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International Journal of Reproducing Kernels; One Century since Bergman, Szegö and Bochner

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100 years ago in summer 1922 two young mathematicians defended their Ph.D. theses at the university in Berlin, Gábor Szegö and Stefan Bergmann (later: Bergman). Szegö at that time had published a couple of papers already and defended his second thesis (habilitation thesis), [3], while Bergman, an assistant with Richard von Mises, [4], p. 472, had just started his publications in this particular year.

Both have started the theory of Reproducing Kernels. It might just have been a case of coincidence that this has happened at the same time and the same location. Probably it was more than that. In the same summer a third Ph.D. thesis was defended at the Berlin university, also on orthogonal systems of analytic functions from Salomon Bochner, [5]. This thesis was influenced by the Bergman thesis and by a shortly before published Szegö paper [1].

In fact three papers of G. Szegö, [1, 2, 3] were devoted to the subject of expanding functions in series of orthogonal polynomials. The third one is his habilitation thesis, but [2] is the one where the kernel function appears explicitly. Szegö is introducing orthogonal polynomials on a rectifiable curve in the complex plane. In case of the unit circle these are just the powers of the variable z, in case of the segment $[-1, 1]$ on the real line the Legendre polynomials. Any regular analytic function inside the curve is expanded in a series of these orthogonal polynomials. The kernel function is developed. Also the relations with the conformal map of the inner and the outer domain of the curve is discussed. The Szegö kernel developed to a popular topic in the theory of several complex variables, G. Szegö became a central figure in the theory of orthogonal polynomials [8]. He has reviewed several of his early publications in Fortschritte der Mathematik and these reviews can be read in Zentralblatt Mathematik which is online accessible. By the way he mentions in a footnote to [2] that he was inspired to his investigations on expanding functions in series of orthogonal polynomials by the Ph.D. thesis [9].

While G. Szegö started with polynomials on curves St. Bergman was considering polynomials in bounded domains with piecewise smooth boundaries. In his thesis the reproducing kernels for analytic functions in the domain were also utilized for harmonic functions and thus extended to three dimensions. Richard von Mises has assigned the topic to Stefan Bergmann, who became his first Ph.D. student. At that time Bergmann was replacing von Mises' assistant who was on maternity leave, [10]. Obviously, the topic was influenced through the presence of Gábor Szegö. Also Ludwig Bieberbach had his impact on the Bergman thesis. Erhard Schmidt acted as expert of the three mentioned theses. In his own thesis [11], published in [12], E. Schmidt has introduced the orthonormalization (Gram-Schmidt) process. This thesis was advised by David Hilbert in connection with his basic work on integral equations [13], where kernel functions are an essential part. The Hilbert-Schmidt works further on have originated the theory of functional analysis.

Nowadays Bergman's kernel function is still a central figure in publications in complex and Clifford analysis, in particular on analytic function (Hardy, Bergman, Fock) spaces. For relations to harmonic (Green, Neumann, Robin) functions and application to pdes see e.g. [6, 7].

Bergman, Szegö and some of their colleagues from Berlin university made to escape from Germany and ended up at Stanford University, [14].

The theory of reproducing kernels is very fundamental, beautiful and has many applications in analysis and numerical analysis. Reproducing kernels are as important and central in analysis as is the Pythagorean theorem in geometry.

G. Szegö and St. Bergman gave great impacts in complex analysis. In this one century, the theory was expanded very widely and deeply through the general theory of reproducing kernels by N. Aronszajn [15] and its essence will be seen in the first paper of the first volume:

1. **Tsutomu Matsuura and Saburou Saitoh:** What is a Reproducing Kernel? - Some Essences for the New Journal. (63 pages).

2. **Hatem Mejjaoli:** Deformed Stockwell Transform and Applications

on the Reproducing Kernel theory. (39 pages).

3. **Heinrich Begehr, Hanxing Lin, Hua Liu and Bibinur Shupeyeva:** Harmonic Neumann Function for a Planar Circular Rectangle - a case study. (33 pages).

4. **Elsabetta Barletta, Sorin Dragomir and Francesco Esposito:** Kostant-Souriau-Odzijewicz Quantization of a Mechanical System Whose Classical Phase Space is a Siegel Domain. (22 pages).

