Int. Jr. of Contributions to Mathematics Teaching
Vol. 1 No. 1 (June, 2019)

Some Remarks on Understanding the Modern Conception of Function and its Role in Mathematics and Science

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Abstract

This paper addresses the problem of understanding modern mathematics in tertiary and upper secondary education, by discussing specific questions related to the teaching of functions. The idea of function is fundamental in contemporary mathematics and science, but its role is often obscured by blind computational routines or hidden by traditional (formalist) teaching expositions. Some of the pedagogical questions raised here concern the teaching of calculus while some other are more closely related to understanding of basic structures (such as arithmetic modulo m) behind undergraduate algebra. Present Greek university textbooks for the two first years in departments of mathematics are used in analyzing the problem without reducing, we hope, its generality and depth. Finally, a duality scheme of representation of functions, permitting unifying teaching approaches to algebra, analysis and geometry, is discussed as an alternative to formal axiomatic presentation.

Keywords: *Algebra, Calculus, Functions, Number Theory, Tertiary Mathematics Education, Textbooks at Undergraduate Level.*

MSC2000: 97D20

1 Introduction

The modern (generalized) conception of function is the cornerstone of contemporary mathematics and perhaps of the whole construction of Modern Scientific Reason. According to the philosophers Deleuze and Guattari [2], sciences in general do not deal with concepts (which are the object of philosophy), but their true objects and tools are functions, which also permit interdisciplinary communication.

Education, on the other hand, cannot be limited to an instrumental understanding or formal treatment of mathematical objects. It needs to enter into their conceptual understanding, in order to permit students a negotiation of mathematical meaning. Students are not machines to execute algorithms and have a right in conceptual understanding, even in an early phase of evolution of their mathematical thought.

In the case of functions, teaching must take into account their historical appearance and their role in science and mathematics. Also the limitations of the early (or “classical”) conception of function need to be thematized and discussed, at least in the upper secondary or the university mathematics classroom; otherwise the modern (set-theoretic) definition sounds like a meaningless declaration.

In this paper we start from an historical and epistemological distinction between the classical and modern conception of function. Then we pass to an examination of the undergraduate teaching of algebra, elementary number theory and calculus of one or many variables. The lack of understanding the modern role of functions is made evident by particular examples. Our pedagogical remarks are based on analysis of contents of textbooks following the two first years program of mathematics departments in Greece; it is a common experience that students have problems with these contents, but we do not use any selected empirical data from classroom observation or interviews. Our aim is to show the limitations that possibly follow, from the structure of contents only, on understanding the modern conception of function, despite official didactical intentions and goals.

2 From "Classical" to "Modern" Conception of Function

"The concept of function can be defined in a formal symbolic way, almost without using words", says Anna Sierpinska. "The logical sense of the concept is confined to just what this definition says (…) But at the very moment the notion is applied in a context, mathematical or mathematized, informal language is being used, and this informal language brings about meanings that transcend the mere logic of the definition." ([5], p.29, our emphasis).

Sierpinska emphasizes that the notion function acquires different meanings depending upon the contexts of use of the notion. It is our conviction, however, that, in spite of this diversity of meanings being present at the same historical time, there has been a significant historical evolution of the concept of function through the last 400 years. This historical knowledge may help understanding at the tertiary level. The need for generalizations and (new) definitions can be at least made evident to students through the limitations of the former conceptions.

There is a radical difference in Western scientific and mathematical thought, between the "era of Baroque" (17th and 18th centuries) and the 19th century. The mathematics of Fermat, Leibnitz, Euler and the Bernoullis differ from the mathematics of Galois, Riemann, Cantor and Hilbert in both epistemic assumptions and method. Sometimes the names "early modern" or "classical" are used for the former historical era, while the latter is simply called "modern", in order to denote a radical change in conception or interpretation.

This change is well represented in the evolution of the idea of function. During the 18th century functions were conceived by mathematicians as formulas linking together some variable magnitudes, or as the variable magnitudes themselves (usually depending on other variable magnitudes). This conception is still appearing in the context of "applications" of mathematics in everyday life problems.

