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TWO PARAMETRIC MEASURE OF WEIGHTED INFORMATION IMPROVEMENT

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Abstract — In the present paper we have obtained measure of weighted information improvement corresponding to sixmeasures of entropy.

Key words - Entropy, Information improvement, Measure of directed divergence, Weighted information improvement.

1. INTRODUCTION

Probabilistic entropy plays an important role in statistical physics. Many problems were clarified by information entropy, which is a measure of uncertainty as well as of information applied by a probabilistic experiment. The measure of entropy introduced by Shannon takes into account only the probabilities associated with the events and not their importance. But there exist many fields dealing with random events where it is necessary to take into account both these probabilities and some qualitative characteristics of the events. For instance, in two-handed game one should keep in mind both the probabilities of different variants of the game, that is, the random strategies of the players and the wins corresponding in these variants. Thus, it is necessary to associate with every elementary event both the probability with which it occurs and its weight.

The object of the present paper is to investigate the correct measures of directed divergence, correct measures of weighted directed divergence, and correct measures of weighted information improvement. In this paper, we extend the result of weighted information improvement $I_k(P{:}Q.W)$ in going from Q to R when the true distribution is P and weight function W(x) and our obtained information measures includes several well-known results.

2. PRELIMINARIES

Let P= $(p_1, p_2, p_3, \dots, p_n)$ be non-degenerate probability distribution and

Let W = $(w_1, w_2, w_3, \dots, w_n)$ be a set of weights associated with the n outcomes, then corresponding to Shannon's [10] measure of entropy

$$S(P) = -\sum_{i=1}^{n} p_i \ln p_i \tag{1}$$

Guiasu [3] defined a measure of weighted entropy as,

$$S(P{:}W) = -\sum_{i=1}^{n} w_i p_i \ln p_i \tag{2}$$

Naturally $p_i \geq 0, w_i \geq 0$

$$\sum p_i = 1, \sum w_i = 1$$

Now if Q= $(q_1, q_2, q_3, \dots, q_n)$ be another non-degenerate probability distribution then the well know Kullback Leibler [9] measure D_1 (P: Q) gives a measure of directed divergence of P and Q as, Where $D_1(P:Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}$ (3)

And Taneja and Tuteja [11] gave the corresponding measure of weighted directed divergence as,

$D_1(P:Q,W) = \sum_{i=1}^{n} w_i\, p_i \ln \frac{p_i}{q_i}$ (4)

Measure (3) is a correct measure of directed divergence, since it has following properties,

(i) $D_1(P:Q) \geq 0$
(5)

(ii) $D_1(P:Q) = 0$ iff $P = Q$ (6)

(iii) $D_1(P:Q)$ is a convex function of P and Q
(7)

However $D_1(P:Q;W)$ is not a correct measure of weighted directed divergence because it does not satisfy (5) and (6)

To see this consider some following cases as,

Where (i) $P = \left(\frac{2}{5}, \frac{3}{5}\right)$, $Q = \left(\frac{3}{5}, \frac{2}{5}\right)$, $W = \left(\frac{3}{5}, \frac{2}{5}\right)$
(8)

Where (ii) $P = \left(\frac{1}{3}, \frac{2}{3}\right)$, $Q = \left(\frac{2}{3}, \frac{1}{3}\right)$, $W = \left(\frac{4}{5}, \frac{1}{5}\right)$
(9)

Then
$$D_1(P:Q,W) = \sum_{i=1}^{n} w_i\, p_i \ln \frac{p_i}{q_i} = \frac{1}{5}\left[3w_2 - 2w_1\right] \ln \frac{3}{2}$$
(10)

Here we can say that both greater than zero and also vanish when $P \neq Q$ so it not correct to recognize $D_1(P:Q.W)$ as correct measure of weighted directed divergence and one object of paper is to find correct measure of weighted information improvement corresponding to six measure of entropy and new two parametric measure of weighted information improvement is obtain, in fact here we have to find more correct measure of weighted directed divergence corresponding to Csiszers's measure of directed divergence,

$D_2(P:Q) = \sum_{i=1}^{n} q_i \phi\left(\frac{p_i}{q_i}\right)$ where $\phi(.)$ is twice differentiable convex function with $\phi(1) = 0$.

