

DOMINATION BY STRONG ARCS IN FUZZY GRAPHS**Dr. SELVAM.A**

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Abstract: In the present work, the concept of strong arc domination is introduced in fuzzy graphs. We have determined the strong arc domination number for some types of fuzzy graphs and we present some common bounds related to it. Further, this new domination parameter is discussed in the Cartesian product of fuzzy graphs. We have determined the strong arc domination number for Cartesian product of complete fuzzy graphs.

Keywords: fuzzy graph, strong arc, non strong arc, strong arc domination number, strong fuzzy graph, complete fuzzy graph.

1. INTRODUCTION

A Mathematical framework to describe uncertainty in real life situation was first suggested by Zadeh [1]. Rosenfeld [6] introduced fuzzy graph and several other fuzzy analogs of graph theoretic concepts such as paths, cycles and connectivity. Fuzzy graph is the generalization of the ordinary graph. Fuzzy graph model can represent a complex, imprecise and uncertain problem, where classical graph model may fail. Moreover, a fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset. Fuzzy graphs have many more applications in modeling real time systems, where the level of information inherent in the system varies with different level.

A. Somasundaram and S. Somasundaram present the concept of domination in fuzzy graphs [4]. K.R. Bhutani and A. Rosenfeld introduced the concept of strong arcs in fuzzy graphs [10]. O.T. Manjusha and Sunitha discussed some concepts in domination and total domination in fuzzy graphs using strong arcs [3]. In this paper, we present the characterization of strong arcs and the concept of domination in fuzzy graphs using strong arcs. For graph-theoretic terminology, we refer to Harary, 1969 [7].

2. PRELIMINARIES

Fuzzy graph $G = (\sigma, \mu)$ is a pair of function $\sigma: V \rightarrow [0,1]$ and $\mu: V \times V \rightarrow [0,1]$ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all u, v in V . The fuzzy graph $H(\tau, \rho)$ is called a fuzzy subgraph of $G = (\sigma, \mu)$ if $\tau(u) \leq \sigma(u)$ for all u in V and $\rho(u, v) \leq \mu(u, v)$ for all u, v in V . The underlying crisp graph of a fuzzy graph $G = (\sigma, \mu)$ is denoted by $G^* = (\sigma^*, \mu^*)$, where $\sigma^* = \{u \in V \mid \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V \mid \mu(u, v) > 0\}$. A fuzzy graph $G = (\sigma, \mu)$ is said to be bipartite if the node set V can be partitioned into two non-empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further if $\mu(v_1, v_2) > 0$ for all $v_1 \in V_1$ and $v_2 \in V_2$ then G is called complete bipartite graph. A complete bipartite graph on 'n' vertices is denoted by $F(K_n)$. An arc (u, v) of a fuzzy graph $G = (\sigma, \mu)$ is called an effective edge if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ and effective edge neighborhood of $u \in V$ is $N_e(u) = \{v \in V: \text{edge}(u, v) \text{ is effective}\}$. A path ρ in a fuzzy graph $G = (\sigma, \mu)$ is a sequence of distinct nodes $v_0, v_1, v_2, \dots, v_n$ such that $\mu(v_{i-1}, v_i) > 0$ where $1 \leq i \leq n$ and n is called the length of ρ . The strength of the path ρ is defined to be $\bigwedge_{i=1}^n \mu(v_{i-1}, v_i)$. A single vertex considers as a path of length zero. The strongest path joining any two nodes u, v is a path corresponding to maximum strength between u and v . The strength of the strongest path is denoted by $\mu^\infty(u, v)$ and it is defined by $\mu^\infty(u, v) = \sup\{\mu^k(u, v) \mid k = 1, 2, 3, \dots, \infty\}$. An arc (u, v) of a fuzzy graph $G = (\sigma, \mu)$ is called a strong arc if $\mu^\infty(u, v) = \mu(u, v)$, otherwise it is called a non-strong arc. Strong neighborhood of $u \in V$ is $N_s(u) = \{v \in$

V : $\text{arc}(u, v)$ is strong}. $N_s[u] = N_s(u) \cup \{u\}$ is the closed neighborhood of u .

