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THE DYNAMICS OF A MATHEMATICAL MODEL CONSISTING TWO PREDATOR HAVING INTERSPECIFIC AND INTRASPECIFIC COMPETITION OVER SUSCEPTIBLE AND VIRUS INFECTED PHYTOPLANKTON POPULATION

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1.Abstract: *In this research paper, it is considered that two predators self-competing and competing with each other are feeding over susceptible and virus-infected phytoplankton populations. Predators are free from virus infection in nature i.e. Viruses do not affect the predator population. A mathematical model is comprised of phytoplankton and grazer populations like zooplankton, another predator that depends on infected as well as susceptible phytoplankton. The growth of the four species phytoplankton, infected phytoplankton, zooplankton, and a predator is given by ordinary non-linear differential equations with a set of parameters. The mathematical system is analyzed analytically. Equilibrium points and their stability are obtained.*

Keywords: *Non-linear ordinary equations, local, global stability and numerical simulation.*

2. Introduction: The dynamics of the non-linear model are rich and sensitive concerning parameters. Viruses typically infect the body by targeting specific proteins on the surface of host cells. For example, the spike proteins of the coronavirus bind to the ACE2 receptor in human cells to enter and replicate. Once inside, the virus hijacks the cell's machinery to produce more virus particles, leading to cell damage or death. These processes interfere with normal protein functions in cells, leading to disease symptoms and triggering the body's immune response to fight the infection. The length of time a virus remains in the body (the duration of infection) depends on several factors, including the strength of the immune response and the nature of the virus. Some viruses cause acute infections, lasting days to weeks, while others may establish chronic infections that last for months. Viruses can be recurring, especially if the immune system doesn't completely eliminate the virus or if the virus mutates (e.g., seasonal flu). Recurring infections can lead to long-term health effects, weakening the immune system, and several environmental changes. Many researchers have discussed the infected population. It has been seen that a non-linear differential equation system under certain parameters is obtained. Nature is nonlinear. From time-to-time various diseases arise and researchers try to find solutions to them by doing research. When we analytically solve these nonlinear differential equations under a certain feasible range of parameters, a rich dynamic is obtained. In this chapter, we tried to consider such a mathematical model in which a susceptible population is infected at a rate under certain parameters by disease based on previous research work listed in references. Zooplankton is taking food from susceptible plankton. A predator is taking food from infected susceptible plankton. We shall consider an epidemiological system consisting of four species, namely, the prey (phytoplankton) (which is susceptible) denoted by $'S'$, the infected prey (which becomes infective by some viruses) denoted by $'I'$ and the zooplankton called P_1 and a predator P_2. Before making the mathematical model, we made some assumptions based on previous research papers listed in references [1-27].

3. Formulation of Mathematical Model

(1) In the absence of virus disease, the phytoplankton cells $S(t)$ grow to a logistic function with a carrying capacity K, with an Intrinsic birth rate r is given by the relation

$$\frac{dN}{dt} = rS(1 - \frac{S}{K}) \tag{3.1}$$

(2) In the presence of viruses, we assume that the total concentration of phytoplankton cell N is divided into two classes, namely, susceptible phytoplankter, denoted by $S(t)$. Therefore, at any time t the total (concentration) of the phytoplankton population is given by the relation

$$N(t) = S(t) + I(t) \tag{3.2}$$

(3) The disease is spreading among the plankton population only. The infected populations do not recover. We assume that susceptible phytoplankton S is capable of reproducing again with logistic law (3.1) and the infective phytoplankton I, is reproducing by infecting the susceptible population at a certain rate.

(4) A susceptible phytoplankton $S(t)$ becomes infected $I(t)$ under the attack of many viruses. Let λ be the rate of force of infection. From the assumptions (3) and (4), the equation (3.1) can be written as:

$$\frac{ds}{dt} = rS\left(1 - \frac{S+I}{K}\right) - \lambda SI \tag{3.3}$$

(5) A grazer zooplankton population P_1 predates the susceptible phytoplankton at a rate a.

Then the equation (3.3) takes the form

$$\frac{ds}{dt} = rS\left(1 - \frac{S+I}{K}\right) - \lambda SI - aSP_1 \tag{3.4}$$

(6) The dynamics of grazer population zooplankton P_1, predator P_2, and infected phytoplankton $I(t)$ may be represented as follows

$$\frac{dI}{dt} = I(\lambda S - kP_1 - h) - bIP_2 \tag{3.5}$$

Where k denotes the rate of capturing of infected prey by the zooplankton P_1 and b the rate of capturing of infected prey by predator P_2, h is the death rate of infected phytoplankton.

Now we consider in this mathematical model zooplankton P_1 and predator P_2, are self-competitor and competing with each other which are shown by the relations:

$$\frac{dP_1}{dt} = P_1(-d_1 + cS + k_1 I) - h_1 P_1^2 P_2 \tag{3.6}$$

Where d_1 is the death rate of P_1, c is the growth rate of predators due to predation of susceptible phytoplankton, k_1 is the growth rate of predators due to predation of infected phytoplankton

$$\frac{dP_2}{dt} = P_2(-d_2 + \alpha S + \beta I) - h_2 P_2^2 P_1 \tag{3.7}$$

Where d_2 is the death rate of P_2 , α is the attacking rate of predator P_2 to susceptible phytoplankton and β is the attacking rate of predator P_2 due to the predation of I(t).

4. The Mathematical Model

Comprising the above equations from (3.1) to (3.7), the mathematical model can be written by the following differential equations describing the time evolution of the prey-predator system.

$$\frac{dS}{dt} = rS\left(1 - \frac{S+I}{K}\right) - \lambda SI - aSP_1$$

$$\frac{dI}{dt} = I(\lambda S - kP_1 - h) - bIP_2$$

$$\frac{dP_1}{dt} = P_1(-d_1 + cS + k_1 I) - h_1 P_1^2 P_2$$

$$\frac{dP_2}{dt} = P_2(-d_2 + \alpha S + \beta I) - h_2 P_2^2 P_1 \quad (4.1)$$

Here the parameters h_1 and h_2 are used as the parameter functions contain interspecific and intraspecific competition parameters self and among both predators. Both zooplankton and predators have self-competition also.