Weighted information improvement $I_k(P:Q.W)$ in going from Q to R when the true distribution is P and weight function W(x) is defined by Kapur [8] is,

$I_k(P:Q,R,W) = D_k(P:Q,W) - D_k(P:R,W)$ (11)

2.1 Correct Measure Of Weighted Directed Divergence Corresponding To Csiszer's [3] Measure

Since $\phi(x)$ is convex function, $\phi'(x) = 0$ is always increasing function also $\phi(1) = 0$ at x =1 and $\phi(x)$ may or may not be positive, negative or zero is defined (From figure 2.1(a), 2.2(b), 2.3(c)

Case I - In the case (a) and (b) $\phi\left(\frac{p_i}{q_i}\right)$ can be positive or negative $D_1(P:Q)$ is always ≥ 0, but $D_1(P:Q,W)$ can be negative or positive

Case II - In case 3.1(c) $\phi\left(\frac{p_i}{q_i}\right)$ is always ≥ 0 and vanishes only when $p_i = q_i$ so that in this case Kapur [6] has obtained accurate measure of weighted directed divergence as ,

$D_2(P:Q,W) = \sum_{i=1}^{n} w_i q_i \phi\left(\frac{p_i}{q_i}\right)$
(12)

It is ≥ 0 and vanishes when $p_i = q_i\ \forall\, i$ Now the accurate measure of weighted directed divergence corresponding to Csiszer's[2] measure of directed divergence is given by (12), provided $\phi(x)$ is convex function twice differentiate
(13)

$\phi(x) = 0$ (14)

$\phi'(x) = 0$ (15)

The condition (15) is the additional condition we impose on $\phi(x)$ has graph $\phi\left(\frac{p_i}{q_i}\right)$ is always ≥ 0 and vanishes only when $p_i = q_i$

2.2 Measure Of Weighted Information Improvement

Now suppose during the course of investigation Q revised to R = $\{r_1, r_2, \ldots, r_n\}$ then measure of information improvement is given by

$I(P:Q,R) = D(P:Q) - D(P:R)$ (16)

The measure $I_k(P:Q:R,W)$of weighted information improvement is going from Q to R when the true distribution P is and the weight function W(x) is associated then Kapur [8] is defined as,

$I_k(P:Q:R,W) = D_k(P:Q,W) - D_k(P:R,W)$ Where,

R = $\{r_1, r_2, \ldots, r_n\}, r_i \geq 0, \sum_{i=1}^{n} r_i = 1$

(17)

3. CORRECT MEASURES OF DIRECTED DIVERGENCE AND WEIGHTED INFORMATION IMPROVEMENT

3.1 Correct Measure Of Directed Divergence And Measure Of Weighted Information Improvement Corresponding To Kullback Leibler [9] Measure

Let

$\phi(x) = x \ln x - x + 1$

(18)

$\phi(x)$ is convex function and it is twice differentiable so that, by(12) the correct measure of weighted directed divergence corresponding to Kullback Leibler [9] measures is as follows,

$D_3(P:Q,W) = \sum_i^n w_i\{ p_i \ln (\frac{p_i}{q_i}) - p_i + q_i\}$ (19)

$D_3(P:Q,W) = \sum_i^n w_i\{ p_i \ln (\frac{p_i}{r_i}) - p_i + r_i\}$ (20)

From (17),

$I_3(P:Q:R,W) = \sum_i^n w_i\{ p_i \ln (\frac{r_i}{q_i}) + q_i - r_i \}$

(21)

$I_3(P:Q:R,W)$ is the measure of weighted information improvement corresponding to Kullback Leibler [9] measure of entropy.