Let u be a vertex in a fuzzy graph $G = (\sigma, \mu)$. The neighborhood of $u \in V$ is

$N(u) = \{v \in V / \mu(u, v) > 0\}$. The degree of the vertex u is defined to be $d(u) = |N(u)|$. A vertex u in a fuzzy graph $G = (\sigma, \mu)$ is said to be isolated if $d(u) = 0$. Let $G = (\sigma, \mu)$ be a fuzzy graph and u, v be two nodes of G . We say that u dominates v if the arc (u, v) is strong.

3. DOMINATION USING STRONG ARCS

Definition 3.1 A subset D of V is called a strong arc dominating set of $G = (\sigma, \mu)$ if for every $v \in V - D$, there exists $u \in D$ such that u dominates v . A strong arc dominating set D is called a minimal strong arc dominating set if no proper subset of D is a strong arc dominating set. The minimum cardinality taken over all minimal strong dominating set is called strong arc domination number, denoted by $\gamma_s(G)$ and the corresponding dominating set is called minimum strong arc dominating set.

Definition 3.2 A fuzzy graph $G = (\sigma, \mu)$ is said to be a strong fuzzy graph if all of its edges are strong arcs

Definition 3.3 Let v be a vertex in a fuzzy graph $G = (\sigma, \mu)$. We define $\sigma^s(v) = \max\{\mu(v, x) / x \in V\}$, that is $\sigma^s(v)$ is the maximum of the weights of the edges incident at v .

Definition 3.4 Let x be a vertex in a fuzzy graph $G = (\sigma, \mu)$ and a vertex y in $V(G)$ is called strong vertex of x if $\sigma^s(x) = \mu(x, y)$. That is strong vertex of x , $S_x = \{y \in V / \sigma^s(x) = \mu(x, y)\}$. It is evident that, for any vertex u in a fuzzy graph $G = (\sigma, \mu)$, $S_u \subseteq N_s(u)$

Theorem 3.5 Let $G = (\sigma, \mu)$ be fuzzy graph and if arc (x, y) in G such that $\sigma^s(x) = \mu(x, y)$ then (x, y) is a strong arc.

Proof: Suppose (x, y) is not strong, then there exist a path $p: x = x_0 = x_1 = x_2 = \dots = x_m = y$ such that $s_p > \mu(x, y)$. This implies $\mu(x_i, x_{i+1}) > \mu(x, y)$ for all $i = 0, 1, 2, \dots, m-1$. In particular $\mu(x_0, x_1) = \mu(x, x_1) > \mu(x, y)$. This implies $\sigma^s(x) > \mu(x, y)$. This is a contradiction.

Theorem 3.6 If (x, y) be an arc in a fuzzy graph $G = (\sigma, \mu)$ such that x is an end vertex, then it is a strong arc

Proof: Let (x, y) be an arc and x is an end vertex. Therefore $\sigma^s(x) = \mu(x, y)$. By Theorem 3.5, (x, y) is a strong arc

Theorem 3.7 If an edge is a strong arc in a fuzzy graph $G = (\sigma, \mu)$ then it is not the weakest arc of any cycle.

Proof: Let C_n be any cycle of $G = (\sigma, \mu)$ and let $x_1, x_2, \dots, x_n$ be the edges of C_n . Let (x_i, x_{i+1}) be an edge of C_n such that $\mu(x_i) < \mu(x_j)$, $i = 1, 2, \dots, n$. Now $C_n - \{x_i\}$ is a $u - v$ path in $G = (\sigma, \mu)$ and it has a strength $> \mu(x_i)$. This implies x_i cannot be a strong arc.