The system has to be analyzed with the following conditions:

$$S(0) > 0, I(0) \geq 0, P_1(0) \geq 0, P_2(0) \geq 0$$

5. Boundedness of the Mathematical Model

Theorem: 5.1 Prove that the trajectories of non-linear dynamic models are bounded.

Proof: Let us consider

$$\eta(t) = S(t) + I(t) + P_1(t) + P_2(t)$$

$$\frac{d\eta(t)}{dt} = \frac{dS(t)}{dt} + \frac{dI(t)}{dt} + \frac{dP_1(t)}{dt} + \frac{dP_2(t)}{dt}$$

$$\frac{d\eta(t)}{dt} + \eta(t) = rS\left(1 - \frac{S+I}{K}\right) - \lambda SI - aSP_1 + I(\lambda S - kP_1 - h) - bIP_2 + P_1(-d_1 + cS + k_1 I) - h_1 P_1^2 P_2 + P_2(-d_1 + \alpha P_1 + \beta I) - h_2 P_2^2 P_1$$

$$\frac{d\eta(t)}{dt} + \eta(t) = rS\left(1 - \frac{S+I}{K}\right) - \lambda SI - aSP_1 - IkP_1 - hI - bIP_2 - P_1 d_1 + cSP_1 + k_1 P_1 I - d_2 P_2 + \alpha P_1 P_2 + \beta IP_2 - P_1 P_2(h_1 P_1 + h_2 P_2) + S + I + P_1 + P_2$$

$$\frac{d\eta(t)}{dt} + \eta(t) = rS\left(1 - \frac{S+I}{K}\right) + S - SP_1(a - c) - P_1 I(k - k_1) - I(h - 1) - IP_2(b - \beta) - P_1(d_1 - 1) - P_2(d_2 - 1) - P_1 P_2(h_1 P_1 + h_2 P_2 - \alpha)$$

$$\frac{d\eta(t)}{dt} + \eta(t) = S\left[r\left(1 - \frac{S}{k}\right) + 1\right] - SP_1(a - c) - P_1 I(k - k_1) - I(h - 1) - \frac{r}{k}SI - IP_2(b - \beta') - P_1(d_1 - 1) - P_2(d_2 - 1) - P_1 P_2(h_1 P_1 + h_2 P_2 - \alpha)$$

$$d_1 > 1, d_2 > 1, b > \beta', h > 1, k > k_1, a > c$$

$$\frac{d\eta(t)}{dt} + \eta(t) \leq S\left[r\left(1 - \frac{S}{k}\right) + 1\right] - P_1 P_2(h_1 P_1 + h_2 P_2 - \alpha)$$

$$\frac{d\eta(t)}{dt} + \eta(t) \leq \max\left\{S\left[r\left(1 - \frac{S}{k}\right) + 1\right]\right\}$$

$$\frac{d\eta(t)}{dt} + \eta(t) \leq \frac{(r+1)^2 k}{4r} = M \Rightarrow \frac{d\eta(t)}{dt} + \eta(t) \leq M \Rightarrow \eta(t) < M$$

$$0 \leq S(t) + I(t) + P_1(t) + P_2(t) \leq M$$

Hence the system is bounded and therefore is dissipative.

Theorem 5.2 (Positivity of the Solution of the Mathematical Model): The solution (S, I, P_1, P_2) is positive for all t greater than and equal to zero.

Proof: We have from the mathematical model equations

$\frac{dS}{dt} \geq -S(\lambda I + aP_1) \Rightarrow \frac{dS}{S} \geq -M_1 dt \Rightarrow S(t) \geq 0$. Where max $(\lambda I + aP_1) = M_1$

$\frac{dI}{dt} \geq -I(kP_1 + h + bP_2) \Rightarrow \frac{dI}{I} \geq -M_2 dt \Rightarrow I(t) \geq 0$. Where max $(kP_1 + h + bP_2) = M_2$

$\frac{dP_1}{dt} \geq -P_1(d_1 + h_1 P_1 P_2) \Rightarrow \frac{dP_1}{P_1} \geq -M_3 dt \Rightarrow P_1(t) \geq 0$. Where max $(d_1 + h_1 P_1 P_2) = M_3$

$\frac{dP_2}{dt} \geq -P_2(d_2 + h_2 P_2 P_1) \Rightarrow \frac{dP_2}{P_2} \geq -M_4 dt \Rightarrow P_2(t) \geq 0$. Where max $(d_2 + h_1 P_2 P_1) = M_4$

Therefore, the solution (S, I, P_1, P_2) is positive for all t greater than and equal to zero.

6. Equilibria and Stability theory of Mathematical Model

We have modelled a non-linear mathematical model (4.1), which has the following feasible equilibriums:

$E_0(0,0,0,0)$, $E_1(S^*, 0,0,0)$, $E_2(S^*, I^* 0,0)$, $E_3(S^*, I^*, P_1^*, 0)$, $E_4(S^*, I^*, P_1^*, P_2^*)$

We will discuss the nature of equilibriums E_0, E_1, E_2, E_3 and E_4.

Theorem 6.1. The $E_0(0,0,0,0)$ is a trivial case.

Proof: From the first isocline of the system (4.1)

$$\frac{dS}{dt} = rS\left(1 - \frac{S+I}{K}\right) - \lambda SI - aSP_1 = 0 \Rightarrow \frac{dS}{dt} = Sg_1(S, I, P_1) = 0$$

Since we know that $S = 0$ therefore, we have $g_1(S, I, P_1) \neq 0$.

From the second isocline of the system (4.1)

$$\frac{dI}{dt} = I((\lambda S - kP_1 - h) - bP_2) = 0 \Rightarrow \frac{dI}{dt} = Ig_2(S, P_1, P_2) = 0$$

Since $I = 0$ therefore we have $g_2(S, P_1, P_2) \neq 0$.

