APPLICATIONS OF FRACTIONAL CALCULUS

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ABSTRACT:
Fractional Calculus is the field of mathematical analysis dealing with the investigation and applications of integrals and derivatives of arbitrary order. Starting from the works of G.W. Leibniz (1695, 1697) and L.Euler (1730) it has been developed up to nowadays. Still there are a number of non-local phenomena unexplored and waiting to be discovered. Most important application of fractional order derivatives is to the prediction of the groundwater flow, investigation in the fields of Physics, Mechanics, Biology, Engineering, Economics, Finances and many areas. Recently their applications are found in viscoelasticity, chaos and fractals, heat transfer, signal processing, robotics, in electric transmission lines, in the formation of autochronous; ultra-sonic wave propagation in human cancellous bones. Modelling of speech signals using fractional calculus; Modelling the Cardiac Tissue Electrode Interface Using Fractional Calculus; Application of Fractional Calculus to the sound Waves Propagation in Rigid Porous Materials; Using Fractional Calculus for Lateral and Longitudinal Control of Autonomous Vehicles; Application of FC to Fluid Mechanics; Fractional Differentiation in edge detection are the main applications of Fractional Calculus to be analyzed. The new fractional order models are satisfying than previous integer-order models. Also, fractional derivatives are excellent tools for describing the memory and hereditary properties of different materials and processes. The formation of bio-heat transfer in one dimension in terms of fractional order differentiation with respect to time is described. Fractional mathematical economics can be considered as the branch of applied mathematics which can deal with economic problems. Fractional Calculus can be used in computer vision for the enhancement, better detection selectivity, developing robust denoising models, and dealing with discontinuities.

Keywords: Chao and Fractals, Bioheat Transfer Thiem' Groundwater Flow Equation, Cancellous bone, Sinc-Fractional Derivative, autochronous problem, Hydraulic conductivity, Hooke's law, elasticity, robotics, Tautochrone. Fractional mathematical economics.

1. INTRODUCTION:

The main subject of the analysis in the field of fractional calculus are fractional derivatives, fractional integrals and their properties. By [9], after the development of classical calculus, Fractional Calculus was formulated in 1695.This Fractional Calculus is closely related to the dynamical complicated real-world problems. Really speaking Fractional Calculus and its applications are in rapid development process. In fractional calculus the research is found to be inherently multi- disciplinary while their applications are performed in different aspects and situations like continuum and quantum mechanics, bioengineering, biomedicine, pollution control, population growth and dispersal, medical imaging and other branches of pure and applied mathematics. From the Riemann- Liouville definition of fractional integral of order α in [4], fractional differentiation and integration was originated historically in the form

$$aD_x^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt.$$

(1) In above (1), $aD_x^{-\alpha} = aJ_x^{-\alpha}$ is known as the Riemann- Liouville integral operator. If $a = -\infty$, above (1) represents the Liouville definition of fractional integral. The paper analyses about the new trends of FC to dynamical systems in control theory, electrical circuits with fractance, tracer fluid flows, numerical computation of fractional derivatives and integration and model of neurons in biology, generalized voltage divider, viscoelasticity. Recently fractional calculus is becoming an interesting new mathematical method of solution of diverse problems in mathematics, science and engineering. Mathematical tools for modelling physical and biological processes can be benefited by fractional calculus. The paper in [12] analyses the applications of fractional calculus for the bioheat transfer problems. In one dimension, the formation of bio- heat transfer in terms of fractional order differentiation with respect to time is described and this bioheat equation in biological systems is used frequently as a first order model of heat transfer. The methods of fractional calculus are developed as the basis for formation and solution of bioheat transfer problem in the borderline tissue regions. As an example of cryogen spray cooling is applied for cooling the skin surface during the laser skin surgery. For the temperature T(x,t) in the tissue developed by Pennes, H.H., on " Analysis of tissue and arterial blood temperatures in the resting human forearm, the generalised one- dimensional bioheat transfer equation can be expressed as

$$\rho c\frac{\partial T(x,t)}{\partial t} = K\frac{\partial^2 T(x,t)}{\partial x^2} + \omega_b\rho_b c_b\,(T_a - T(x,t)) + Q_m + Q_r(x,t)$$

(2) where ρ, c, and K are the density , specific heat and thermal conductivity of the tissue and ρ_b, c_b the density and specific het of the blood temperature (constant), Q_m is the metabolic het generation , ω_b is the blood perfusion, T_a is the arterial blood temperature (constant) and $Q_r(x,t)$ is the heat generation due to spatial heating in the medium. Fractional differentiation's memory property can make improvement the accuracy of processing large-sized input images. In [3], for modelling the groundwater problem into the mathematical formulation is taken as one of the most difficult real-world tasks till now. But understanding the nature of the medium through which the flow takes is important because this medium can alter its direction from one point to another as well as from one period to another.