Theorem 3.8

Let $G = (\sigma, \mu)$ be a fuzzy graph with $|V(G)| = 2n + 1$, $n \in \mathbb{Z}$. If $G = (\sigma, \mu)$ has no isolated vertices then

there exists a vertex u in G such that $|N_s(u)| \geq 2$

Proof: Suppose each vertex u of $G = (\sigma, \mu)$ has $|N_s(u)| = 1$, then $\sum_{u \in V} N_s[u] = 2n$. Therefore $|V(G)| = \frac{1}{2} \sum_{u \in V} N_s[u] = n$. This is a contradiction.

Theorem 3.9 Let $G = (\sigma, \mu)$ be a fuzzy graph. If $N_s(u) = \emptyset$ for some $u \in V[G]$, then $id(u) = 0$.

Proof: Suppose $id(u) > 0$ then we can find an edge in $G = (\sigma, \mu)$ such that $\sigma^s(u) = \mu(u, x)$. This implies (u, x) is a strong arc and hence $N_s(u) \neq \emptyset$.

Theorem 3.10 Let u be a vertex in a fuzzy graph $G = (\sigma, \mu)$, then u dominates any vertex in S_u

Proof: Let $x \in S_u$. This implies that $\sigma^s(u) = \mu(u, x)$. Suppose u does not dominate the vertex x , then (u, x) is not a strong arc. Therefore there exists a path $\rho: u = u_0, u_1, u_2, \dots, u_n = x$ such that $s_\rho > \mu(u, x)$. This implies $\mu(u_i, u_{i+1}) > \mu(u, x)$ for all $i = 1, 2, \dots, n-1$. In particular $\mu(u_0, u_1) = \mu(u, u_1) > \mu(u, x)$. This implies $\sigma^s(u) > \mu(u, x)$ which is a contradiction. Therefore (u, x) is a strong arc. Consequently u dominates any vertex in S_u

Theorem 3.11 Let x be a non isolated vertex in a fuzzy graph $G = (\sigma, \mu)$, then the set $S_x \neq \emptyset$.

Proof: Let x be a non isolated vertex and let $N(x) = \{y \in G / \mu(x, y) > 0\}$. Since x is non isolated, $N(x) \neq \emptyset$. Let $y_k = \text{Max}\{\mu(x, y) / y \in N(x)\}$. It is clear that $\sigma^s(x) = \mu(x, y_k)$ and therefore $y_k \in S_x$

Corollary 3.12 Let $G = (\sigma, \mu)$ be fuzzy graph without isolated vertices, then $\gamma_s(G) \leq \left\lceil \frac{|V(G)|}{2} \right\rceil$

Definition 3.13 The crown graph $C_n.K_1$ is the graph obtained from a cycle C_n by attaching pendant edges at each vertex of the cycle

Theorem 3.14 Let G be a crown graph $C_n.K_1$. Then for the fuzzy graph $G = (\sigma, \mu)$, $\gamma_s(G) = n$.

Proof: By the definition of $G = (\sigma, \mu)$, it has exactly n end vertices namely $u_1, u_2, \dots, u_n$. By theorem 3.7 each end vertex u_i strongly dominates a unique vertex of C_n . Therefore $D = \{u_1, u_2, \dots, u_n\}$ is a minimal strong arc dominating set. Therefore $\gamma_s[G] = n$.

Definition 3.15 The star graph S_n of order n is a tree on n nodes with one node having a vertex of degree $n-1$ and the other $n-1$ having vertex of degree 1. That is the star graph S_n is the complete bipartite graph $K_{1,n}$

Theorem 3.16 Let G be a star graph S_n . Then for the fuzzy graph $G = (\sigma, \mu)$, $\gamma_s[G] = 1$.

