,W) - 2nk\eta(X)\eta(W)\}\Big] = 0$ (8.5)

Contracting Y and W and using (3.7) we get

$\sigma(U,X)[2nk(3-2n) - 2(n-1) + n\mu \pm 2n\lambda\mu] = 0$ (8.6)

Hence $\sigma(U,X) = 0$, provided

$$[2nk(3-2n) - 2(n-1) + n\mu \pm 2n\lambda\mu] \neq 0$$

Therefore, the manifold is totally geodesic. The converse part of the theorem is trivial. Therefore, the theorem is proved.

Corollary 8.1. Let N be an invariant submanifold of a Sasakian manifold M of dimension odd satisfying $Q(\sigma, W_0^*) = 0$ if and only if N is totally geodesic.

Corollary 8.2. Let N be an invariant submanifold of a N(K)-contact manifold M of dimension odd satisfying $Q(\sigma, W_0^*) = 0$ if and only if N is totally geodesic, provided $2nk(3-2n) - 2(n-1) \neq 0$.

9. INVARIANT SUBMANIFOLDS OF $(\mathbf{k},\boldsymbol{\mu})$-CONTACT MANIFOLD SATISFYING $\mathbf{Q}(\boldsymbol{\sigma},\mathbf{W_1^*}) = \mathbf{0}$

This section establishes the necessary and sufficient condition for the invariant submanifolds of (k,μ)-contact manifold satisfying $Q(\sigma, W_1^*) = 0$ to be totally geodesic.

Theorem 9.1. Let N be an invariant submanifold of a (k,μ)-contact manifold M satisfying $Q(\sigma, W_1^*) = 0$. Then N is totally geodesic if and only if $2(n-1) - n\mu - 2nk \pm \lambda\mu \neq 0$.

Proof. Let N be an invariant submanifold of a (k,μ)-contact manifold M satisfying $Q(\sigma, W_1^*) = 0$. Therefore

$0 = Q(\sigma, W_1^*) = Q(\sigma, W_1^*)(X,Y,Z;U,V) = ((U \wedge_\sigma V).W_1^*)(X,Y)Z = -W_1^*\big((U \wedge_\sigma V)X, Y\big)Z - W_1^*(X,(U \wedge_\sigma V)Y)Z - W_1^*(X,Y)(U \wedge_\sigma V)Z$ (9.1)

Using (4.2) in (9.1) we get

$-\sigma(V,X)W_1^*(U,Y)Z + \sigma(U,X)W_1^*(V,Y)Z - \sigma(V,Y)W_1^*(X,U)Z + \sigma(U,Y)W_1^*(X,V)Z - \sigma(V,Z)W_1^*(X,Y)U + \sigma(U,Z)W_1^*(X,Y)V = 0$ (9.2)

Putting $Z = V = \xi$ in (9.2) and using (3.2), we get

$\sigma(U,X)W_1^*(\xi,Y)\xi + \sigma(U,Y)W_1^*(X,\xi)\xi = 0$ (9.3)

Using (2.16) in (9.3) we obtain

$\sigma(U,X)\Big[k\{\eta(Y)\xi - Y\} - \mu hY - \frac{1}{2n}\{2nk\eta(Y)\xi - QY\}\Big] + \sigma(U,Y)\mu hX = 0$ (9.4)

Taking inner product with W, we get

$\sigma(U,X)\Big[k\{\eta(Y)\eta(W) - g(Y,W)\} - \mu g(hY,W) - \frac{1}{2n}\{2nk\eta(Y)\eta(W) - g(QY,W)\}\Big] + \sigma(U,Y)\mu g(hX,W) = 0$ (9.5)

Contracting Y and W and using (3.7) we get

$$\sigma(U,X)[2(n-1)-n\mu-2nk\pm\lambda\mu]=0 \quad (9.6)$$

Hence $\sigma(U,X)=0$, provided

$$[2(n-1)-n\mu-2nk\pm\lambda\mu]\neq 0$$

Therefore, the manifold is totally geodesic. The converse part of the theorem is trivial. Thus, the theorem is proved.

Corollary 9.1. Let N be an invariant submanifold of a Sasakian manifold M of dimension odd satisfying $Q(\sigma,W_1^*)=0$ if and only if N is totally geodesic.

Corollary 9.2. Let N be an invariant submanifold of a N(K)-contact manifold M of dimension odd satisfying $Q(\sigma,W_1^*)=0$ if and only if N is totally geodesic, provided $\{n(1-k)-1\}\neq 0$.

References:

[1] A. Bejancu and N. Papaghuic, Semi-invariant submanifolds of a Sasakian manifold, An Sti. Univ. “Al. I. Cuza” Iasi, 27 (1981) 163-170.

[2] A. Kumari and S. K. Chanyal, On The T Curvature Tensor of Generalized Sasakian Space Form, IOSR Journal of Mathematics, 11 (3) (2015) 61-68.

[3] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Math., 509, Springer-Verlag, Berlin (1976).

[4] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, Contact metric manifolds satisfying a nullity condition, Israel J. of Math., 19 (1995) 189-214.

[5] D. Nirmala and C.S. Bagewadi, A Note on Invariant Submanifolds of (LCS)n Manifold, International Journal of Mathematics Trends and Technology, 43 (1) (2017) 63-67.

[6] G. P. Pokhariyal and R. S. Mishra, Curvature Tensors and Their Relativistic Significance II, Yokohama Math. J., 19 (1971) 97-103.

[7] H. G. Nagaraja and G. Somashekhara, τ-curvature tensor in (k,μ)-contact manifolds, Mathematica Aeterna, 2 (6) (2012) 523-532.

[8] Jae-Bok, Ahmet Yildiz and Uday Chand De, On ϕ-recurrent (k,μ)- contact metric manifolds, Bull. Korean Math. S, 45 (4) (2008) 689-700.

[9] M. M. Tripathi and P. Gupta, τ-curvature tensor on a semi-Riemannian manifold, J. Adv. Math, Stud, 4 (1) (2011) 117-129.

[10] M. M. Tripathi and T. Sasahara and J.-S. Kim, On invariant submanifolds of contact metric manifolds, Tsukuba J. Math., 29 (2) (2005) 495-510.

[11] M. S. Siddesha and C. S. Bagewadi, On some classes of invariant submanifolds of (k,μ)-contact manifold, Journal of Informatics and Mathematical Science, 9 (1) (2017) 13-26.

[12] M. S. Siddesha and C. S. Bagewadi, Totally geodesic submanifolds of (k,μ)-contact manifold, IOSR Journal of Mathematics, 12(6) (2016) 84-89.

[13] R. S. Mishra, Structure on a differentiable manifold and their applications, Chandrama Prakashan, 50-A, Balrampur House, Allahabad, India, 1984.

[14] S. Tanno, Ricci curvatures of contact Riemannian manifolds, Tohoku Mathematical Journal, 40(3) (1988) 441-448.

[15] U. C. De and P. Majhi, On invariant submanifolds of Kenmotsu manifolds, J. Geom., 106(1) (2015) 109-122.