SOME CORDIAL LABELING ON KALAMI MODULAR STAR GRAPH

Piyush V. Patel, Roopal M. Gajjar

(piyushpatel002@gmail.com, roopalgajjar@gmail.com)
Silver Oak University, Ahmedabad, Gujarat, India

Abstract: *In this paper, we described a new graph family called Kalami Modular Star, which is derived from a new origami modular design. We studied different cordial labeling approaches on the Kalami Modular star graph family, including cordial, E-cordial, sign product cordial, and sum cordial labeling.*

Keywords: *Cordial graph, E –cordial graph, signed product cordial graph, even sum cordial graph, Kalami Modular star graph.*

AMS Subject classification (2010): 05C78

1. INTRODUCTION

We begin with simple, finite, undirected graph $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ denotes the vertex set and the edge set respectively. For all other terminology we follow Gross [11]. We provide some useful definitions for the present work.

1.1 Definitions

Definition 1.1: The graph labeling is an assignment of numbers to the vertices or edges or both subject to certain condition(s). A detailed survey of various graph labeling is explained in Gallian [2].

Definition 1.2: [2] For a graph $G = \big(V(G), E(G)\big)$, a mapping $f: V(G) \rightarrow \{0, 1\}$ is called a binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f. For an edge $e = uv$, the induced edge labeling $f^*: E(G) \rightarrow \{0,1\}$ defined as $f^*(uv) = |f(u) - f(v)|$.

Let $v_f(0)$ and $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$ and $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.3: A binary vertex labeling f of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is said to be cordial if it admits cordial labeling. The notion of cordial labeling was introduced by Cahit [1].

Definition 1.4: [3] Let G be a graph and let $f: E(G) \rightarrow \{0, 1\}$. Define f^* on $V(G)$ by $f^* = \sum\{f(uv)/uv \in E(G)\}\,(\mathrm{Mod}2)$. The function f is called an E- cordial labeling of G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called E-cordial if it admits E-cordial labeling.

Definition 1.5: A signed product cordial labeling of a graph G is a function $f: V(G) \rightarrow \{-1,\ 1\}$ such that each edge uv is assigned the label $f(u)\,f(v)$, the number of vertices with label -1 and the number of vertices with label 1 differ by atmost 1 and the number of edges with label -1 and the number of edges with label 1 differ by atmost 1. A graph which admits signed product cordial labeling is called a signed product cordial graph. The notion of signed product cordial labeling was introduced by Babujee and Loganathan [4].

Definition 1.6: Let $G = (V, E)$ be a simple graph and $f: V \rightarrow \{1, 2, 3 \ldots . |V|\}$ be a bijection. For each edge uv, assign the label 1 if $f(u) + f(v)$ is even and the label 0 otherwise. f is called an even sum cordial labeling if $|e_f(0) - e_f(1)| \leq 1$, where $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and not labeled with 0 respectively. A graph with an even sum cordial labeling is called an even sum cordial graph. S.Abhirami et al. introduced Even Sum cordial graph [12].

Definition 1.7: The join of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is the graph with vertex set and edge set as follows: $V(G_1 + G_2) = V(G_1) \cup V(G_2)$, $E(G1 + G2) = E(G1) \cup E(G2) \cup J$, where $J = \{uv : u \in V(G1), v \in V(G2)\}$. Thus J consists of edges which join every vertex of G1 to every vertex of G2.

Definition 1.8: The wheel W_n is join of the graphs n-cycle graph C_n and 1-complete graph K_1. i.e $W_n = C_n + K_1$. Here vertices corresponding to C_n are called rim vertices and C_n is called rim of W_n while the vertex corresponding to K_1 is called apex vertex.

Definition 1.9: The helm H_n, $n \geq 3$ is the graph obtained from the wheel W_n by adding a pendant edge at each rim vertex.

Definition 1.10: The closed helm CH_n, is the graph obtained from a helm by joining each pendent vertex to form a cycle, here vertices corresponding to this cycle are called outer rim vertices and vertices corresponding to wheel except the apex vertex are called inner rim vertices.