It is very instructive to consider the change of the above conception in the writings of one and the same person, Richard Dedekind, who was one of the leading figures in the movement of Arithmetization of Analysis during 19th century. Looking, first, at Dedekind's introduction to his pamphlet with title "Continuity and Irrational Numbers" (containing the invention of Dedekind cuts) we read:

"In discussing the notion of the approach of a *variable magnitude* to a fixed limiting value, and especially in proving the theorem that every *magnitude* which *grows continually*, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences."

([1], p.1; our emphasis)

Dedekind, here, refers to his lectures at the Polytechnic School in Zurich, in the year 1858, when he first noticed "the lack", as he says, "of a really scientific foundation of arithmetic.". In this historical phase functions still appear in Analysis courses as "variable magnitudes" and particularly as "magnitudes growing continually". However, in a later text titled "The Nature and Meaning of Numbers" (which is essentially another formulation of Peano's Arithmetic), Dedekind introduces the idea of a "transformation of a system", which is in full accordance with the modern (set-theoretic) conception of function:

"By a *transformation* (*Abbildung*) φ of a system S we understand a law according to which to every determinate element s of S there *belongs* a determinate thing which is called *the transform* of s and denoted by φ(s); we say also that φ(s) *corresponds* to the element s, that φ(s) *results* or *is produced* from s by the transformation φ, or that s *is transformed into* φ(s) by the transformation φ."

([1], p.50; emphasis already present in Dover edition).

It is remarkable, here, that the conception of function is implicitly thematized by the symbolic letter "φ". This indicates a conscious choice to adopt the new conception of function as playing a role in funding arithmetic.

3 Textbooks of basic undergraduate mathematics

The mathematical content taught, concerning algebra and elementary number theory (in Greece and elsewhere) usually starts from Peano axioms (or simply mathematical induction) and proofs of simple theorems on divisibility, prime numbers, greatest common divisor etc. until the arithmetic modulo a given integer. The textbooks aiming to an introduction to (modern) algebra contain also an initial chapter with basic set-theoretical language, operations on sets and an abstract set-theoretic treatment of the notion of function, but they hardly use this notion to throw "new light" on arithmetic. The modern conception of function appears in use for the first time in formal definitions of operations in groups and rings. In this way there is a lack of understanding of its important role in restructuring arithmetic and algebra. For example, the theorem that two integers a and b are congruent modulo a positive integer m if and only if they leave the same remainder when they are divided by m appears rather as an accidental fact; the students, as textbooks readers, hardly understand that the congruence

$$a \equiv b \pmod{m}$$

can be expressed in a modern functional form, as an equality of remainders,

$$\mathrm{r}_m(a) = \mathrm{r}_m(b),$$

which are a kind of "special representatives" of integers a, b. The function

$$a \mapsto \mathrm{r}_m(a)$$

is a ring homomorphism provided that remainders are added and multiplied modulo m.

A somewhat different situation holds with textbooks of calculus, especially those concerning functions of one real variable. There are interesting examples, as the Dirichlet function, but there seems to be an overdose of formalism. Set-theoretic definitions and inverse images of functions are not essentially used in proving theorems. Sequences of real numbers are defined as functions

$$\mathbb{N} \to \mathbb{R},$$

but a function of this kind is not interesting by itself as an object of mathematical analysis, since $\mathbb{N}$ is a set of isolated points in $\mathbb{R}$. Of course such functions (but not all of them) could be useful in analytic number theory, which is a special area of advanced mathematics.

Second year calculus textbooks, treating functions of many variables, follow a more ambitious program: they are intended to provide knowledge tools for students of mathematics, physics, economics and polytechnic schools (as e.g. optimization methods on non-linear functions). Despite, again, the set-theoretical language introduced at the beginning, the exposition is mostly classical and full of routine computations. Sometimes there is an obvious orientation of examples towards 3-dimensional analytic geometry and/or the theory of surfaces. Some topological theorems on real functions of many variables are stated without proof (as, e.g. that a continuous function defined on a compact set is necessarily bounded). In this way a little “mixing” between classical and modern conceptions takes place, without offering a better understanding of either category.

Historical elements in tertiary mathematics textbooks are of a rather decorative character. There is hardly a thematization of limitations of the one or the other conception. We do find, however, some refutations of false conjectures by counterexamples.