From the third isocline of the system (4.1)

$$\frac{dP_1}{dt} = P_1(-d_1 + cS + k_1 I) - h_1 P_1^2 P_2 = 0 \Rightarrow \frac{dP_1}{dt} = P_1 g_4(S, I, P_1, P_2) = 0$$

Since $P_1 = 0$ therefore we have $g_3(S, I, P_1, P_2) \neq 0$.

From the fourth isocline of the system (4.1) we have

$$\frac{dP_2}{dt} = P_2[(d_2 + \alpha S + \beta I) - h_2 P_2 P_1] = 0 \Rightarrow \frac{dP_2}{dt} = P_2 g_4(S, I, P_1, P_2) = 0$$

Since $P_2 = 0$ therefore we have $g_4(S, I, P_1, P_2) \neq 0$.

Theorem 6.2. Prove that the point E_0 is a saddle point.

Proof: The variational matrix about E_0 is given by

$$V_{E_0} = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & -h & 0 & 0 \\ 0 & 0 & -d_1 & 0 \\ 0 & 0 & 0 & -d_2 \end{bmatrix}$$

Since the eigenvalues are

$\lambda_1 = r > 0, \lambda_2 = -h < 0, \lambda_3 = -d_1 < 0, \lambda_4 = -d_2 < 0$

Thus E_0 is a saddle point. Since the eigenvalues are positive and negative, therefore E_0 also shows bifurcation.

Theorem 6.3. Prove the equilibrium $E_1(S^*, 0,0,0)$ is the point $E_1(K, 0,0,0)$.

Proof: From the first isocline of the system (4.1) we have

$$\frac{dS}{dt} = S\left[r\left(1 - \frac{S+I}{K}\right) - \lambda I - aP_1\right] \Rightarrow \frac{dS}{dt} = Sg_1(S, I, P_1) = 0$$

Since $S \neq 0$ therefore we have

$$g_1(S, I, P_1) = r\left(1 - \frac{S+I}{K}\right) - \lambda I - aP_1 = 0$$

Putting $P_1 = 0$ and $I = 0$ we have

$$(1 - \frac{S^*}{K}) - 0 - 0 = 0 \Rightarrow S^* = K$$

From the second isocline of the system

$$\frac{dI}{dt} = I((\lambda S - kP_1 - h) - bP_2) \Rightarrow \frac{dI}{dt} = Ig_2(S, P_1, P_2) = 0$$

Since $I = 0$ therefore we have $g_2(S, P_1, P_2) \neq 0$.

From the third isocline of the system, we have

$$\frac{dP_1}{dt} = P_1(-d_1 + cS + k_1 I) - h_1 P_1^2 P_2 \Rightarrow \frac{dP_1}{dt} = P_1 g_4(S, I, P_1, P_2) = 0$$

Since $P_1 = 0$ therefore we have $g_3(S, I, P_1, P_2) \neq 0$.

From the fourth isocline of the system, we have

$$\frac{dP_2}{dt} = P_2[(d_2 + \alpha S + \beta I) - h_2 P_2 P_1] \Rightarrow \frac{dP_2}{dt} = P_2 g_4(S, I, P_1, P_2) = 0$$

Since $P_2 = 0$ therefore we have $g_4(S, I, P_1, P_2) \neq 0$.

Hence the equilibrium $E_1(S^*, 0,0,0)$ is the point $E_1(K, 0,0,0)$.

Theorem 6.4. Prove that equilibrium $E_1(K, 0,0,0)$ is asymptotically stable if the following conditions are satisfied $K < Min(\frac{h}{\lambda}, \frac{d_1}{c}, \frac{d_2}{\alpha})$.

Proof: The variational matrix about $E_1(K, 0,0,0)$ is given by

$$V_{E_1} = \begin{bmatrix} S^*\left(-\frac{r}{K}\right) & S^*(-\frac{r}{K} - \lambda) & -aS^* & 0 \\ 0 & \lambda S^* - h & 0 & 0 \\ 0 & 0 & -d_1 + cS^* & 0 \\ 0 & 0 & 0 & -d_2 + \alpha S^* \end{bmatrix}$$

$$= \begin{bmatrix} -r & -r - \lambda K & -aK & 0 \\ 0 & \lambda K - h & 0 & 0 \\ 0 & 0 & -d_1 + cK & 0 \\ 0 & 0 & 0 & -d_2 + \alpha K \end{bmatrix}$$

Since the eigenvalues are

$\lambda_1 = -r < 0,\ \lambda_2 = \lambda K - h < 0,\ \lambda_3 = -d_1 + cK < 0,\ \lambda_4 = -d_2 + \alpha K < 0$

The equilibrium $E_1 \equiv (K, 0,0,0)$ is asymptotically stable if

$(i)\ K < \frac{h}{\lambda}$ $(ii)\ K < \frac{d_1}{c}$ (iii) $K < \frac{d_2}{\alpha}$

Combing(i)-(iii), we get $K < Min(\frac{h}{\lambda}, \frac{d_1}{c}, \frac{d_2}{\alpha})$

The equilibrium becomes a saddle point if the one of condition (i)-(iii) violates.

Theorem 6.5. Prove that equilibrium $E_2(S^*, I^*, 0,0)$ will exist and will be positive if $K > \frac{h}{\lambda}$, where $S^* = \frac{h}{\lambda}, I^* = \frac{r(\lambda K - h)}{\lambda(r + \lambda k)}$.

Proof: After putting $P_2 = 0\ and\ P_1 = 0$ The first, second, and third isocline of the system (4.1) were reduced to

$$\frac{dS}{dt} = S\left[r\left(1 - \frac{S+I}{K}\right) - \lambda I\right] \qquad (i)$$

$$\frac{dI}{dt} = I(\lambda S - h) \qquad (ii)$$

$$\frac{dP_1}{dt} = P_1(-d_1 + cS + k_1 I) \qquad (iii)$$

Since $S \neq 0$ therefore from (i) we have $r(1 - \frac{S+I}{K}) - \lambda I = 0$ (iv)

Since $I \neq 0$ then from (ii) we have $\lambda S - h = 0$ (v)

Using (iv) and (v) we get $S^* = \frac{h}{\lambda}, I^* = \frac{r(\lambda K - h)}{\lambda(r + \lambda k)}$.