3.2 Corrected Measure Of Weighted Directed Divergence Corresponding To Harvda And Charvat's [4] Measure Of Entropy And Its Measure Of Weighted Information Improvement

Take

$\phi(x) = \left(\frac{x^\alpha - \alpha x + \alpha - 1}{\alpha(\alpha-1)}\right), \alpha \neq 0, 1$

(22)

$\phi'(x) = \frac{1}{\alpha-1}\{(\alpha-1)x^{\alpha-2}\}, \alpha \neq 0, 1$

(23)

$\phi''(x) = x^{\alpha-2}$

(24)

From it is clear that (24) $\phi(x)$ is twice differentiate convex function and from (22) and (23) it is seen that $\phi(1) = 0$ and $\phi'(1) = 0$ Again by (12) the correct measure of directed divergence and measured weighted information improvement corresponding to Harvda and Charvats [4] measure of entropy

$D_4(P{:}Q,W)=\frac{1}{\alpha(\alpha-1)}\sum_{i=1}^{n} w_i\,[p_i^{\alpha}q_i^{1-\alpha}-\alpha p_i+\alpha q_i-q_i]$
(25)

$D_4(P{:}R,W)=\frac{1}{\alpha(\alpha-1)}\sum_{i=1}^{n} w_i\left[p_i^{\alpha}r_i^{1-\alpha}-\alpha p_i+\alpha\, r_i-\ r_i\right]$
(26)

Again from (17)

$I_4(P{:}Q{:}R,W)=\ D_4(P{:}Q,W)\ -\ D_4(P{:}R,W)$ $I_4(P{:}Q{:}R,W)\ =\ \frac{1}{\alpha(\alpha-1)}\ \sum_{i=1}^{n} w_i[\ p_i^{\alpha}\left(q_i^{1-\alpha}-r_i^{1-\alpha}\right)+(\alpha-1)(q_i-\ r_i)\right]$
(27)

∴ $I_4(P{:}Q{:}R,W)$ is the measure of weighted information improvement corresponding to Havrda and Charvats [4] measure of entropy
Now we will obtain the particular cases of $I_4(P{:}Q{:}R,W)$ as if $\alpha \to 1$ in (25) we get (19) Kullback – Zeibler [7] measure

as $\alpha \to 0$ measure (25) approaches (19)

∴ $D_4(P{:}Q,W) = D_3(P{:}Q,W)$
(28)

And $D_4(P{:}R,W) = D_3(P{:}R,W)$ (29)

And so $I_4(P{:}Q{:}R,W) = I_3(P{:}Q{:}R,W)$ (30)

3.3 Corrected Measure of Weighted Directed Divergence And Measure Of Weighted Information Improvement Corresponding To Kapur's [5], [6] Measure Of Entropy

$\phi(x)\ =\ x\ \ln\ x\ -\ \frac{(1+ax)}{a}\ \ln\ (1+ax)\ +\ x\ \ln\ (1+a)\ +\ \frac{(1+a)}{a}\ \ln\ (1+a)\ -\ \ln\ (1+a),\ a\ \geq\ 0$
(31)

$\phi'(x) = \ln x - \ln (1+ax) + \ln (1+a)$
(32)

$\phi''(x) = \frac{1}{x} - \frac{a}{1+ax}$ (33)

It is clearly seen from (32) and (33) $\phi(x)$ is a twice differentiable convex function $\phi(1) = 0$ and $\phi'(1) = 0$ By using (12) the correct measure of directed divergence and measure of weighted information improvement corresponding to Kapur [6] and [7] measure of entropy as,