Special attention is paid to the existence of limits of functions of two variables. A typical example, in most Greek textbooks, of a function with no limit at the point (0, 0), is

$$f(x,y)=\frac{2xy}{x^2+y^2} \;, \quad (x,\, y) \neq (0,\, 0)$$

The textbook authors stress the fact that, although there is a limit of f(x, y) as (x, y) tends to zero along any ray

y = □x (x>0 or x<0)

with l fixed, this limit depends on l and thus the $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. However, the above discussion could suggest to some students that, if the limit of a function f(x, y) is supposed to be the same along any ray ending at (0, 0), then the $\lim_{(x,y)\to(0,0)} f(x,y)$ must necessarily exist. A counterexample to this false conjecture is furnished by the function (defined in polar coordinates)

$$f(r,\theta)=\begin{cases}1, if \quad r \geq \theta \\ 0, if \; 0<r<\theta\end{cases}$$

Obviously for every θ_0 with $0<\theta_0\leq 2$□, the limit of f(*r*,□) tends to zero as *r* along the ray □= θ_0, equals zero, and yet the $\lim_{(r,\theta)\to(0,0)} f(r,\theta)$ does not exist! This is intuitively understood if we “move” towards (0, 0) along the spiral curve

r = □,

on which the function f(*r*,□) has the constant value 1.

There is, however, at least one Greek calculus textbook in which it is proved that a necessary and sufficient condition for the existence of the limit of a function f, of n variables, at a point x_0 in ¡ ^{n}, is the existence and equality of the limits of f along *all continuous curves* ending at x_0 ([4], p.47).

4 A duality scheme of representation

An exceptional textbook addressed to students as introduction to “mathematical thought”, written in Greek by C. Drossos [3], emphasizes that the notion of function is always related to the idea of change. This idea is incorporated, in the book, in a “dynamical” representation of a function

$$f: T \to A$$

in two ways, which are dual to each-other:

i) f may be represented as a “varying element” of the set A, i.e. as the family of elements $f=\left(f(t)\right)_{t\in T}$;

ii) f may be characterized by the family of sets (inverse images of one-element sets) $T_a = \{t \in T : f(t) = a\}$ $(\alpha \in A)$.

In the first representation, T may be interpreted as "time" and f(t) as a "mobile point" as t "runs over" T. In the second one, A is interpreted as a "space of states" and each set T_a $(\alpha \in A)$ expresses "the total time spent" by a system in the state *α*. Such a double scheme offers, on the one hand, a unification of the classical and modern conceptions of function and, on the other, perhaps a deeper understanding of the mathematical modeling of (natural) systems.

As far as we know, this representation of functions in a duality scheme as above has not been systematically used in tertiary mathematics teaching. The following examples are intended to give, as a conclusion, an indication of how the preceding theoretical discussion can be actually implemented in tertiary education. Suppose that the teaching target is trigonometry at university level, as a concrete synthesis of algebra, analysis and geometry (which is an alternative of formal axiomatic presentation). This amounts to teaching of just Euler's extension of the exponential function to complex values,

$$t \to e^{it} = \cos t + i \sin t \qquad (0 \le t < +\infty)$$

By applying the above "dynamical" duality scheme, we can imagine the curve

$$(e^{it})_{0 \le t < +\infty}$$

as the orbit of a material point starting to rotate on the unit circle. The dual representation of the same function, then, is:

$$T_\theta = \{\theta + 2k\pi \,/\, k = 0,1,2,...\}$$

which, for each given angle θ $(0 \le \theta < 2\pi)$, expresses the time instances in which our mobile point is at an angle θ from its original position.

Suppose, once more, that the teaching target is the theory of surfaces, and let the equation of certain surface be given as a real function

$$z = f(x, y), \quad (x,y) \in \mathbb{R} \times \mathbb{R}$$

Then, for each fixed value $\lambda \in \mathbb{R}$, the set

$$T_\lambda = \{(x,y) \in \mathbb{R} \times \mathbb{R} : f(x,y) = \lambda\}$$

is identical to the locus of points situated at a height λ from the horizontal plane.

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Note: The content of the following university textbooks has been analysed, typically representing the mathematics textbooks used in the first two years of studies in Greece:

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