Theorem 6.6. Prove that equilibrium $E_2(S^*, I^*, 0,0)$ is asymptotically stable provided the following conditions are satisfied $cS^* + k_1 I^* < d_1,\ \beta I^* + \alpha P_1^* < d_2,\ m < S^* r$ with

$$m^2 = r^2 S^{*2} - 4S^* I^* \lambda K^2 \left(\frac{r}{K} + \lambda\right) > 0$$

If the above conditions violate then we get E_2 as a saddle point.

Proof: The variational matrix about E_2 is given by

$$\begin{bmatrix} S^*\left(-\frac{r}{K}\right) & S^*\left(-\frac{r}{K} - \lambda\right) & -aS^* & 0 \\ \lambda I^* & 0 & -I^* k & -bI^* \\ 0 & 0 & -d_1 + cS^* + k_1 I^* & 0 \\ 0 & 0 & 0 & -d_2 + \beta I^* + \alpha P_1^* \end{bmatrix}$$

The eigenvalues of the above variational matrix are given by

$$\sigma_1, \sigma_2 = -\frac{S^* r}{2K} \pm \frac{1}{2K}\sqrt{r^2 S^{*2} - 4S^* I^* \lambda K^2 \left(\frac{r}{K} + \lambda\right)}\ , \sigma_3 = -d_1 + cS^* + k_1 I^*, \sigma_4 = -d_2 + \beta I^* + \alpha P_1^*.$$

$\sigma_1, \sigma_2 = -\frac{S^* r}{2K} \pm \frac{m}{2K}$, m= $\sqrt{r^2 S^{*2} - 4S^* I^* \lambda K^2 \left(\frac{r}{K} + \lambda\right)}$, $\sigma_3 = -d_1 + cS^* + k_1 I^*, \sigma_4 = -d_2 + \beta I^* + \alpha P_1^*$.

Theorem 6.7. Prove that equilibrium $E_3(S^*, I^*, P_1^*, 0)$ will exist and will be positive if the following conditions are satisfied (i) $\frac{rk}{K} + a\lambda > c\left(\frac{r}{K} + \lambda\right)$, (ii) $ah + rk > d_1\left(\frac{r}{K} + \lambda\right)$, (iii) $ahc + rkc > \frac{rkd_1}{K} + a\lambda d_1$ (iv) $\lambda hc + \frac{rhc}{K} + rk\lambda > \frac{rkh}{K} + \lambda^2 d_1 + \frac{r\lambda d_1}{K}$.

Proof: After putting $P_2 = 0$ the first, second, and third isoclines of the system (4.1) reduced to

$$\frac{dS}{dt} = S\left[r\left(1 - \frac{S+I}{K}\right) - \lambda I - aP_1\right] \qquad (i)$$

$$\frac{dI}{dt} = I(\lambda S - kP_1 - h) \qquad (ii)$$

$$\frac{dP_1}{dt} = P_1(-d_1 + cS + k_1 I) \qquad (iii)$$

Since $S \neq 0$ therefore from (i) we have $r(1 - \frac{S+I}{K}) - \lambda I - aP_1 = 0$ $\qquad (iv)$

Since $I \neq 0$ then from (ii) we have $\lambda S - kP_1 - h = 0$ $\qquad (v)$

Since $P_1 \neq 0$ then from (iii) we have $(-d_1 + cS + k_1 I) = 0$ $\qquad (vi)$

Applying the Cramer rule to find the positive equilibrium from (iv) -(vi), we get

$$S^* = \frac{-kd_1\left(\frac{r}{K}+\lambda\right) + ahk + rk^2}{D}, I^* = \frac{-\frac{rkd_1}{K} - a\lambda d_1 + ahc + rkc}{D},$$

$$P_1^* = \frac{-\frac{rkh}{K} - \lambda^2 d_1 + \lambda hc - \frac{r\lambda d_1}{K} + \frac{rhc}{K} + rk\lambda}{D}, D = \frac{rk^2}{K} - ck(\frac{r}{K} + \lambda) + a\lambda k$$

The Positive equilibrium E_3 will be positive if the following conditions are satisfied

$$D > 0 \Rightarrow \frac{rk}{K} + a\lambda > c\left(\frac{r}{K} + \lambda\right), S^* > 0 \Rightarrow ah + rk > d_1\left(\frac{r}{K} + \lambda\right), I^* > 0 \Rightarrow ahc + rkc > \frac{rkd_1}{K} + a\lambda d_1$$

$$P_1^* > 0 \Rightarrow \lambda hc + \frac{rhc}{K} + rk\lambda > \frac{rkh}{K} + \lambda^2 d_1 + \frac{r\lambda d_1}{K}.$$

Theorem 6.8. Prove that equilibrium $E_3(S^*, I^*, P_1^*, 0)$ is asymptotically stable provided the following conditions are satisfied. $\frac{rS^{*2}}{K}\left\{I^*\lambda\left(\lambda + \frac{r}{K}\right) + acP_1^*\right\} > \{I^* P_1^* k(c + a\lambda S^*\}$ and $\beta I^* + \alpha S^* < d_2$

Proof: The variational matrix about E_3 is V_{E_3}

$$= \begin{bmatrix} S^*\left(-\frac{r}{K}\right), & S^*\left(-\frac{r}{K} - \lambda\right) & -aS^* & 0 \\ \lambda I^* & 0 & -I^* k & -bI^* \\ cP_1^* & k_1 P_1^* & 0 & -h_1 {P_1^*}^2 \\ 0 & 0 & 0 & -d_2 + \beta I^* + \alpha S^* \end{bmatrix}$$