Proof: By the definition of $G = (\sigma, \mu)$, there exist a vertex u in $G = (\sigma, \mu)$ such that $id(u) = n-1$ and also $G = (\sigma, \mu)$ has $n-1$ end vertices. By theorem 3.7, $G = (\sigma, \mu)$, has exactly $n-1$ strong arcs. But $G = (\sigma, \mu)$ has exactly $n-1$ edges. Therefore $D = \{u\}$ is a strong arc dominating set

4. DOMINATION IN CARTESIAN PRODUCT OF FUZZY GRAPHS

Definition 4.1 Let σ_i be a fuzzy subset of V_i and let μ_i be a fuzzy subset of X_i , $i = 1, 2$. Define the fuzzy subsets $\sigma_1 \times \sigma_2$ of V and $\mu_1 \times \mu_2$ of X as follows:

$$(\sigma_1 \times \sigma_2)(u_1, u_2) = \min\{\sigma_1(u_1), \sigma_2(u_2)\} \quad \forall (u_1, u_2) \in V$$

$$(\mu_1 \times \mu_2)((u, u_2), (u, v_2)) = \min\{\sigma_1(u), \mu_2(u_2, v_2)\} \quad \forall u \in V_1 \text{ and } \forall (u_2, v_2) \in X_2 \text{ and } (\mu_1 \times \mu_2)((u_1, w), (v_1, w)) = \min\{\sigma_2(w), \mu_1(u_1, v_1)\} \quad \forall w \in V_2 \text{ and } \forall (u_1, v_1) \in X_1, \text{ where } V = V_1 \times V_2.$$

Then the fuzzy graph $G(\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ is said to be the Cartesian product of $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$. We simply denote $G(\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ by $G_1 \times G_2$ for our convenient.

Definition 4.2 Let $G_1(\sigma_1, \mu_1)$ be a fuzzy graph with vertex set $u_1, u_2, u_3, \dots, u_m$ and $G_2(\sigma_2, \mu_2)$ be a fuzzy graph with vertex set $v_1, v_2, v_3, \dots, v_n$. Let $k_1 = \min\{\sigma_1(u_i) / i = 1 \text{ to } m\}$ and $k_2 = \min\{\sigma_2(v_j) / j = 1 \text{ to } n\}$. If $k_1 \leq k_2$ then the vertex u_s in $G_1(\sigma_1, \mu_1)$ such that $\sigma_1(u_s) = k_1$ is called m -vertex and it is denoted by u^* .

Definition 4.3 A fuzzy graph $G = (\sigma, \mu)$ is said to be a strong fuzzy graph if all of its edges are strong arcs. The number of strong arcs in $G_1 \times G_2$ is denoted by $N_s[G_1 \times G_2]$

Definition 4.4 Let (u_i, v_j) be any vertex in $P_m \times P_n$. $\sigma^s[(u_i, v_j)]$ is defined as the maximum of the weights of the edges incident at (u_i, v_j) .

Result 4.5 Let $e = (u_i, u_k)$ be an effective edge in $G_1(\sigma_1, \mu_1)$. Let v_j be a vertex in $G_2(\sigma_2, \mu_2)$ then the edge connecting the points (u_i, v_j) and (u_k, v_j) is an effective edge in $G_1 \times G_2$

Theorem 4.6 Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be fuzzy graphs and let $u^* \in G_2(\sigma_2, \mu_2)$. If there exists an edge (u_i, u_{i+1}) in $G_1(\sigma_1, \mu_1)$ such that $\mu_1[(u_i, u_{i+1})] \geq \sigma_2(u^*)$ then the edge connecting the points (u_i, u^*) , (u_{i+1}, u^*) in $G_1 \times G_2$ is a strong arc.

Proof: $\mu[(u_i, u^*), (u_{i+1}, u^*)] = \Lambda[\sigma_2(u^*), \mu_1(u_i, u_{i+1})] = \sigma_2(u^*) \dots (1)$. If u_n be any vertex adjacent to u_i in $G_1(\sigma_1, \mu_1)$, then $\mu[(u_i, u^*), (u_n, u^*)] = \Lambda[\sigma_2(u^*), \mu_1(u_i, u_n)] \leq \sigma_2(u^*) \dots (2)$. Let v_n be any vertex adjacent to u^* in $G_2(\sigma_2, \mu_2)$, $\mu[(u_i, v_n), (u_i, u^*)] = \Lambda[\sigma_1(u_i), \mu_2(v_n, u^*)]$. By definition of fuzzy graph and u^* we have $\mu_2(v_n, u^*) \leq \sigma_2(u^*) \leq \sigma_1(u_i)$. This implies $\mu[(u_i, v_n), (u_i, u^*)] = \mu_2(v_n, u^*) \leq \sigma_2(u^*) \dots (3)$. From (1), (2) and (3) $\sigma^s[(u_i, u^*)] = \sigma_2(u^*) = \mu[(u_i, u^*), (u_{i+1}, u^*)]$. Therefore, the edge connecting the points (u_i, u^*) , (u_{i+1}, u^*) in $G_1 \times G_2$ is a strong arc.