Definition 1.11: The web Wb_n, is the graph obtained by joining the pendent points of a helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle, here vertices corresponding to this outer cycle are called outer rim vertices and vertices corresponding to wheel except the apex vertex are called inner rim vertices.

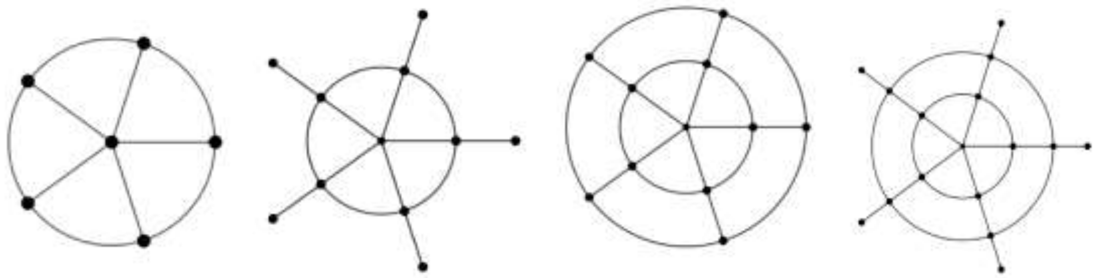

Figure 1: Wheel W_5, Helm H_5, Closed helm CH_5 and Web Wb_5

Origami is the ancient Japanese technique of folding paper. The term origami is derived from two Japanese words: "ori" (to fold) and "kami" (paper). Traditionally, origami models are created just by folding paper. There's no cutting or gluing involved. Origami, while primarily an artistic product, has piqued the curiosity of mathematicians due of its intriguing algebraic and geometrical features. Origami's effect on mathematics extends to graph theory, which can express origami folds as graphs. In origami, each fold or crease can be represented as an edge, and each intersection of creases can be represented as a vertex. The entire origami model can be represented as a graph, with the creases (edges) dividing the paper's surface into regions and the vertices representing the intersections of these creases. To map the concept of origami to graph cordial labeling, first represent an origami model as a graph and then apply the cordial labeling criteria to this graph representation. By modelling origami models as graphs and using cordial labeling, we may investigate novel mathematical characteristics while ensuring that the labels fulfil the specific balance requirements. This approach creates a novel link between the art of origami and the rigor of graph theory, perhaps leading to new insights and applications in both domains. In [1], Cahit developed cordial labelling on graphs as a weaker variant of harmonic and graceful labeling and investigated several graph families in later research. In [5, 6, 7, 8, 9, 10], Gajjar and Prajapati investigated various cordial labels of several origami-inspired graph families. Various mathematicians have studied many variant of cordial labelling on several graph families.

We provide a new graph family modeled after the Kalami Modular Star origami model. We investigated them for various graph cordial labels like cordial labellng, E-cordial labelling, signed product cordial labelling and even sum cordial labeling.

1.2 Construction of Kalami Modular Star (KMS_n):

The Kalami Modular Star graph KMS_n is the graph obtained from web Wb_n by joining each pendant vertex to form a cycle, and then by subdividing each path of the outer cycle with the vertex.

Let $v_1, v_2, \dots, v_n$ be consecutive n vertices of cycle graph C_n. Let $W_n = K_1 + C_n$ be a Wheel Graph having Apex vertex u_0. We form a closed Helm CH_n from W_n by adding rim vertices $w_1, w_2, \dots, w_n$ and joining them together to form an outer cycle. Let Web graph Wb_n be formed by joining n pendent vertices $x_1, x_2, \dots, x_n$. to CH_n.