Thiem' Groundwater Flow Equation:

This ordinary differential equation of the Theim fractional groundwater flow can express the water level's change as a function of distance during the pumping test of

$$D_{rr}^{\alpha}\,\Phi(r) + \frac{1}{r}\,\Phi(r) = 0,\ 1 < \alpha \leq 2$$

(3) Subject to the condition, $Q = 2\pi T D_r(\Phi(r_b))$. To derive the analytical solution of this equation, the Laplace transform is used. Lastly the solution of the fractional- Thiem groundwater flow equation is in the form of

$$\Phi(r) = a_1 r^{\alpha-1}\ {}_0\psi_1[(\alpha,\alpha-1)\,|\frac{x^{\alpha-1}}{\alpha-1}\,] + a_2\sum_{m=2}^{l} b_m(m-1)\,\Gamma\frac{[\frac{(\alpha-m)}{(\alpha-1)}]}{(m-1)}\ r^{\alpha-m}$$

$$X\ {}_1\Psi_2[(\alpha+1-m,\alpha-1),\left(\frac{\alpha-m}{\alpha-1},1\right)|\frac{x^{\alpha-1}}{\alpha-1}\,]$$

(4) Here in this case l = 2.

Time- Fractional Theis Groundwater Flow Equation: -

For developing a formula of unsteady state flow that introduces the time factor and the storativity first in 1935 was Theis. In his work he got the knowledge that the influence of the discharge can enlarge s outward with time when a well- penetrating extensive confined aquifer is pumped at a constant rate. Then from the analogy between the flow of groundwater and the conduction of heat, an unsteady- state (or Theis) equation was derived. This unsteady – state of equation is possibly the most widely used partial differential equation in groundwater investigations.

$$SD_t\Phi(r,t) = TD_{rr}\Phi(r,t) + \frac{1}{r} D_r\Phi(r,t) .$$

(5)

3. APPLICATIONS OF FRACTIONAL CALCULUS:

Fractional Order Derivative is used to predict the Groundwater Flow as analysed in [3]: It is taken that Groundwater problem is one of the most difficult real-world problems for modelling into mathematical formulation. For modelling such problems accurately, the analysis of behaviour of the medium through the water is moving is important. This medium can change from one point to another and also from one period to another. The hydraulic conductivity of an aquifer can differ from one direction to another. Though many scholars tried for an effective model for predicting the movement of water through the aquifer there are some lacks. Recently real problems modelled via fractional order derivative can present better results when matching their mathematical representations with experimental data. J.F. Botha and A.H. Cloot on " A generalized groundwater flow equation using the concept of non- integer order derivatives" presented some good results by popularising the groundwater flow equation to the concept of fractional order derivatives. A. Atangana on "Numerical solution of space-time fractional order derivative of groundwater flow equation" surveyed a suitable solution of the generalized groundwater flow equation via the Frobenius methods showing better prediction. So far there is no appropriate analytical mathematical expression which can describe the solution of the fractional groundwater flow equation [3]. An aquifer or pumping test is conducted for evaluating an aquifer by stimulating the aquifer through constant pumping and noting the aquifer's response in observation wells. Generally aquifer tests are interpreted by using an analytical model of aquifer flow to match the observed data in the real world.

In the field of fractional calculus, fractional derivatives, fractional integrals and their properties are studied mainly [1]. The physical and geometric interpretations of fractional order integration and differentiation are not clear in general. For more than 300 years, there was no any reliable geometric and physical interpretation of these operations from the origin of concept of arbitrary order differentiation and integration. After this, Igor Podlubny had shown that geometric interpretation of fractional integration is "shadows on the walls" and its physical interpretation is "shadows of the past". The continued research gave their uses in the studies of viscoelastic materials as well as in many fields of science and engineering including fluid flow, rheology, electrical networks, electromagnetic theory and probability. The first application of semi derivative that means derivative of order $\frac{1}{2}$ was done by Abel which is related with the solution of the integral equation for the tautochrone problem. This tautochrone problems are dealing with the determination of the slope of the curve with time of gravity is independent of the starting point. The applications of fractional calculus are conveniently analysed as the following.

(i) In Autochronous Problems:

Application of fractional calculus is found in the formation of the autochronous problem dealing with the resolution of a frictionless plane curved shape through the origin in a vertical plane. Let a particle of mass m be fallen along such plane during a time which is independent of the starting position. If the sliding time T constant, then the Abel integral equation turns out into the fractional integral equation

$$T\sqrt{2g} = \Gamma\left(\frac{1}{2}\right)_0 D_0^{-\frac{1}{2}} f'(\eta)$$

(6) where g is the acceleration due to gravity, η is the initial position and s = f(y) is the sliding curve equation.

(ii) Electric transmission lines: The concept of fractional derivatives in the study of electric transmission lines was developed and introduced successfully with the introduction of symbol ‘p’ for the differential operator $\frac{d}{dt}$ by Heaviside giving the solution of the diffusion equation $\frac{\partial^2 u}{\partial x^2} = a^2 p$. (7)

(iii) Ultrasonic wave propagation in human cancellous bone: Fractional calculus is used for describing the viscous interactions between fluid and solid construction. For a slab of cancellous bone in the elastic frame, reflection and transmission scattering operators are obtained by using Blot’s theory.

(iv) Shaping the Cardiac Tissue Electrode Interface Using Fractional Calculus: To all forms of biopotential recording like ECG, EMG, EEG and functional electrical stimulation such as pacemaker, cochlear input, deep brain stimulation, the tissue electrode interface is very common. The fractional order models extended by generalization of the order of differentiation through modification of the defining current- voltage relationships can provide a better description of observed bio electrode behaviour.