In the first paper, some general essences are introduced:

1. Introduction and Some Global Viewpoint on Reproducing Kernels
2. What Is a Reproducing Kernel ?
3. What Is the Theory of Reproducing Kernels ?
4. Why Reproducing Kernels are Fundamental and Important ?
5. Generalized Reproducing Kernels and Generalized Delta Functions
6. Probability Theory and Support Vector Machines
7. Random Fields Estimations
8. Differential Equations and Integral Equations
9. Green's function, Delta Function, Differential Equations and Reproducing Kernels
10. General Nonlinear Transforms and Reproducing Kernels
11. Eigenfunctions, Initial Value Problems and Reproducing Kernels
12. Inversion Formulas for Linear Mappings
13. General Integral Transforms
14. Inversion From Many Types Data
15. The Aveiro Discretization Method
16. Representation of Inverse Functions
17. Representation of Implicit Functions
18. Best Approximations
19. The Tikhonov Regularization
20. Approximations in Sobolev Spaces by Tikhonov Regularization
21. General Inhomogeneous PDEs on Whole Spaces
22. PDEs and Inverse Problems
23. Practical Applications to Typical Inverse Problems
24. Numerical Experiments

As basic general concepts of the applications of reproducing kernels, the next topics are also referred:

General Fractional Functions - Division by Zero,
Convolutions, Integral Transforms and Integral Equations,
Operator Equations With a Parameter,
Sampling Theory, Kramer -Type Lemma and Loss Error,
Membership Problems for RKHSs,
Graphs and Reproducing Kernels,
Natural Outputs and Global Inputs of Linear Systems,
Identifications of Nonlinear Systems,
Band Preserving and Phase Retrieval,
Singular Integral Equations and Reproducing Kernels,
and other concepts in analytic function theory.

In the second paper, the generalized translation operator associated with the deformed Hankel transform on $\mathbf{R}$ is studied. Firstly, it proves the trigonometric form of the generalized translation operator and derives the positivity of this operator on a suitable space of even functions. Making use of the positivity of the generalized translation operator it studies the deformed Stockwell transform. By using reproducing kernels, it investigates this transform. In particular, it investigates some applications of the Tikhonov regularization for the generalized Sobolev spaces and it studies some time-frequency concentration problems.

In the third paper, first recall that an application of the parqueting-reflection principle leads to the harmonic Green function for a circular rectangle in the complex plane and to the Poisson kernel function. The same procedure produces the harmonic Neumann function explicitly. As the Green function it is also at the same time the Neumann function for any of the countably many domains of the parqueting of the complex plane from the reflection process starting from the original circular rectangle. This is worked out here. The Neumann function constructed is in so far particular as it fails to be symmetric in its variables due to the necessity of inserting convergence generating factors in the infinite product determining the function. This results in a difference between the piecewise continuous boundary values of the normal derivatives of the function with respect to the two variables.

In the last paper, they adopt the Kostant-Souriau-Odzijewicz quantization scheme for quantizing both the quantizable observables and the classical states of a mechanical system whose classical phase space is the Siegel domain $\Omega_n = \left\{\zeta \in \mathbb{C}^n \,:\, \mathrm{Im}(\zeta_1) > |\zeta'|^2\right\}$. They compute the transition probability amplitude $a_{0\bar{0}}(\zeta, z)$ from the state $z \in \Omega_n$ to the state $\zeta \in \Omega_n$. When the system interacts with weak external fields ϵB, $B \in L^\infty(\Omega)$, $0 < \epsilon << 1$, they show that the corresponding transition probability amplitudes are $a_{0\bar{0}}(\zeta, z) + \mathrm{O}(\epsilon)$. They refute A. Odzijewicz's assumption that the measure on phase space [associated to the reproducing kernel of $L^2H(\Omega_n, \gamma)$] should coincide, up to a multiplicative constant, with the Liouville measure.

Roughly the topics of reproducing kernel theories can be classified as:
1. One Complex Variable CV
2. Several Complex Variable SCV
3. Several Real Variables and Harmonic Functions SRVH
4. Abstract Theory and Operator Theory ATOT
5. Integral Transforms and Integral Equations ITIE
6. Kernel Methods KM
7. Probability and Statistics Theory PST
8. Numerical Analysis NA

and others.

The topics in the worldwide unique journal may be identified as the reproducing kernels. Therefore there is no problem for the unified content with the importance and great impact to mathematical science. We accept any substantial paper for reproducing kernels. Positive definite functions, concrete reproducing kernels with deep properties, and abstract theory of reproducing kernels are welcome for the journal. Random fields estimations, many topics in filtering and estimation theory in signal and image processing. Support vector machines and probability theory for data analysis, powerful computational methods for solving learning and function estimating problems such as pattern recognition, density and regression estimation and operator inversion. Statistical learning theory, as the hypothesis space in the approximation of regression functions. Those applications of reproducing kernels are welcome. Furthermore, applications to physics and other fields are, in particular, greatly welcome.

The spirit of the journal is:

fundamental, beautiful and good impact to human beings.

Love, passion and fairness

are important to the journal.

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