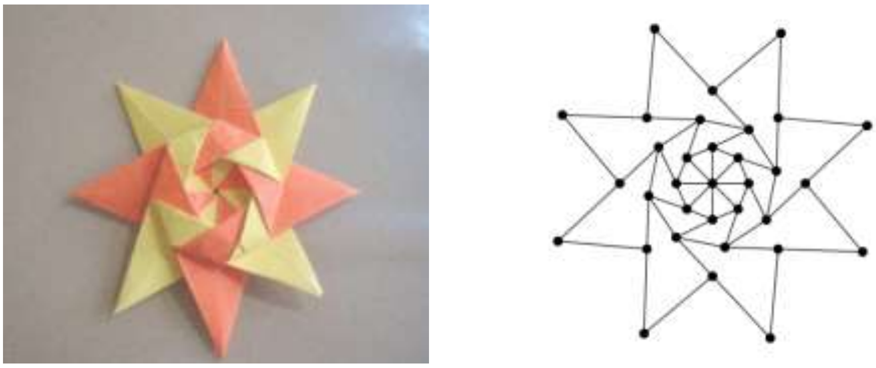

Figure 2: Kalami Modular Star design and Graph

We now form a **Kalami Modular Star graph** KMS_n by joining outer rim vertices of Web graph Wb_n to form a cycle and then by subdividing each our rim edge by adding vertices $y_1, y_2, \dots, y_n$.

Definition 1.12: A graph $G = KMS_n = (V, E)$, with vertex set $V(KMS_n) = \{u_0, v_i, w_i, x_i, y_i / \ i \in [n]\}$ and edge set $E(KMS_n) = \{u_0v_i, v_iw_i, w_iy_i, x_iy_i, v_iv_{i+1}, w_iw_{i+1}, y_ix_{i+1} \ / \ i \in [n]\}$, where the indices i are taken as modulo n, is called Kalami modular star graph.

Note that, order $|V(KMS_n)| = 4n + 1$ and size $|E(KMS_n)| = 7n$.

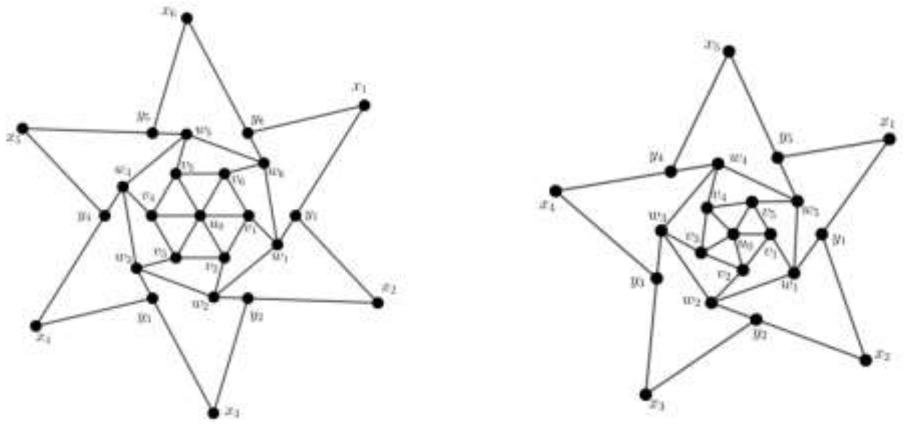

Figure 3: Kalami Modular Star Graph KMS_6 and KSM_5

2. RESULT AND DISCUSSION

We discuss the various cordiality of this graph family. Firstly let's investigate its cordial labelling.

Theorem 2.1: $G = KMS_n$ **is a cordial graph.**

Proof: We define a mapping $f: V(G) \to \{0,1\}$, a binary vertex labelling, such that $f(v) = 0$ or 1 for all vertices $v \in V(G)$ as follows:

$$f(u_0) = 1, \qquad f(v_i) = 0, \qquad f(w_i) = 1,$$

$$f(x_i) = \begin{cases} 0, \text{if i is odd.} \\ 1, \text{if i is even} \end{cases}, \qquad f(y_i) = \begin{cases} 0, \text{if i is odd.} \\ 1, \text{if i is even} \end{cases}, \qquad 1 \le i \le n.$$

Now consider an induced edge labelling by a mapping $f^*: E(G) \to \{0, 1\}$ such that $f^*(e) = |f(u) - f(v)|$ for all $e = uv \in E(G)$ which assigns the labels on edges as follows:

$f^*(u_0v_i) = |f(u_0) - f(v_i)| = 1,$

$f^*(viv_{i+1}) = |f(v_i) - f(v_{i+1})| = 0,$

$f^*(v_iw_i) = |f(v_i) - f(w_i)| = 1,$

$f^*(w_iw_{i+1}) = |f(w_i) - f(w_{i+1})| = 0,$

$f^*(w_iy_i) = |f(w_i) - f(y_i)| = \begin{cases} 1, \text{if i is odd} \\ 0, \text{if i is even} \end{cases},$

$f^*(x_iy_i) = |f(x_i) - f(y_i)| = 0,$

$f^*(y_ix_{i+1}) = f(y_i) \cdot f(x_{i+1}) = 1$, for all $i \in [n]$ where indices are taken as modulo n.