(v) Modelling of speech signals using fractional calculus: The novel approach for speech signal modelling using fractional calculus is provided which is found to be contrasted with the celebrated Linear Predictive Coding (LPC) approach based on integer order models. By using a few integrals of fractional orders as basic functions, the speech signal can be modelled correctly.

(vi) Fractional Calculus for Lateral and Longitudinal Control of Autonomous Vehicles: Fractional Order Controllers (FOC) can be applied to the path tracking problems in an autonomous electric vehicle. For implementing conventional and fractional order controllers, a lateral dynamic model of an industrial vehicle has been considered.

(vii) Fractional Calculus to the sound waves propagation in Rigid Porous Materials: Asymptotic expressions of stiffness and damping inporous materials are proportional to fractional powers of frequency and hence time derivatives of fractional order might describe the behaviour of sound waves in such materials covering relaxation and frequency dependence.

(viii) Fractional differentiation for edge detection: Edge detection in image processing often uses of integer- order differentiation operators. But using an edge detector based on fractional differentiation can improve the criterion of thin detection, or detection selectivity for parabolic luminance transitions, and the immunity criterion to noise.

(ix) Fractional Calculus to Fluid Mechanics: To the solution of time-dependent, viscous-diffusion fluid mechanics problems, fractional calculus can be applied. The application of Fractional calculus with the Laplace transform method to classical transient viscous- diffusion equation in a semi- infinite space can be shown for yielding explicit analytical solutions for the shear stress and fluid speed anywhere in the domain. The fractional methodology is found to be validated, simpler and powerful than the prevailing tools when the fractional results obtained for boundary shear- stress and fluid speed is compared with the prevailing analytical results for first and second stokes problems.

(x) Fractional Calculus in the theory of viscoelasticity: In the studies of complex moduli and impedances for various models of viscoelastic substances methods and tools of fractional derivatives are used. Also, in the theory of viscoelasticity, fractional derivatives can afford the possibilities for getting constitutive equations for elastic complex modules of viscoelastic materials. The application of fractional derivative in [5] is explained in a realistic model example arising in mechanics. For explaining the behaviour of certain materials under the influence of external force is the main target. For dealing such the laws of Hooke and Newton are used. The relation between stress σ(t) and strain ε(t) where they are functions of time t is the interested part. For viscous liquids, Newton's law

$\sigma(t) = \eta D^{\prime}\varepsilon(t)$

(8) is the chosen tool. η is the viscosity of the material.

And the Hooke's law

$\sigma(t) = ED^{0}\varepsilon(t)$

(9) where E is modulus of elasticity of material and D^0 is the identity operator. In an experiment where the strain is manipulated in a controlled model that $\varepsilon(t) = t$ for $t \in [0, T]$ and $T > 0$ then stress becomes $\sigma(t) = Et$

(10) In elastic solid and $\sigma(t) = \eta$ = constant

(11) for a viscous liquid. Summarising (10) and (11),

$\psi_k = \frac{\sigma(t)}{\varepsilon(t)} t^k$

(12) where $\psi_0 = E$ and $\psi_1 = \eta$.And K = 0 refers to the Hooke's law for solids and k = 1 refers to Newton's law for liquids. It is common to find viscoelastic materials exhibiting a behaviour somewhere between the pure viscous liquid and the pure elastic solids that a relationship of the form (12) with $0 < k < 1$. In addition to φ_k, k is second material constant. Polymers and some types of biological tissue may possess such property along with metals like aluminium as an example under particular temperature and pressure conditions.

By a formula $\sigma(t) = const.t^{-k}$

(13) For a constant strain ε, the stress in such material will be developed converging to 0 for very long observation times. It lies once between viscous liquid for $\sigma \to 0$ and an elastic solid whose stress σ is non zero constant. Due to the new possibilities of fractional calculus which can bring into the modelling of many different problems it has been demanding its applications in engineering, physics, and economics from the last two decades.

Tomas Kisela Four applications are discussed [10]: tautochrone problems, hydrogeology applications, fractional oscillator and fractional viscoelasticity as

(1) The Tautochrone problem: It is about the finding of a curve in the (x,y)- plane where the time of sliding down of a particle on a curve to its lowest point is free of its initial placement on the curve. In a homogenous gravity field without friction , the lowest point of the curve at origin and initial point (x,y)

as the position of a curve in the first quadrant and (x^*,y^*) as any point between (0,0) and (x,y). If y , m and g be the length along the curve measured from the origin, mass of the particle and the acceleration due to gravity, then by the energy conservation law

$$\frac{m}{2}\left(\frac{d\sigma}{dt}\right)^2 = mg\,(y - y^*),$$

(14) For $\frac{d\sigma}{dt} < 0$ and $\sigma = \sigma(y^*(t))$ .Then the formula is rewritten as

$$\frac{1}{2}\left(\frac{d\sigma}{dt}\right)^2 = g\,(y - y^*),$$

$$\Rightarrow \left(\frac{d\sigma}{dt}\right)^2 = 2g(y - y^*)$$

and also operating derivative $\sigma = \sigma(y^*(t))$ w.r.t 't'

$$\Rightarrow \sigma^* \frac{dy^*}{dt} = -\sqrt{2g(y - y^*} \quad \text{as } \frac{d\sigma}{dt} < 0.$$