$D_5(P{:}Q,W) =\sum_{i=1}^{n} w_i\, p_i \ln(\frac{p_i}{q_i}) + \sum_{i=1}^{n} w_i\ (p_i - q_i) \ln (1+a) + \frac{(1+a)}{a} \ln (1+a) \sum_{i=1}^{n} w_i q_i - \frac{1}{a}\sum_{i=1}^{n} w_i(q_i + ap_i) \ln (1+ \frac{ap_i}{q_i})$
(34)

$D_5(P{:}R,W) =\sum_{i=1}^{n} w_i\, p_i \ln (\frac{p_i}{r_i}) + \sum_{i=1}^{n} w_i\ (p_i - r_i) \ln (1+a) + \frac{(1+a)}{a} \ln (1+a) \sum_{i=1}^{n} w_i r_i - \frac{1}{a}\sum_{i=1}^{n} w_i (r_i + ap_i)$

$\ln\left(1+\frac{ap_i}{r_i}\right)$
(35)

$I_5(P:Q:R,W) = D_5(P:Q,W) - D_5(P:R,W)$

$= \sum_{i=1}^{n} w_i \{p_i \ln\left(\frac{r_i}{q_i}\right) + (r_i - q_i)\ln(1+a) + \frac{1+a}{a}\ln(1+a)\ (q_i - r_i) - \frac{1}{a}[(q_i + ap_i)\ln\left(1+\frac{ap_i}{q_i}\right) - (r_i+ap_i)\ln\left(1+\frac{ap_i}{r_i}\right)]\}$
(36)

$I_{4,1-\alpha}(P:Q,W) = I_{4,\alpha}(Q:P,W)$
(37)

4. SOME ANOTHER CLASS OF MEASURE OF WEIGHTED DIRECTED DIVERGENCE AND ITS MEASURE OF WEIGHTED INFORMATION IMPROVEMENT

$D_6(P:Q,W) = \sum_{i=1}^{n} w_i\, q_i\, \phi\left(\frac{p_i}{q_i}\right) + \sum_{i=1}^{n} w_i\, (1+aq_i)\, \phi\left(\frac{1+ap_i}{1+aq_i}\right)$
(38)
(38) Provided
(i) $\phi(x)$ is twice differentiable convex function
(ii) $\phi(1) = 0$
(iii) $\phi'(1) = 0$
Then $D_6(P:Q,W)$ is convex function of $(p_1,p_2,\ldots\ldots p_n)$ which has minimum value zero when each $p_i=q_i$
So that $D_6(P:Q,W) \geq 0$ and vanishes iff P = Q
When we put a = 0 in (38)
We get $D6(P:Q,W) = D_2(P:Q,W)$
(39)
$\therefore D6(P:Q,W) is$ a valid measure of weighted directed divergence
(39) is true for a > 0 or a < 0
If a < 0, since we want $1 + ap_i > 0$ and $a \geq -1$ by using (12) taking limit as $\alpha \to 1$ or any other suitable measure we can get a large number of correct measure of weighted directed divergence and weighted information improvement

4.3 Some Another Class Of Measure Of Weighted Directed Divergence And Its Measure Of Weighted Information Improvement Corresponding To Kullback Leibler [9] Measure Of Entropy

Let $\phi(x) = \ln x - x + 1$
(40)

$\phi'(x) = 1/x - 1$
(41)

$\phi''(x) = -1/x^2$ (42)

It is easily seen from (41), (42)

$\therefore$ (38) Gives the following measure of weighted directed divergence as

$D_7(P:Q,W) = \sum_{i=1}^{n} w_i\,[p_i \ln(\frac{p_i}{q_i}) - p_i + q_i] + \sum_{i=1}^{n} w_i\,[(1+ap_i)\ln\left(\frac{1+ap_i}{1+aq_i}\right) - ap_i + aq_i]$ (43)

$D_7(P:R,W) = \sum_{i=1}^{n} w_i\,[p_i \ln(\frac{p_i}{r_i}) - p_i + r_i] + \sum_{i=1}^{n} w_i\,\{(1+ap_i)\ln\left(\frac{1+ap_i}{1+ar_i}\right) - ap_i + ar_i\}$ (44)