Case 1: When n is even, we have $v_f(0) = n + \frac{n}{2} + \frac{n}{2} = 2n$ and $v_f(1) = 1 + n + \frac{n}{2} + \frac{n}{2} = 2n + 1$. Therefore, $|v_f(0) - v_f(1)| = |2n - 2n - 1| = 1 \leq 1$.

We observe that the number of edges labeled 0 and 1 are $e_f(0) = n + n + \frac{n}{2} + n = \frac{7n}{2}$ and $e_f(1) = n + n + \frac{n}{2} + n = \frac{7n}{2}$ respectively. Therefore, $|e_f(0) - e_f(1)| = 0 \leq 1$.

Case 2: When n is odd, we have $v_f(0) = n + \frac{n+1}{2} + \frac{n+1}{2} = 2n + 1$ and $v_f(1) = 1 + n + \frac{n-1}{2} + \frac{n-1}{2} = 2n$. Therefore, $|v_f(0) - v_f(1)| = |2n + 1 - 2n| = 1 \leq 1$

We observe that the number of edges labeled 0 and 1 are $e_f(0) = n + n + \frac{n-1}{2} + (n+1) = \frac{7n+1}{2}$ and $e_f(1) = n + n + \frac{n+1}{2} + (n-1) = \frac{7n-1}{2}$ respectively. Therefore, $|e_f(0) - e_f(1)| = 1 \leq 1$.

Thus, in both the balance conditions of vertices and edges of a graph to be cordial are verified.

Hence, we prove the graph $G = KMS_n$ is cordial. ■

Example 2.2: KMS_5 **and** KMS_6 **are cordial graphs.**

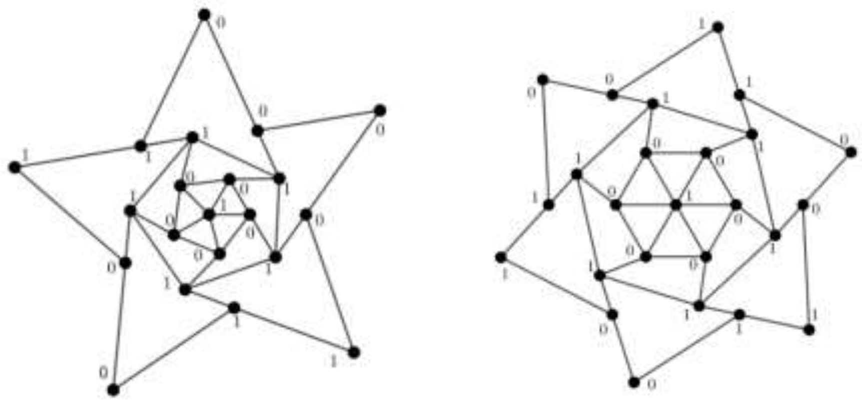

Figure 4: KMS_5 and KMS_6 are cordial graphs

We discuss now another cordial labeling called E-cordial for this graph family.

Theorem 2.3: $G = KMS_n$ **is an E-cordial graph.**
Proof: We define a binary edge labeling mapping $f: E(G) \rightarrow \{0,1\}$, such that $f(e) = 0 \text{ or } 1$ for all edges $e \in E(G)$ as follows:

$f(u_0v_i) = 0$, $f(v_iv_{i+1}) = 0$, $f(v_iw_i) = 1$, $f(w_i, w_{i+1}) = 0$, $f(w_iy_i) = \{1, \text{ if i is odd } 0, \text{if i is even},$, $f(y_ix_i) = 1$, $f(y_ix_{i+1}) = 1$, for $i \in [n]$, where indices are taken as modulo n.