(15) Operating integration from $y^*= y$ to $y^*= 0$ and from $t = 0$ to $t = T$ and after further calculation the following integration equation can be attained.

$$\int_0^y \frac{\sigma\prime(y^*)}{\sqrt{y-y^*}}\,dy^* = \sqrt{2g}\,T$$

(16) Here the Caputo differintegral can easily be recognized and is written as

$${}^{C}D_0^{\frac{1}{2}}\,\sigma(y) = \frac{\sqrt{2g}}{\Gamma(\frac{1}{2})}\,T$$

(17) where T is the time of decent and hence constant. A relation between the length along curve and the initial position in y direction can be obtained by applying composition formula of Caputo differintegrals as

$$\sigma(y) = \frac{\Gamma(1)\sqrt{2g}}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}T\,y^{\frac{1}{2}} = \frac{2\sqrt{2g}T}{\pi}\,y^{\frac{1}{2}}$$

(18)

By $\frac{d\sigma}{dy} = \sqrt{1 + (\frac{dx}{dy})^2}$,

coordinates of points generating the curve is obtained . Substituting $\sigma(y)$, $\frac{dx}{dy} = \sqrt{\frac{2gT^2}{\pi^2} - 1}$. Solution of this equation is tautochrone whose parametric equations are $X = \frac{A}{2}[\,u + \sin(u)]$, $y = \frac{A}{2}[\,1 - \cos(u)]$, where $A = \frac{2gT^2}{\pi^2}$.

In special case for $T = \frac{\pi}{\sqrt{2g}}$ or A = 1, the tautochrone is drawn in the figure. Thus the knowledge of the rules of fractional calculus is very useful for solving this type of integral equations.

(2) Fractional Oscillator : In classical mechanics, the harmonic oscillator is the basic model and generalization to fractional derivative in the equation of classical oscillator is in progress.

$$D_{-\infty}^{\alpha} x(t) + \omega^{\alpha} x(t) = f_0 \omega^{\alpha-2} g(t),$$

(19) where $\alpha \in (1,2)$ is the order of the derivative, ω is the vibration eigen frequency, x(t) the displacement from the equilibrium, g(t) is the source function and f_0 is a constant. For two bodies linked by a spring with dissipation shows an interesting behaviour of the system resulted by using the fractional derivatives.

(3) In Viscoelasticity: Viscoelasticity belonging to application fields of differintegrals and having two main quantities stress and strain is scientific discipline explaining the behaviour of materials.From the lecturer notes of Dr. Prakash P. [7] on "Fractional Differential Equation: Challenges, Importance and Applications" , some applications of fractional calculus are analysed. Calculating time- fractional derivative of a function f(t) at some $t = t_1$ takes all f(t) from $t = 0$ to $t = t_1$. Fractional derivatives can be applied for modelling systems with memory. Calculating space- fractional derivative of a function f(x) at $x = x_1$ requires all non-local f(x) values. Fractional derivatives can be used for modelling distributed parameter systems. Applications of Fractional Calculus from this lecture notes are discussed Fractional order model of neurons in biology.

Electro chemistry and tracer fluid flow.

Fractional order multi poles in electro magnetism.

Generalized voltage divider.

Electrical circuits with fractance.

Modeling of systems with memory.

Dynamical processes with self-similar structures.

Acoustic wave propagation in inhomogeneous porous material.

As a new generalization of the classical PID controller, the idea of $PI^{\lambda}D^{\mu}$ controller, involving fractional order integration and fractional order differentiator has been found to be a more efficient control of fractional order dynamical systems.

Two applications are discussed [11] as:

One is mortgage problem: The equivalence fractional differential equation with RL-derivative and its solution using LT is written by applying the mathematical framework.

Here a problem for a mortgage of amount Rs. P is given at fixed interest rate of r% per month which is to be paid back with interest in n months. The main work is to find the monthly instalment, say Rs. X with condition that the total due is to be cleared within a specified time. Supposing the total debt at the end of t months is denoted by F(t), then this will satisfy the following difference equation

$$F(t + \Delta t) = f(t)(1 + r\,\Delta t) - X.\,\Delta t$$

(20) where X. Δt is the instalment amount to be paid in Δt fraction of time. Taking Δt→0 and after rearranging the terms, we get

$$Df(t) = r\,f(t) - X;\ \ f(0) = p\ ;\ D = \frac{d}{dt}$$. Then solution of this is

$$f(t) = pe^{rt} - \frac{x}{e}\,(e^{rt} - 1)$$

(21) Debt at time t = n is zero or f(n) = 0. Monthly instalment amount is as follow:

$$X = \frac{pre^{rn}}{[e^{rn} - 1]}$$

(22) Considering $e^{r} \approx (1 + r)$ for small value of r. From (22) the following relation is obtained as:

$$X = \frac{pr(1+r)^{n}}{(1+r)^{n} - 1}.$$

$$D^{\alpha} f(t) = r\,f(t) - X \text{ for } 0 < \alpha < 1$$

(23)

$$f_{\alpha}(t) = pt^{\alpha-1} E_{\alpha,\alpha}(rt^{\alpha}) - Xt^{\alpha} E_{\alpha,\alpha+1}(rt^{\alpha})$$

(24) It can be verified easily that the solutions of

$f(t) = pe^{rt} - \frac{x}{e}\,(e^{rt} - 1)$ and $f_{\alpha}(t) = pt^{\alpha-1} E_{\alpha,\alpha}(rt^{\alpha}) - Xt^{\alpha} E_{\alpha,\alpha+1}(rt^{\alpha})$ from (21) and (24) are the same for $\alpha = 1$.