We use the Routh Hurwitz Criterion to find the stability by using the above variational matrix. For this, we have the following characteristic equation,

$$(-d_2 + \beta I^* + \alpha S^* - \mu)\ [\mu^3 + S_1\mu^2 + S_2\mu + S_3] = 0$$

Now we find the value of S_1, S_2 and S_3 as follow.

$$S_1 = \frac{rS^*}{K} > 0, S_2 = I^*P_1^*k^2 + S^*I^*\lambda\left(\lambda + \frac{r}{K}\right) + acS^*P_1^* > 0,$$

$$S_3 = \frac{rS^*}{K}I^*P_1^*k^2 + I^*P_1^*kc + a\lambda kS^*P_1^*I^* > 0$$

Now by Routh-Hurwitz Criterion, $S_1 > 0, S_2 > 0, S_3 > 0,\ S_1S_2 > S_3$.

$$S_1S_2 > S_3 \Longrightarrow \frac{rS^{*2}}{K}\left\{I^*\lambda\left(\lambda + \frac{r}{K}\right) + acP_1^*\right\} > \{I^*P_1^*k(c + a\lambda S^*\}$$

Then E_3 is asymptotically stable.

Theorem 6.9. Existence of Positive equilibrium $E_4(S^*, I^*, P_1^*, P_2^*)$ when predators competing parameter to each other over Susceptible and Virus Infected Phytoplankton Populations are not equal. The positive equilibrium $E_4(S^*, I^*, P_1^*, P_2^*)$ will exist and positive if susceptible population S will have fixed value in the following range domain

$Min(F_1, F_2) < S < \frac{r}{BL}$ *provided* $h_2d_1 > h_1d_2,\ h_2c > h_1\alpha,$

where $L = \left(\frac{r}{BK} + \frac{(h_2c - h_1\alpha)}{(\beta h_1 - kh_2)}\right), B = \lambda + \frac{r}{K}$

Proof: From the first isocline of the system (3.8), we have

$$\frac{dS}{dt} = rS\left(1 - \frac{S+I}{K}\right) - \lambda SI - aSP_1 = 0 \Rightarrow \frac{dS}{dt} = Sg_1(S, I, P_1) = 0$$

Since $S \neq 0$ therefore, we have $g_1(S, I, P_1) = 0$

$$\Rightarrow r\left(1 - \frac{S+I}{K}\right) - \lambda I - aP_1 = 0 \qquad (i)$$

From the second isocline of the system (3.8), we have

$$\frac{dI}{dt} = I((\lambda S - kP_1 - h) - bP_2) = 0 \Rightarrow \frac{dI}{dt} = Ig_2(S, P_1, P_2) = 0$$

Since $I \neq 0$ therefore, we have $g_2(S, P_1, P_2) = 0$

$$\Rightarrow (\lambda S - kP_1 - h) - bP_2 = 0 \qquad (ii)$$

From the third isocline of the system (3.8), we have

$$\frac{dP_1}{dt} = P_1[(-d_1 + cS + k_1I) - h_1P_1P_2] = 0 \Rightarrow \frac{dP_1}{dt} = P_1g_3(S, P_1, P_2, I) = 0$$

Since $P_1 \neq 0$ therefore, we have $g_3(S, P_1, P_2, I) = 0$

$$\Rightarrow (-d_1 + cS + k_1I) - h_1P_1P_2 = 0 \qquad (iii)$$

From the fourth isocline of the system (3.8), we have

$$\frac{dP_2}{dt} = P_2[(d_2 + \alpha S + \beta I) - h_2P_2P_1] = 0 \Rightarrow \frac{dP_2}{dt} = P_2g_4(S, I, P_1, P_2) = 0$$

Since $P_2 \neq 0$ therefore, we have $g_4(S, I, P_1, P_2) = 0$

$$\Rightarrow (d_2 + \alpha S + \beta I) - h_2P_2P_1 = 0 \qquad (iv)$$

From the isoclines (iii) and (iv), taking the competing predator parameters and subtracting them from each other, then we have the equation

$$-(h_2d_1 - h_1d_2) + S(h_2c - h_1\alpha) + I(\beta h_1 - kh_2) = 0 \Rightarrow I^* = \frac{-(h_2d_1-h_1d_2)+S(h_2c-h_1\alpha)}{(\beta h_1-kh_2)} \quad (v)$$

$$I^* > 0 \; when \; S > \frac{(h_2d_1 - h_1d_2)}{(h_2c - h_1\alpha)} = F_1(say), provided \;\; h_2d_1 > h_1d_2 \; and \; h_2c > h_1\alpha$$

From the isoclines (i) and (v), we get

$$P_1^* = \frac{1}{Ba}\left\{\frac{r}{B} - S\left(\frac{r}{BK} + \frac{(h_2c - h_1\alpha)}{(\beta h_1 - kh_2)}\right)\right\} = \frac{1}{Ba}\left\{\frac{r}{B} - SL\right\}$$

$$Thus \; P_1^* > 0 \; when \; \frac{r}{SB} > L \;, where \; L = \left(\frac{r}{BK} + \frac{(h_2c - h_1\alpha)}{(\beta h_1 - kh_2)}\right), B = \lambda + \frac{r}{K}$$

Substitute the value of P_1^* In the isoclines (ii), we get

$$P_2^* = \frac{1}{b}\left\{S\left(\lambda + \frac{Lk}{Ba}\right) - \left(h + \frac{kr}{B^2a}\right)\right\}, thus \; P_2^* > 0 \; when \; S > \frac{\left(h + \frac{kr}{B^2a}\right)}{\left(\lambda + \frac{Lk}{Ba}\right)} = F_2(say)$$