Now consider an induced vertex labeling function $f^*: V(G) \to \{0,1\}$ such that $f^*(v) = \sum\{f(uv)/uv \in E(G)\}(\text{mod } 2)$, for all $v \in V(G)$. This induced labeling on vertices f^*assigns the labels on vertices of G as follows:

$f^*(u_0) = 0,\ f^*(v_i) = 1,\ f^*(w_i) = \begin{cases} 0, \text{if i is odd} \\ 1, \text{if i is even} \end{cases},\ f^*(x_i) = 0, f^*(y_i) = \begin{cases} 1, \text{if i is odd} \\ 0, \text{if i is even} \end{cases}.$

We have, $e_f(0) = n + n + n + \lfloor\frac{n}{2}\rfloor = \lfloor\frac{7n}{2}\rfloor$ and $e_f(1) = n + \lceil\frac{n}{2}\rceil + n + n = \lceil\frac{7n}{2}\rceil$. Therefore, $|e_f(0) - e_f(1)| = \begin{cases} 0, \text{if n is even} \\ 1, \text{if n is odd} \end{cases} \le 1$, that is, number of edges labeled 0 and labeled 1 differs by almost 1.

Now we observe that count of vertices labeled 0 and 1 respectively are as follows:

$v_f(0) = 1 + \lceil\frac{n}{2}\rceil + \lfloor\frac{n}{2}\rfloor + n = 2n + 1$ and $v_f(1) = n + \lfloor\frac{n}{2}\rfloor + \lceil\frac{n}{2}\rceil$=2n.

Therefore, $|v_f(0) - v_f(1)| = 1 \le 1$. Thus $v_f(0)$ and $v_f(1)$ differs by almost 1.

Hence we prove, $G = KSM_n$ is an E-Cordial graph. ■

Example 2.4: KMS_5 **and** KMS_6 **are E-Cordial graphs.**

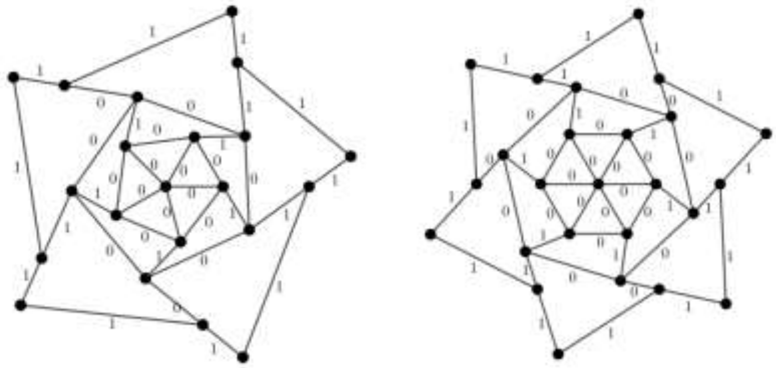

Figure 5: KMS_5 and KMS_6 are E-cordial graphs

We investigate one more cordial labeling called signed product cordial labeling of the Kalami modular star graph family.

Theorem 2.5: $G = KMS_n$ **is a signed product cordial graph.**
Proof: Define a signed vertex labeling map $f: v(G) \to \{-1,1\}$ such that $f(u_0) = 1$, $f(v_i) = -1$, $f(w_i) = 1$, $f(x_i) = \begin{cases} 1, \text{if i is odd} \\ -1, \text{if i is even} \end{cases}$ and $f(y_i) = \begin{cases} 1, \text{if i is odd} \\ -1, \text{if i is even} \end{cases}$ for all $i \in [n]$, where indices are taken as modulo n. We verify that $|v_f(1) - v_f(-1)| \le 1$.

Now observe that the induced edge labeling map $f^*: E(G) \to \{-1,1\}$ assigns edge labeling as follows:

$f^*(u_0v_i) = f(u_0) \cdot f(v_i) = -1,$
$f^*(viv_{i+1}) = f(v_i) \cdot f(v_{i+1}) = 1,$
$f^*(v_iw_i) = f(v_i) \cdot f(w_i) = -1,$
$f^*(w_iw_{i+1}) = f(w_i) \cdot f(w_{i+1}) = 1,$
$f^*(w_iy_i) = f(w_i) \cdot f(y_i) = \begin{cases} 1, \text{if i is odd} \\ -1, \text{if i is even} \end{cases},$
$f^*(x_iy_i) = f(x_i) \cdot f(y_i) = 1,$
$f^*(y_ix_{i+1}) = f(y_i) \cdot f(x_{i+1}) = -1$, for all $i \in [n]$ where indices are taken as modulo n.