(43) and (44) gives measure of weighted information improvement as,

$I_7(P:Q:R,W) = \sum_{i=1}^{n} w_i\,[p_i \ln(\frac{r_i}{q_i}) + (q_i - r_i)] + \sum_{i=1}^{n} w_i\,[(1+ap_i)\ln\left(\frac{1+ar_i}{1+aq_i}\right) + a(q_i - r_i)]$ (45)

$I_7(P:Q:R,W)$ is the measure of weighted information improvement corresponding to Kullback Leibler [9] measure of entropy.

If we put a = 0 in (43) and (45) we get (19) and (21), if we put a = -1 in (43) and (45) we get the following measures as,

$D_7(P:Q,W)=\sum_{i=1}^{n} w_i \left\{p_i ln(\frac{p_i}{q_i}) + (1-p_i) ln\left(\frac{1-p_i}{1-q_i}\right)\right\}$ (46)

Similarly,

$D_7(P:R,W)=\sum_{i=1}^{n} w_i \left\{p_i ln(\frac{p_i}{r_i}) + (1-p_i) ln\left(\frac{1-p_i}{1-r_i}\right)\right\}$ (47)

Which is measure of weighted directed divergence corresponding to Fermi-Dirac measure of directed divergence and its measure of weighted information improvement is as,

When a = -1 in (45) we will get weighted information improvement measure as,

$I_7(P:Q:R,W) = \sum_{i=1}^{n} w_i\,[p_i \ln(\frac{r_i}{q_i}) + (1-p_i)\ln\left(\frac{1-r_i}{1-q_i}\right)]$ (48)

4.4 The Measure Of Weighted Directed Divergence Corresponding To Havrda And Charvat [4] Measure Of Entropy And Its Measure Of Weighted Information Improvement

Let $\phi(x) = \frac{x^{\alpha}-\alpha x+\alpha-1}{\alpha(\alpha-1)}$, $\alpha \neq 0,\ \ \alpha \neq 1$ or $\alpha > 1$ (49)

$\phi'(x) = \frac{x^{\alpha-1}-1}{\alpha-1}$ (50)

$\phi''(x) = x^{\alpha-2}$ (51)

It is easily seen from (50) and (51)

$\phi(1) = 0, \quad \phi'(1) = 0$

Since $\phi(x)$ satisfies (13), (14) and (15) so (38) gives

$D_8(P:Q,W) = \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i \{p_i^{\alpha} q_i^{1-\alpha} - \alpha p_i + \alpha q_i - q_i) + (1+ap_i)^{\alpha} (1+aq_i)^{1-\alpha} - \alpha a(p_i - q_i) - q_i\}$
(52)

$D_8(P:R,W)= \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i \{p_i^{\alpha} r_i^{1-\alpha} - \alpha p_i + \alpha r_i - r_i + (1+ap_i)^{\alpha} (1+ar_i)^{1-\alpha} - \alpha a(p_i - r_i) - r_i\}$
(53)

$I_8(PQ:R,W)=\frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i\{p_i^{\alpha} (q_i^{1-\alpha} - r_i^{1-\alpha})+(\alpha+\alpha a-2)(q_i - r_i) + (1+ap_i)^{\alpha}[(1+aq_i)^{1-\alpha} - (1+ar_i)^{1-\alpha}]\}$
(54)

$I_8(P:Q:R,W)$ is the measure of weighted information improvement corresponding to Havrda and Charvats [4] measure of entropy.
We put a = -1 in (52) we get
$D_8(P:Q,W) =$
$\frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i[p_i^{\alpha}q_i^{1-\alpha} + (1-p_i)^{\alpha}(1-q_i)^{1-\alpha} - 2q_i]$ (55)
Similarly,
$D_8(P:R,W) = \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i[p_i^{\alpha}r_i^{1-\alpha} + (1 - p_i)^{\alpha}(1 - r_i)^{1-\alpha}-2r_i]$
(56)
And (54) becomes