Theorem 4.7 Let $e = (u_i, u_k)$ be an effective edge in $G_1(\sigma_1, \mu_1)$. Let v_j be a vertex in $G_2(\sigma_2, \mu_2)$ then the edge connecting the points (u_i, v_j) and (u_k, v_j) is a strong arc in $G_1 \times G_2$.

Proof: By result 4.5 the edge connecting the points (u_i, v_j) and (u_k, v_j) is an effective edge. Therefore $\mu[(u_i, v_j), (u_k, v_j)] = \Lambda[\sigma(u_i, v_j), \sigma(u_k, v_j)]$. Assume $\sigma(u_i, v_j) < \sigma(u_k, v_j)$. Therefore $\mu[(u_i, v_j), (u_k, v_j)] = \sigma(u_i, v_j)$. We have $\sigma^s(u_i, v_j) \leq \sigma(u_i, v_j)$. Therefore, for any path ρ from (u_i, v_j) to (u_k, v_j) we have $s_\rho \leq \sigma(u_i, v_j)$. Also the strength of the direct path from (u_i, v_j) to (u_k, v_j) is $\mu[(u_i, v_j), (u_k, v_j)] = \sigma(u_i, v_j)$. Therefore $\mu^\infty[(u_i, v_j), (u_k, v_j)] = \sigma(u_i, v_j) = \mu[(u_i, v_j), (u_k, v_j)]$.

Corollary 4.8 If $G_1(\sigma_1, \mu_1)$ has n effective edges then $G_1 \times G_2$ has at least np_2 strong arcs.

Corollary 4.9 If $G_2(\sigma_2, \mu_2)$ has n effective edges then $G_1 \times G_2$ has at least np_1 strong arcs

Theorem 4.10 If $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ are complete fuzzy graphs then $G_1 \times G_2$ is a strong fuzzy graph.

Proof: Let $G_1(\sigma_1, \mu_1)$ be a fuzzy graph with p_1 vertices and q_1 edges. Let $G_2(\sigma_2, \mu_2)$ be a fuzzy graph with p_2 vertices and q_2 edges. Since $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ are complete, they have q_1, q_2 effective edges respectively. By corollary 4.8 & 4.9, $G_1 \times G_2$ has at least $p_1 q_2 + p_2 q_1$ strong arcs. Therefore, $Ns[G_1 \times G_2] \geq p_1 q_2 + p_2 q_1$. But $G_1 \times G_2$ has at most $p_1 q_2 + p_2 q_1$ edges. Therefore, $Ns[G_1 \times G_2] = p_1 q_2 + p_2 q_1$.

Result 4.11 Let $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ be complete fuzzy graphs with $|V(G_1(\sigma_1, \mu_1))| = m$, $|V(G_2(\sigma_2, \mu_2))| = n$. If $m < n$ then $\gamma_s(G_1 \times G_2) = \min\{\sum_{i=1}^m \sigma(x'_i)\}$ where x'_i is a vertex of $G_1 \times G_2$.

Conclusion

The concept of domination in graphs is very rich both in theoretical developments and applications. This paper presents the domination parameter using strong arcs. The strong arc domination number of some fuzzy graphs are obtained. We extend our work on this domination parameter in the Cartesian product of fuzzy graphs and we determined the strong arc domination number of Cartesian product of complete fuzzy graphs

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