Therefore, the number of edges labeled -1 and 1 are $e_f(-1) = n + n + \lfloor\frac{n}{2}\rfloor + n = \lfloor\frac{7n}{2}\rfloor$ and $e_f(1) = n + n + \lceil\frac{n}{2}\rceil + n = \lceil\frac{7n}{2}\rceil$ respectively.

Thus, $|e_f(-1) - e_f(1)| = \begin{cases} 1, & \text{if n is odd} \\ 0, & \text{if n is even} \end{cases} \leq 1.$

Hence, we prove $G = KMS_n$ is signed product cordial graph. ■

Example 2.6: KMS_5 **and** KMS_6 **are signed product cordial graphs.**

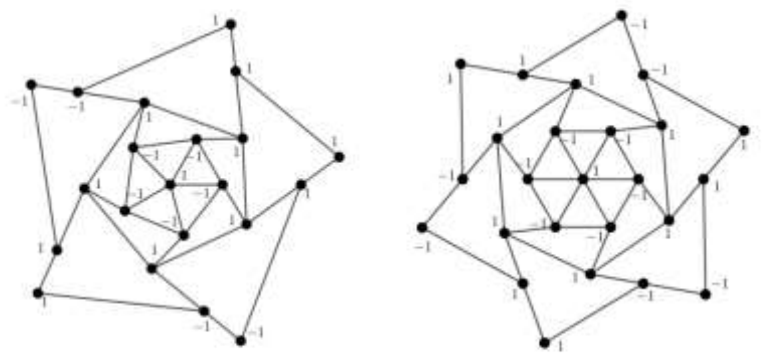

Figure 6: KMS_5 and KMS_6 are signed product cordial graphs

Finally we study even sum cordial labeling on this graph family and we show that its even sum cordial in the following result.

Theorem 2.7: $G = KMS_n$ **is an even sum cordial graph.**
Proof: We define a mapping $f: V(G) \to \{1, 2, \ldots, |V|\}$, a vertex labeling, as follows:

$$f(v) = \begin{cases} 2i, & v = v_i \\ 2i - 1, & v = w_i \\ 2n + i, & v = x_i \\ 3n + i, & v = y_i \\ 4n + 1, & v = u_0 \end{cases}, i \in [n]$$

Now consider an induced edge labeling by a bijection $f^*: E(G) \to \{0, 1\}$ such that $f^*(e) = f(u) + f(v))$ for all $e = uv \in E(G)$. It assigns the edge labels as follows:
$f^*(u_0v_i) = f(u_0) + f(v_i) = odd + even = odd = 0,$
$f^*(v_iw_i) = f(v_i) + f(w_i) = even + odd = odd = 0,$
$f^*(w_iy_i) = f(w_i) + f(y_i) = \begin{cases} odd + odd = even, \text{if i is odd} \\ odd + even = odd, \text{if i is even} \end{cases} = \begin{cases} 1, \text{i is odd} \\ 0, \text{i is even} \end{cases},$

$f^*(y_i x_i) = f(x_i) + f(y_i) = \begin{cases} odd + even = odd, & \text{if i is odd} \\ even + odd = odd, & \text{if i is even} \end{cases} = 0,$
$f^*(y_i x_{i+1}) = f(y_i) + f(x_{i+1}) = even + even = even = 1,$
$f^*(v_i v_{i+1}) = f(v_i) + f(v_{i+1}) = even + even = even = 1,$
$f^*(w_i w_{i+1}) = f(w_i) + f(w_{i+1}) = odd + odd = even = 1$, $i \in [n]$, where indices are taken as modulo n.