Also the debt $f_{\alpha}(t)$ must be zero at time t = n for any $\alpha \epsilon$ (0,], i.e., $f_{\alpha}(n) = 0$ then from (24), the monthly instalment amount as :

$$X = \frac{pE_{\alpha,\alpha}(rn^{\alpha})}{nE_{\alpha,\alpha+1}(rn^{\alpha})}$$

(25) Here also the two instalment amounts from (22) and (25) are the same for $\alpha = 1$.

Another application is in the motion of the fractional oscillation: I. In quantum physics, electrodynamics, lattice vibrations and phonons, this equation can be used as an approximation of many phenomenon. The equation of motion of the simple harmonic oscillator is:

$$m\frac{d^{2}x}{dt^{2}} + kx = 0.$$

(26) It is equivalent to integral equation:

$$x(t) = x(0) + x^{l}(0)t - \omega^{2} \int_{0}^{1} (t - u)\,x(u)du, \quad 1 < \alpha$$

Here x(0) and $x^{l}(0)$ are the initial displacement and initial velocity and $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency.

$Generalization\ of\ RHS\ of\ above\ integral\ equation\ to\ fractional\ integral\ of\ order\ \alpha > 0\ \ gives$ the equation of motion of fractional oscillator as:

$$x(t) = x(0) + x^{I}(0)t - \frac{\omega^2}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1} x(u)du, \quad 1< \alpha \leq 2.$$

(27)Application of fractional calculus in modelling and solving the bioheat equation by [12]: Mathematical tools for modelling physical and biological processes is provided by fractional calculus. In one dimension, the formation of bio- heat transfer in terms of fractional order differentiation with respect to time is described and this bioheat equation in biological systems is used frequently as a first order model of heat transfer. The methods of fractional calculus are developed as the basis for formation and solution of bioheat transfer problem in borderline tissue regions. As an example of cryogen spray cooling is applied for cooling the skin surface during the laser skin surgery. The generalized one-dimensional bioheat transfer equation for the temperature T(x,t) in the tissue developed by Pennes, H.H can be expressed as

$$\rho c\,\frac{\partial T(x,t)}{\partial t} = K\,\frac{\partial^2 T(x,t)}{\partial x^2} + \omega_b \rho_b c_b\,(T_a - T(x,t)) + Q_m + Q_r(x,t)$$

(28) where ρ, c, and K are the density , specific heat and thermal conductivity of the tissue and ρ_b, c_b the density and specific het of the blood temperature (constant), Q_m is the metabolic het generation , ω_b is the blood perfusion, T_a is the arterial blood temperature (constant) and Q_r(x,t) is the heat generation due to spatial heating in the medium. FC has emerged as a powerful and efficient mathematical instrument during the past six decades, mainly due to its demonstrated applications [9]. In this journal applications of fractional calculus are analysed as

(1) Application of FC in physics: In the case of applied physics, several techniques of fractional calculus are found to be applied in the statement of chaotic systems and random walk problems, in biophysics, in polymer material science etc.

(2) FC technique in random optimal search: Using the scheme of optimizing random search is a common approach to the movement patterns of animals. On the study of different animals' foraging movements deep attentions are taken by many researchers. Most animals become a Levy flight when foraging and their movements are found to fit closely to a Levy distribution (power law distribution) due to law of density of food items.

(3) Sinc- Fractional Derivative on Shannon wavelets:For computing the fractional derivative of the L2(R) – functions of Hilbert space by Shannon wavelet, the sinc- functional operator will be generalized. It takes special part in applications.

(4) Linear visco elastic response functions and the Caputo- Fabrizio fracctional operator:
Motivated with exponential kernel of fractional operator and the efforts for showing the existing knowledge and techniques of data treatment in the framework of linear visco elasticity can creat the Caputo- Fabrizio fractional operator. There are cases where the experimental behaviors of viscoelastic materials can exhibit strong deprtures from the power- law.

(5) Fractional Calculus and Electrical spectro scopy impedence:
The electrical spectroscopy impedance technique plays experimentally an important role to obtain information about the electrical properties of different materials particularly for liquids. To solve the problem created by low frequency limits and describe the experimental nature in all frequency range, well-established features of fractional calculus as well as performing suitable changes in boundary conditions can be applied.

(2) Application of Fractional Calculus in mechanics:

Though the analysis of applications of complex number order derivatives are in rare case, there are some application areas it can be adopted in engineering mechanics.