The positive equilibrium $E_4(S^*, I^*, P_1^*, P_2^*)$ will exist and be positive if susceptible population S will have a fixed value in the following range domain

$$Min(F_1, F_2) < S < \frac{r}{BL}$$

Corollary 6.10. Existence of Positive equilibrium $E_4(S^*, I^*, P_1^*, P_2^*)$ when predators compete with each other over Susceptible and Virus Infected Phytoplankton Population have equal competition. The positive equilibrium $E_4(S^*, I^*, P_1^*, P_2^*)$ will exist and be positive if susceptible population S will have a fixed value in the following range domain

$$Min\left\{\frac{(d_1 - d_2)}{(c - \alpha)}, \frac{\left(\frac{kr}{a}\right) + \frac{kB(d_1 - d_2)}{(\beta - k)}}{\lambda - k\left\{\left(\frac{r}{aK}\right) + \frac{B(c - \alpha)}{\beta - k}\right\}}\right\} < S < K\left\{1 + \frac{Ba(d_1 - d_2)}{r(\beta - k)}\right\} / \left\{1 + \frac{Ka(c - \alpha)}{r(\beta - k)}\right\}$$

$provided \;\; d_1 > d_2 \; and \; c > \alpha$

Proof: In the theorem 3.5.6 substituting $h_2 = h_1$.

Theorem 6.11 Prove that equilibrium $E_4(S^*, I^*, P_1^*, P_2^*)$ is asymptotically stable provided the following conditions are satisfied $S_1 = \frac{rS^*}{K} + h_1P_1^*P_2^* + h_2P_1^*P_2^* > 0$,

$$S_2 = \frac{rS^*}{K}P_1^*P_2^*(h_1 + h_2) + S^*I^*\lambda\left(\lambda + \frac{r}{K}\right) + acS^*P_1^* + b\,\beta P_2^*I^* + kk_1P_1^*I^* > 0, S_3 = L - M > 0,$$

$S_4 = Q - R > 0$, $Q{S_1}^2 + L^2 + M^2 < {S_1}^2({S_2}^2 + R)$. Where $L = \frac{rS^*}{K}(I^*P_1^*kk_1 + bI^*P_2^*\beta) + bI^*\beta {P_2^*}^2h_1P_1^* + k_1{P_1^*}^2h_2P_2^*I^*k + S^*\lambda h_2P_1^*I^*P_2^*\left(\lambda + \frac{r}{K}\right) + S^*\lambda h_1P_1^*I^*P_2^*\left(\lambda + \frac{r}{K}\right) + acS^*h_2P_2^*{P_1^*}^2 + aS^*\lambda\, I^*k_1P_1^*$,

$$M = I^*{P_1^*}^2kh_1P_2^*\beta + bI^*{P_2^*}^2kh_2P_1^* + S^*\alpha P_2^*bI^*\left(\lambda + \frac{r}{K}\right) + S^*cP_1^*I^*k\left(\lambda + \frac{r}{K}\right) + a\alpha h_1S^*P_1^*P_2^*,$$

$Q = \frac{rS^*}{K}\left(I^*P_2^*kk_1h_2{P_1^*}^2 + bI^*P_1^*\beta h_1{P_2^*}^2\right) + S^*\left(\lambda + \frac{r}{K}\right)\left\{\alpha h_1kI^*P_2^*{P_1^*}^2 + bch_2I^*P_1^*{P_2^*}^2 + b\alpha h_1I^*P_1^*{P_2^*}^2\right\} + aS^*\{\lambda k_1h_2{P_1^*}^2I^*P_2^* + bc\beta I^*P_1^*P_2^*\},$

$R = \frac{rS^*}{K}\left(bI^*P_1^*k_1h_2{P_2^*}^2 + I^*P_2^*\beta h_1k{P_1^*}^2\right) + S^*\left(\lambda + \frac{r}{K}\right)\left\{kch_2I^*{P_1^*}^2P_2^*\right\} + aS^*\{\beta h_1\lambda {P_1^*}^2I^*P_2^* + b\alpha k_1I^*P_1^*P_2^*.$

Proof: The variational matrix about $E_4(S^*, I^*, P_1^*, P_2^*)$ is V_{E_4}

$$\begin{bmatrix} -\frac{r}{K}S^*, & S^*\left(-\frac{r}{K} - \lambda\right) & -aS^* & 0 \\ \lambda I^* & 0 & -I^*k & -bI^* \\ cP_1^* & k_1P_1^* & -h_1P_1^*P_2^* & -h_1{P_1^*}^2 \\ \alpha P_2^* & \beta P_2^* & -h_2{P_2^*}^2 & -h_2P_1^*P_2^* \end{bmatrix}$$

We use the Routh Hurwitz Criterion to find the stability by using the above variational matrix. For this, we have the following characteristic equation,

$$\mu^4 + S_1\mu^3 + S_2\mu^2 + S_3\mu + S_4 = 0$$

Now we find the value of S_1 ,S_2, S_3 and S_4 as follow. $S_1 = \frac{rS^*}{K} + h_1P_1^*P_2^* + h_2P_1^*P_2^* > 0,$

$S_2 = \frac{rS^*}{K}P_1^*P_2^*(h_1 + h_2) + S^*I^*\lambda\left(\lambda + \frac{r}{K}\right) + acS^*P_1^* + b\,\beta P_2^*I^* + kk_1P_1^*I^* > 0,$

$S_3 = L - M > 0\ if\ L > M$ Where

$L = \frac{rS^*}{K}(I^*P_1^*kk_1 + bI^*P_2^*\beta) + bI^*\beta {P_2^*}^2h_1P_1^* + k_1{P_1^*}^2h_2P_2^*I^*k + S^*\lambda h_2P_1^*I^*P_2^*\left(\lambda + \frac{r}{K}\right) + S^*\lambda h_1P_1^*I^*P_2^*\left(\lambda + \frac{r}{K}\right) + acS^*h_2P_2^*{P_1^*}^2 + aS^*\lambda\, I^*k_1P_1^*$

$M = I^*{P_1^*}^2kh_1P_2^*\beta + bI^*{P_2^*}^2kh_2P_1^* + S^*\alpha P_2^*bI^*\left(\lambda + \frac{r}{K}\right) + S^*cP_1^*I^*k\left(\lambda + \frac{r}{K}\right) + a\alpha h_1S^*P_1^*P_2^*$