Therefore, Edge Count with Label 0 and 1: $e_f(0) = n + n + \lfloor \frac{n}{2} \rfloor + n = \lfloor \frac{7n}{2} \rfloor$ and $e_f(1) = n + n + \lceil \frac{n}{2} \rceil + n = \lceil \frac{7n}{2} \rceil$. Thus we have $|e_f(0) - e_f(1)| = \begin{cases} 1, & \text{if n is odd} \\ 0, & \text{if n is even} \end{cases} \leq 1.$

Hence, we prove the graph $G = KMS_n$ is an even sum cordial. ■

Example 2.8: KMS_5 **and** KMS_6 **are even sum cordial graphs.**

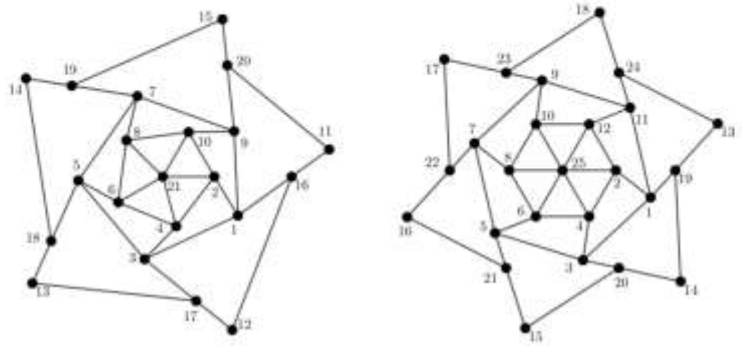

Figure 7: KMS_5 and KMS_6 are even sum cordial graphs

3. CONCLUSION

Using origami, new graph Kalami Modular star graph is introduced. Various graph labeling are applied on it. The graph family admits labeling like cordial, E – Cordial and signed product cordial and even sum cordial labeling. Different cases where these graph family admits or does not admit other labeling is an open problem for the researchers.

4. REFERENCES

4.1. Journal Article

[1] I. Cahit, Cordial Graphs: A Weaker Version of Graceful and Harmonious Graphs, Ars Combinatoria., vol. 23, **(1987)**, pp. 201-207.
[2] J. A. Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics, #DS6, vol. 25, **(2022)**.
[3] R. Yilmaz and I. Cahit, E-cordial graphs, Ars. Combin., Vol. 46 **(1997)**, pp.251-266.
[4] J. B. Babujee and S. Loganathan, On Signed Product Cordial Labeling, Applied Mathematics, vol.2, **(2011)**; pp. 1525-1530.
[5] R.Dhavaseelan, R.Vikramaprasad and S.Abhirami,.A New Notions of Cordial Labeling Graphs, Global Journal Of Pure And Applied Mathematics, vol. 11 no. 4 **(2015)**, pp. 1767—1774.

[6] U. M. Prajapati and R. M. Gajjar, Some labeling techniques of shining alice graph, Journal of Emerging Technologies and Innovative Research, vol. 5, no. 7, **(2018)**, pp. 361-369.

[7] U. M. Prajapati and R. M. Gajjar, Some labeling techniques of holiday star graph, Internat. J. Math. Trends Tech., vol. 49, no. 6, **(2017)**, pp. 339-344.

[8] U. M. Prajapati and R. M. Gajjar Some labeling techniques of Kusudama flower graph, Mathematics Today, vol. 34, no. A, **(2018)**, pp. 211-217.

[9] U. M. Prajapati and R. M. Gajjar, Some labeling techniques of boreale star graph, International Journal of Research and Analytical Reviews, vol. 6, no. 1, **(2019)**, pp. 668-671.

[10] U. M. Prajapati and R. M. Gajjar, Some labeling techniques of christmas star graph, International Journal of Mathematics and Soft Computing, vol. 8, no. 2, **(2018)**, pp. 01-09.

[11] U. M. Prajapati and R. M. Gajjar, Some labeling techniques of braided star graph, International Journal of Mathematics And its Applications, vol. 5, no. 4-C, **(2017)**, pp. 361-369.

[12] R.Dhavaseelan, R.Vikramaprasad and S.Abhirami.A New Notions of Cordial Labeling Graphs, Global Journal Of Pure And Applied Mathematics, Vol. 11 no. 4, (**2015**), pp.1767—1774.

4.2. Book

[1] J. L. Gross and J. Yellen, Handbook of Graph Theory, CRC Press. Boca Raton, **(2004)**.