$I_8(P:Q:R,W) = \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i\{p_i^{\alpha} (q_i^{1-\alpha}- r_i^{1-\alpha}) + (1-p_i)^{\alpha} [(1-q_i)^{1-\alpha} - (1-r_i)^{1-\alpha} - 2(q_i - r_i)]\}$
(57)

4.5 Measure Of Weighted Directed Divergence And Its Measure Of Weighted Information Improvement Corresponding To Kapur [8] Measure Of Entropy.

If inequality constraint is imposed on probabilities i.e. $a_i \le p_i \le b_i$, i = 1,2,….,n
(58) the following measure of weighted directed divergence is introduce by Kapur [8]
$D_9(P:Q,W) = \sum_{i=1}^{n} w_i \{(p_i - a_i)\ln\left(\frac{p_i-a_i}{q_i-a_i}\right) +(b_i- p_i) \ln\left(\frac{b_i-p_i}{b_i-q_i}\right)]\}$
(59)
$D_9(P:R,W) =\sum_{i=1}^{n} w_i\{(p_i - a_i)\ln\left(\frac{p_i-a_i}{r_i-a_i}\right) +(b_i- p_i) \ln\left(\frac{b_i-p_i}{b_i-r_i}\right)]\}$
(60)
From (59) and (60) it is easily seen to be a convex function of $p_1,p_2,\ldots.p_n$ whose minimum value zero arises when each $p_i = q_i$
∴ (59) It is used as valid measure of weighted directed divergence.

Therefore the measure of weighted information improvement corresponding to $D_9(P:Q,W)$ is as follows,

$I_9(P:Q:R,W) = \sum_{i=1}^{n} w_i\{(p_i-a_i) \ln\left(\frac{r_i-a_i}{q_i-a_i}\right)+ (b_i - p_i) \ln\left(\frac{b_i-r_i}{b_i-q_i}\right)\}$
(61)
If we put $a_i = 0$ and $b_i = 1$ in (61), i.e. $0\le p_i\le 1$ which is natural constraints
∴ (61) Becomes
$I_9(P:Q:R,W)=\sum_{i=1}^{n} w_i\{p_i \ln\left(\frac{r_i}{q_i}\right) +(1 - p_i) \ln\left(\frac{1-r_i}{1-q_i}\right)\}$ (62)
Which is same as weighted information improvement $I_7(P:Q:R,W)$ corresponding to Kullback Leibler [9] measure of entropy at $a_i= 0$ and $b_i=1$

I_9 (P:Q:R,W) = I_7 (P:Q:R,W)
(63)

5. CONCLUSION

1. All the outcomes weighted directed divergence measure and weighted information improvement measure of one parametric and two parameter mention in section three and four are equally important and open the further space of research, the corrected measures of weighted directed divergence as well as the measures of weighted information improvement are interconnected and out of which all probability distribution satisfying constraints.
2. D_1 (P:Q) has minimum value for distribution P Which outcomes are not equally important for this we can use the modified principal of minimum weighted cross entropy, by using this, out of all probability distribution satisfying given constraints, D_1 (P:Q,W) is minimize for distribution P
3. In the present paper we have given correct directed divergence, weighted directed divergence and information improvement measures for discrete random variable probability distribution and corresponding to this measure we can obtained probability distribution of continuous random variable
4. The term weighted directed divergence is also called as directed divergence with respected to W(x) or useful directed divergence or qualitative -quantitative measure of directed divergence.
5. Here we have not discussed here how to assign the weights partly because such an assignment will depend on the specific application we have in view. Thus the weights may depend on the penalty to be paid due to our being able to meet the demands, these may be due to different returns from different outcomes in stock market or the different losses caused by earthquakes or different intensities and so on.

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