$S_4 = \frac{rS^*}{K}\left(I^*P_2^*kk_1h_2{P_1^*}^2 + bI^*P_1^*\beta h_1{P_2^*}^2\right) - \frac{rS^*}{K}\left(bI^*P_1^*k_1h_2{P_2^*}^2 + I^*P_2^*\beta h_1k{P_1^*}^2\right) + S^*\left(\lambda + \frac{r}{K}\right)\left\{\alpha h_1kI^*P_2^*{P_1^*}^2 + bch_2I^*P_1^*{P_2^*}^2 + b\alpha h_1I^*P_1^*{P_2^*}^2 - kch_2I^*{P_1^*}^2P_2^*\right\} + aS^*\{\lambda k_1h_2{P_1^*}^2I^*P_2^* - \beta h_1\lambda {P_1^*}^2I^*P_2^* + bc\beta I^*P_1^*P_2^* - b\alpha k_1I^*P_1^*P_2^*$

$S_4 = Q - R > 0\ if\ Q > R;$ where

$Q = \frac{rS^*}{K}\left(I^*P_2^*kk_1h_2{P_1^*}^2 + bI^*P_1^*\beta h_1{P_2^*}^2\right) + S^*\left(\lambda + \frac{r}{K}\right)\left\{\alpha h_1kI^*P_2^*{P_1^*}^2 + bch_2I^*P_1^*{P_2^*}^2 + b\alpha h_1I^*P_1^*{P_2^*}^2\right\} + aS^*\{\lambda k_1h_2{P_1^*}^2I^*P_2^* + bc\beta I^*P_1^*P_2^*\}$

$R = \frac{rS^*}{K}\left(bI^*P_1^*k_1h_2{P_2^*}^2 + I^*P_2^*\beta h_1k{P_1^*}^2\right) + S^*\left(\lambda + \frac{r}{K}\right)\left\{kch_2I^*{P_1^*}^2P_2^*\right\} + aS^*\{\beta h_1\lambda {P_1^*}^2I^*P_2^* + b\alpha k_1I^*P_1^*P_2^*$

Now by the Routh-Hurwitz Criterion for stability, $S_1 > 0, S_2 > 0, S_3 > 0, S_4 > 0$, $S_1S_2S_3 > {S_1}^2S_4 + {S_3}^2$. Then $E_4(S^*, I^*, P_1^*, P_2^*)$ is asymptotically stable.

7. Locally asymptotically stable & globally asymptotically stable

Theorem 7.1. The flow of the nonlinear harvesting model (4.1) contracts volume uniformly for positive non-zero equilibrium $E_4(S^*, I^*, P_1^*, P_2^*)$.

Proof: Because the divergence of the vector field for the positive non-zero equilibrium $E_4(S^*, I^*, P_1^*, P_2^*)$ is

$$\frac{\partial}{\partial P_1}\frac{dP_1}{dt} + \frac{\partial}{\partial P_2}\frac{dP_2}{dt} + \frac{\partial}{\partial S}\frac{dS}{dt} + \frac{\partial}{\partial I}\frac{dI}{dt} = -[h_1 P_1 P_2 + h_2 P_1 P + \frac{rS}{K}] < 0$$

Hence the result.

Theorem 7.2. If the positive non-zero equilibrium $E_4(S^*, I^*, P_1^*, P_2^*)$ is locally asymptotically stable, then the positive non-zero equilibrium $E_4(S^*, I^*, P_1^*, P_2^*)$ will be globally asymptotically stable with the condition $c(r + \lambda K) = \frac{\alpha\beta}{\alpha_1 b} aK\lambda$.

Proof: Assume $S = S^* + u,\ I = I^* + v, P_1 = P_1^* + w,\ P_2 = P_2^* + x$, where u, v, w and x are small perturbations.

Let $V(t)$ be a positive definite function for arbitrarily chosen positive constants $D_1, D_2, D_3\ and\ D_4$ such that

$$V(t) = D_1\left(u - S^* \log(1 + \frac{u}{S^*}\right) + D_2\left(v - I^* \log(1 + \frac{v}{I^*}\right) + D_3\left(w - P_1^* \log(1 + \frac{w}{P_1^*}\right) + D_4\left(x - P_2^* \log(1 + \frac{x}{P_2^*}\right)$$

$$\frac{dV(t)}{dt} = \frac{\partial V(t)}{\partial u}\frac{du}{dt} + \frac{\partial V(t)}{\partial v}\frac{dv}{dt} + \frac{\partial V(t)}{\partial w}\frac{dw}{dt} + \frac{\partial V(t)}{\partial x}\frac{dx}{dt} = -\left[\frac{D_1 r u^2}{K} + D_4 h_2 P_1^* x^2 + D_3 h_1 P_2^* w^2\right]$$

$$-(P_2 - P_2^*)(P_1 - P_1^*)[D_3 h_1 P_1^* + D_4 h_2 P_2^* + D_3 h_1 (P_1 - P_1^*) + D_4 h_2 (P_2 - P_2^*) < 0.$$

provided $D_1\left(\lambda + \frac{r}{K}\right) = D_2\lambda,\ D_1 a = D_3 c,\ D_2 k = D_3 k_1, D_2 b = D_4 \beta.$

Therefore, the dynamics of the mathematical model about the non-zero positive equilibrium is globally asymptotically stable under the conditions

(1) The non-zero positive equilibrium is locally asymptotically stable.