(i) Fractional calculus and Newtonian Mechanics: For a point- particle of constant mass m, One-dimensional Newtonian mechanics is based upon Newton's second law of motion of second-order ODE.

$$\frac{d^2x(t)}{dt^2} = \frac{F}{m} \text{………(1).}$$

(29) Using fractional calculus two possible ways of generalizing Newton's second law----

- Changing the order of the time derivative in LHS of (29) to an arbitrary number q.
- Generalizing the expression of the force F on the RHS of (29) to include differintegrals of arbitrary order q.

modelling oil pressure:
To relate fluid motion to pressure and gravitational gradients Darcy's law can be applied. Continuity Equation with Darcy's Law can form heat- conducting D.E in mathematical physics describing the transfer of the fluid.
(iii) Micro flows of visco elastic fluids with fractional constitutive relationships: Recently the micro flows of viscoelastic fluids have been studied extensively Since the importance and applications in micro fluidic systems, the micro flows of viscoelastic fluids have been studied extensively nowadays.
(iv) Unsteady flow towards subsurfaced drains:
By analytically solving Boussines q equation (BE) Glover- Dumm equation (GDE) was produced. Because of having non-locality property, fractional derivatives can reduce the Scale effects on the parameters can be reduced by fractional derivatives due to its non-locality property and as a result better reproduce the hydro- geological processes and hereby a fractional BE is developed.
(3) Application in Biology:
(i) Bio Heat Transfer Equation: The methods of FC are developed as the basis for formulation and solution of the bioheat transfer problem in peripheral tissue areas. Recently a number of investigators have applied the bioheat transfer model to periodic diffusion problems in localized tissue regions. FC is ideally suited to address this kind of periodic heating or cooling.
(ii) Models of bone remodeling and bone tumors using variable order derivatives:-- Cells of bone tissue are always dying and being replaced. The changes in dynamic behaviour when there is a tumor can be modelled by tuning the parameters of au-tocrine and paracrine effects.
(4) Application of fractional calculus in signal and image processing:
(i) A study on fractional calculus applications in image processing: Recently fractional calculus has been significantly examined in computer version . The fractional- order derivative operator has a non-local behavior. For low frequency signal, fractional differential lessens the signal not as much as the integer one and for high- frequency one, fractional differential improves signal not as much as the integer one. Thus, fractional differential can upgrade the high-frequency signals, and reinforce the medium frequency one, while non-linear retain the low- frequency one.
(ii) Application of the GPCF and DGIs for improving the resolution and quality of nano images.
For improving the quality and sharpness of nano images in the range of resolution (10- 1000) mm, the generalized pearson correlation function (GPCF), POLS and discrete geometrical invariants (DGI) are applied.
(iii) NAFAS Sinaction: intermediate fractal model for the fitting of complex systems data:
The NAFASS (Non-orthogonal Amplitude Frequency Analysis of the Smoothed Signals) approach is modernized and opens an alternative way for creation of new fluctuation spectroscopy when the segment of the Fourier series can fit any random signal with trend. (5) Application of Fractional Calculus in Control: (i) Application of D-decomposition technique in solving some controlproblems: Application of D- decomposition technique,

conceived by the Russian scientist Neimark during the 1950s, can be extended for linear fractional order systems and gives powerful tool for the analysis of systems stability and performance. The D-decomposition method can be successfully used to solve a problem of asymptotic stability of inverted pendulum systems controlled by a fractional order controller.

(ii)The application of fractional order control for an air- based precision positioning system:
Precision bandwidth (speed) and stability of motion are the most important performance indexes of any motion system and for this fractional order PID has proven to be very effective for improving the performance. By floating a silicon wafer on a thin film of air, a contactless precision positioning system is designed. The system has been controlled in which two cascade single-input/ single- output controllers are designed. By using only the fractionality, the bandwidths are extended by 14.6 % and 62%, for the inner and outer loops.
(6) Application of fractional calculus in Engineering:
(i)Tuning of PID controllers using Fractional calculus concepts:
The PID controllers are the most commonly used control algorithms in industry. Among the various existent schemes for tuning PID controllers, the Ziegler- Nichols (Z-N) method is the most popular and is still extensively used for the determination of the PID parameters.
(ii) Fractional PD^{α} Control of a Hexapod Robot:
The dynamic model for the hexapod body and foot- ground interaction is presented. It is considered robot body compliance because walking animals have a spine that allows supporting the locomotion with improved stability. The robot body is divided into n identical segments each with mass $M_b n^{-1}$ and a linear spring – damper system is adopted to implement the inter body compliance.
(iii) In Heat Diffusion:
The heat diffusion is governed by a linear one- dimensional partial differential equation (PDE) involving the solution of parabolic type of the form: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ where k is the diffusivity, t is the time, u is the temperature, and x is the space coordinate. In economic models, differential equations with derivatives of integer orders can not describe processes with memory and non-locality [13]. New economic methods as well as models are needed bearing in mind that economic agents may remember the changes of economic indicators and factors in the past affecting the agents' behaviour and their decision making. For describing their behaviour standard mathematical apparatus of differential equations of integer order cannot be used. The new revolution that is "Memory revolution" is intended is included in the modern economic theory and mathematical economics different processes with long memory and non-locality. The theory of fractional calculus, fractional differential and difference equations are the mathematical tools designing to "cure amnesia" in economics providing the emergence of fractional mathematical economics.
(i) ARFIMA: This stage is characterized by models with discrete time and application of the Grunwald-Letnikov fractional differences.
(ii) Fractional Brownian Motion (Mathematical Finance) stage: Financial models and the application of stochastic calculus methods and stochastic differential equations are the characterized stages. The fractional mathematical finance is a field of applied mathematics concerning with mathematical modelling of financial markets by using the fractional stochastic differential equations.
(iii) Econophysics stage: This stage is mainly characterized by the financial models and the application of physical methods and equations. In this stage the fractional calculus was applied mainly for financial processes.
(iv) Deterministic chaos stage: This stage is characterized by financial and economic models and application of methods of nonlinear dynamics.
(v) Mathematical Economics stage: Macroeconomic and microeconomic models with continuous time and generalization of basic economic concepts and notions. By this paper, applications of fractional calculus are analysed as:
(i) Fractional- order dynamical systems in control theory:

New and effective methods for the time- domain analysis of fractional order dynamical systems are required for solving problems of control theory. As a new generalization of the classical PID-Controller, the concept of $PI^{\lambda}D^{\mu}$-Controller, fractional order integrator and fractional-order differentiator has been found to be more efficient control of fractional- order dynamical systems. In the time domain, a dynamical system is expressed by the fractional order differential equation (FDE) as

$$[\sum_{k=0}^{n} a_{n-k} D^{\alpha_{n-k}}]y(t) = f(t)$$

(30) where $\alpha_{n-k} > \alpha_{n-k-1}$ (k = 0,1,2,……,n) are arbitrary real numbers, α_{n-k} are arbitrary constants, and $D^{\alpha} = {}_{0}^{C}D_{t}^{\alpha}$ denotes Caputo's fractional- order derivative of order α . Fractional- order transfer function (FTF) associated with the D.E is given by

$$G_n(s) = [\sum_{k=0}^{n} a_{n-k}\, s^{\alpha_{n-k}}]^{-1}.$$

(31)

(ii) Electrical Circuits with fractance: Resistors, capacitors described by integer-order models are contained in classical electrical circuits. Fractance in circuits represents an electrical element with fractional- order impedance. Two kinds of fractances (i) tree fractance and (ii) chain fractance are considered. The impedance of the fractance is given by

$$Z\,(i\omega) = \sqrt{\frac{R}{C}}\,\omega^{-1/2}\,\exp\left(-\frac{\pi i}{4}\right).$$

(32)

(iii) Generalized voltage divider: Both the tree fractance and chain fractance consist not only of resistors and capacitors properties, but also exhibit electrical properties with non-integer order impedance.Westerlund generalized the classical voltage divider in which the fractional-order impedaances F_1 and F_2 represent impedances of tree fractence and chain fractance. The transfer function of Westerlund's voltage divider circuit is given by $H(s) = \frac{k}{s^{\alpha}+k}$, $-2 < \alpha < 2$ and k is constant depending on the elements of the voltage divisor.

(iv) Fractiona Calculus in viscoelasticity:

Any viscoelastic material may be treated as a linear system with stress (or strain) as excitation function input) and the strain (or stress) as the response function (output).According to Bagley, the unmodified power law is a special case of the general fractional calculus model while modified power law is associated with the fractional calculus model of viscoelasticity.

(v) Fractional- order model of neurons in biology:

The main function of vestibule-ocular reflex (VOR) model is to keep the retinal image stable by producing eye rotations which counter balance head rotations. The fractional –order dynamics of vestibule-oculomotor neurons suggests that fractional-order rather than integer-order forms of signal processing occur in the vestibule- oculomotor system.

(vi)Numerical computation of fractional derivatives and integration:

Using numerical computation of fractional derivatives and integrals is important and necessary to have good approximation of the fractional integral operator J^{α} for the numerical treatment of fractional differential and integral equations.

(vii) Fractional Calculus in electrochemistry and tracer fluid flows:

(viii) Fractional-order multipoles in electromagnetism:

The axial multipole expansion of the electrostatic potential of electric charge distribution in three dimensions is $\Phi_n(r) = \frac{q}{4\pi\varepsilon}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\, p_n\ (\cos\theta)$ (33)

where q = electric monopole moment, ε is constant permittivity of the homogenous isotropic medium $p_n\ (\cos\theta)$ = Legendre function of integer-order n.

The electrostatic potential functions for quadrupole (2^2) is given as

$$\Phi_2(r) = \frac{q}{4\pi\varepsilon}\sum_{n=0}^{\infty}\frac{1}{r^3}\, p_2\ (\cos\theta)$$

(34)

Engheta generalized the idea of the integer-order multipoles related to powers of 2 to the fractional order multipoles called 2^{α}-poles.

The paper [2] presents the applications of fractional calculus in six different domains such as edge detection, optical flow, image segmentation, image de-noising, image recognition, and object detection. In this thorough review of the application of fractional calculus in computer vision is clearly presented. Various integer-order differential image-processing methods re generalized to fractional order.By using Riemann Liouville derivative, GL derivative, Caputo derivative and Caputo- Fabrizio derivative many edge detection operators have been grown.