(2) $c(r + \lambda K) = \frac{\alpha\beta}{\alpha_1 b} aK\lambda.$

8. Numerical Analysis: A numerical simulation of a given mathematical model under the set of certain values of parameters is carried out. The mathematical model is sensitive to the change in parametric values. We draw Figure 1 for the considered set of parameters

r = 0.82; λ = 0.61; a = 0.021; K = 11.6; k = 0.0235; b = 0.037; h = 0.048; c = 0.225; k_1 = 0.42; α = 0.22; d_1 = 0.025; α_1 = 0.015; β = 0.014; h_1 = 0.46; d_2 = 0.031;

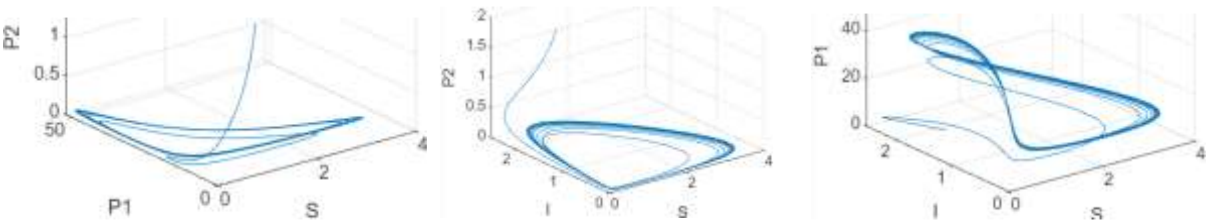


Figure 1

In Figure 1 it is seen that in the 3-D projection of the mathematical model, the trajectories converge to a limit cycle. Thus, we concluded that the system is periodic stable. Consider again a set of parameter values under the derived analytical results and draw Figure 2

r = 0.62; λ = 0.81; a = 0.021; K = 9.6; k = 0.0235; b = 0.037; h = 0.048; c = 0.0225; k_1 = 0.042; α = 0.022; d_1 = 0.025; α_1 = 0.022; β = 0.014; h_1 = 0.026; d_2 = 0.031;

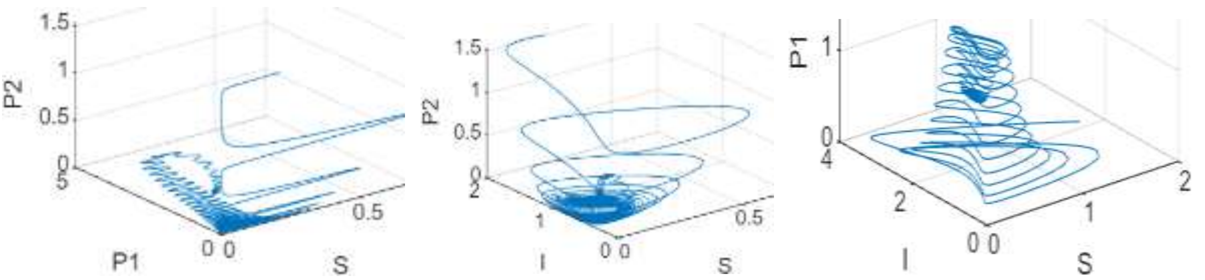
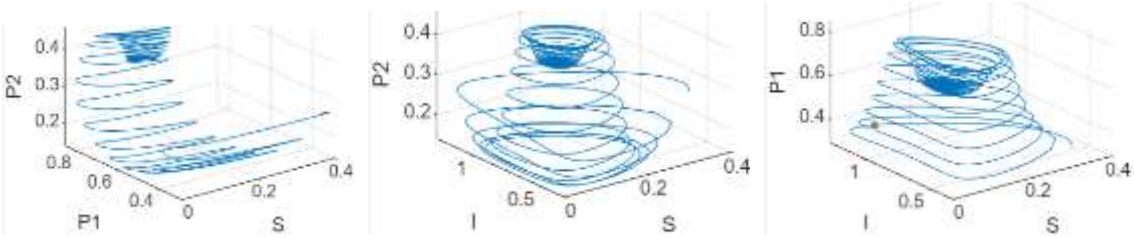


Figure 1

In Figure 2 it is seen that in the 3-D projection of the mathematical model, the trajectories have quasiperiodic solution. Thus, we concluded that the system has no stable solution. Consider again a set of parameter values under the derived analytical results and draw figure 3:

r = 0.62; λ = 0.81; a = 0.021; K = 9.6; k = 0.0235; b = 0.037; h = 0.048; c = 0.0225; k_1 = 0.042; α = 0.022; d_1 = 0.025; α_1 = 0.042; β = 0.014; h_1 = 0.026; d_2 = 0.031;



In Figure 3, in this case, also it is seen that in the 3-D projection of the mathematical model, the trajectories have a quasiperiodic solution. Thus, we concluded that the system has no stable solution.

9. Conclusion: The proposed and analyzed model for four varieties namely, susceptible phytoplankton, infected phytoplankton, predator zooplankton, and second small predator. The phytoplankton grows according to logistic growth. We analyzed in this paper the existence of equilibria. We have analyzed the stability of the model. we have shown that the eigenvalues are positive and negative, therefore E_0 also shows saddle bifurcation. The equilibrium $E_1 \equiv (K, 0,0,0)$ is asymptotically stable if $K < \frac{h}{\lambda}, K < \frac{d_1}{c}$. f the conditions $K < \frac{h}{\lambda}, K < \frac{d_1}{c}$ violates then this equilibrium E_1 become a saddle point. The equilibrium $E_3(S^*, I^*, P_1^*, 0)$ is feasible if $P_1^* > 0$ and $\frac{K(rKc+rd_1+a\lambda Kk)}{rKc+\lambda c+rk+a\lambda Kk} < \min\left(\frac{d_1}{k_1}, \frac{\lambda d_1 - ch}{k\lambda}\right)$. We also analyzed the stability of the system when the predators are assumed to be zero. Then the equilibrium $E_3(S^*, I^*, P_1^*, 0)$ is asymptotically stable under certain conditions. Lastly, we analyzed the existence of positive equilibrium points under certain feasible conditions. Hence, we conclude that the non-linear dynamical model is bounded and has local behavior about E_0, E_1, E_3. The equilibrium E_0 also shows bifurcation. The existence of positive non-zero equilibrium is obtained. A numerical simulation of the derived results is tried to analyze under a certain range of parameters. The range of parameters for global stability of the non-zero equilibrium is found analytically.

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