Applications in [6] are discussed as:

(1). Fractional–order Speech Modeling:

Linear Predictive Coding (LPC) known as integer order model is a method used mostly in audio signal processing and speech processing. Speech signal modelling is totally depend on it. In discrete form, a speech signal can be expressed as a linear combination of its fractional derivatives as $\hat{x}$ (n) = $\sum_{i=1}^{Q}\mu_k\, D^{\alpha_i}\, x$, where x(n) = n

(35) sample long speech analysis frame. $\{\mu_k\}$ = the fractional LP coefficients

$$D^{\alpha}x(t) = \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)j}{j!}\ \text{x(t-jh)}\ ;\ \text{h = the step size.}$$

(36)

(2). Fractional RC Electrical Circuit:

The governing ODE of an RC electrical circuit is

$$\frac{dV(t)}{dt} + \frac{V(t)}{RC} = \frac{\mu(t)}{RC}$$

where V is the voltage, R = resistance, C = capacitance and (t) = the source of volt.

(3). Application to Geo- Hydrology:

Mittag- Leffler function consolidates the effect of memory. This is the basic for investigations of groundwater flow. The memory effect is important for water molecule to 'remember' its path of flow within a fractured network. Thus, for an accurate representation of mathematical model:

$$S^{ABC}D_t^{\infty}H(r,t) = \frac{k}{t^{d-1-\theta}}\frac{\partial}{\partial r}(r^{d-1}\frac{\partial H(r,t)}{\partial r}.$$

(37) With the conditions H(r,0) = 0, $\lim_{t\to\infty} H(r,t) = 0$, d = dimension.

(4). Modelling of children's physical Development:

$$D_x^{\alpha}\ \mathrm{f(x)} = \frac{d^{\alpha}f(x)}{dx^{\alpha}} = \sum_{n=1}^{\infty} a_n\, nx^{n-1}\ .$$

(38) Here f(x) represents the data of children's height, weight, and body mass index with respect to time x. It can give a better chance of predicting expected values of the child for future by using previous data collected in the child development process.

(5). Fractional order mathematical modelling of COVID- 19: $D_tS(t) = \Delta - \lambda S - \frac{\alpha S(I+\beta A)}{N} - \Psi SQ$;

$$D_tE(t) = \frac{\alpha S(I+\beta A)}{N} + \Psi SQ - (1-\phi)\delta E - \phi\mu E - \lambda E\ ;\ D_tI(t) = (1-\phi)\delta E - (\sigma+\lambda)I\ ;$$

$$\mathrm{D_tA(t)} = \phi\mu E - (\rho+\lambda)A;\ \mathrm{D_t\ R(t)} = \sigma I + \rho A - \lambda R$$

$\mathrm{D_tQ(t)} = \kappa I + \nu A - \varepsilon Q.$ Here N = Total number of people.

Hidden dynamics of the infection in the mathematical models of the infectious disease is analyzed by the memory features of fractional derivative.

(6). Analyze Economic Growth Modelling:

From many point of view, different fractional order models are introduced for accounting the nature of financial processes.

$y(t) = \sum_{k=1}^{9} C_k D^{\alpha_k} x_k(t)$, C_k = constant weights, y = the GDP, x_k are the macroeconomic indicators variables. Taking an example as $x_9 = x_5$ represent the impact () of investment in the economy.

(7). Diffusion- Wave Equation of Groundwater Flow:

A groundwater system is concerned with significant heterogeneity. Common mathematical model of the radial groundwater flow to or from a well is

$$a^{ABC}D_t^{\alpha}H(r,t) + b^{ABC}D_t^{\alpha}\,H(r,t) = \frac{D_t}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H(r,t)}{\partial r}\right);$$

(39) Here a and b represents the ratio of immobile and mobile section porosities with total porosity and satisfies the relation between a and b as a + b = 1.The fractional derivative parameters for the mobile and the immobile sections are $0 < \alpha \leq 1$; $0 < \beta \leq 1$ D_f= the fractional diffusivity and H = the normalized groundwater depth.

(8). Sturm- Liouville BVP for integer order system:

These cases are found in the analysis of problems on simple harmonic motion, signal and image processing, heat conduction, vibration of string and wave propagation. As an example of heat equation such as

$$\frac{\partial u}{\partial t} = c\,\frac{\partial^2 u}{\partial x^2}\,;\ u(x,0) = f(x),\ u(0,t) = 0,\ u(L,t) = 0.$$

(40)

9. Fractional derivative models in the biomedical and underwater sediment fields: A concise model for the characterization of the dynamic events occurring in biological tissues can be provided by fractional calculus. Since fractional differential operator (FDO) has the main property such as ‘memory’, it is found to be very effective right now in heart surgery and even to the analysis of the behaviour of blood flow.

10. Other applications: Applications of fractional mass-spring- Damper System stated by Generalized Fractional Order Derivatives, Fractional order Proportional integral derivative (PID) Controllers may be mentioned. Applying in Image processing for edge detection can improve the criterion of this detection.

4.CONCLUSION:

In recent years FC has been a fruitful field of research in science and engineering. After the development of classical calculus, fractional calculus was formulated in 1695. FC has emerged as a powerful and efficient mathematical instrument during the past six decades, mainly due to its demonstrated applications. The dynamical complicated real-world problems is closely related to fractional calculus. Fractional calculus plays important roles in sound waves, describing memory and hereditary properties of different materials, electric transmission lines, modelling the cardiac tissue electrode interface in medical lines, fluid dynamics, prediction of groundwater flows, modelling and solving bioheat equiation. An outline picture for the development of fractional calculus applications in economics, the birth of a new attention towards mathematical economics as well as the future use of fractional calculus in economics and new revolution in economic theory are proposed.

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