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# The Development and Implementation of a Numerical Integrator for the Study of Autonomous IVP Models on Climate Change

Enoch O. O[1], Adeniji A. A[2*], Ngungu M[3], Akofa C. O[1]
opeyemi.enoch@fuoye.edu.ng, adejimi.adejimi.adeniji@gmail.com, mngungu@hsrc.ac.za

[1]Department of Mathematics, Federal University Oye-Ekiti, Ekiti State, Nigeria
[2]Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria, South Africa.
[3]HSRC (Human Sciences Research Council), Pretoria, South Africa.
* Corresponding Author

***Abstract*-We present a class of numerical schemes that are derived from a perturbated interpolant that has a varying order polynomial as the assumed solution to the Ordinary Differential Equation (ODE) models on climate change. These numerical integrators are capable of solving problems arising from chemical kinetics, population models, mechanical oscillations, planetary motions, electrical networks, nuclear reactor control, tunnel switching problems, reversible enzyme kinetics. But in this work, we desire to apply them to the numerical solutions of Autonomous Initial Value Problems (IVPs) in ODEs on climate change. At the end we conduct a numerical experiment, the resulting methods, algorithms, and solutions will be predictive tools for the study of climatic change models with applications to big data.**

*Index Terms - Integrator, Rational Function, Singularity, Environmental, Climate Change, Ordinary Differential Equation, Algorithm, Prediction, Autonomous Initial Value Problems, and Mitigation.*

## 1. INTRODUCTION

Climate change presents the greatest environmental challenge of our time, with scientific evidence pointing to a more uncertain future (IPCC, 2014). While field observations have provided a longtime series of historical data, enabling the understanding of past climate changes, modern modeling techniques and projections are providing renewed insights into the predictions of future climate behavior. This is particularly useful in informing appropriate adaptation and mitigation strategies for our scientific community, policymakers, and other life support systems.

Climate models analyze long time spans of weather. They predict how average conditions will change in a region over the coming decades. The change is premised on scenario paths presented by different human adaptive and mitigation approaches. For instance, a business-as-usual scenario takes effect when present damaging effects continue unabated (IPCC, 2014). Other scenarios assume behavior change that leads to halving or reduction by a certain degree like the emissions causing damaging effects to the recent climate change (Shongwe *et. al.*, 2011, [2]). In all these cases and particular scenarios lead to specific projection outcomes which could either be favorable or more damaging to the ecosystem and human livelihoods.

Modelling and predictions of weather and climate are not new, they were initiated in the early sixties of the 20$^{th}$ Century (Mihailović, *et. al.*, 2014, [1]). Climate change models of first-order ODE with a known initial condition is referred to as an Initial Value

Problem (IVP). A goal motivated by practical demands in linear/ non-linear climate change models is to develop analytical and numerical solutions for independent or autonomous IVPs and discuss the implementation.

In this work, we consider a class of formulae for the numerical solution of,

$$y^{"} = f(x,y);\ y(x) = y \qquad (1)$$

in which the underlying interpolant is a rational function. This is in contrast with the classical formulae which are in general based on polynomial approximation (Enoch, *et.al.,* 2011, 2017; [2,3]). With these formulae, the solution of (1) is locally represented by an interpolant that can possess a simple pole thus affording an improved numerical solution of (1). Two of the parameters in the scheme are intended to control the position and the nature of singularity. The values of these parameters are automatically chosen and revised during the computation (Enoch, *et.al.*, 2011; [1]). Thus, the method produces an algorithm based on a local interpolant which is automatically adjusted to suit the needs of the particular differential equation (Bronson., 2003; [4]) whose solution is sought. The numerical estimate to the theoretical solution $y(x) \in \Re$ is desired in the finite internal $[x_0 : x_0 + x] \in \Re$. It is normally assumed that the function $f(x,y)$ is continuous with respect to $x$ and Lipchitz condition with respect to $y$ (Lambert and Shaw, 1965, 1967; [5,6]).

The significance and contribution of this study is to deliver a mathematical method and an algorithm that can be used to design a computer-friendly interface for numerically solving IVP climatic change models. This would be useful for the scientific community and policymakers in climate change mitigation.

### 1.1 AIM OF THE STUDY

The main aim of this research work is to develop mathematical methods and algorithms for numerical analysis in climate change problems. The study also discusses and tests the methods using numerical techniques of increasing the number of iterations using big data.

## 2. MATERIAL METHOD OF DERIVATION OF THE NUMERICAL INTEGRATOR

The interpolant that will be used for the generation of the numerical integrators (Aashikpelokhai., 1991; [7]), are presented below (Lambert, *et.al*.1965; [5]),

$$F(x) = R_L(x) + b\,|A + x|^N \text{ for } N \notin \{0,1,\ldots,L\} \qquad (2)$$

Such that

$R_L(x) = \sum_{n=0}^{L} a_n x^n$, where $a_n, b, A$ and $N$ , $L$ is a positive integer.

Assuming that $f(x_n) = y_n;\ f(x_{n+1}) = y_{n+1}$ (3)

and

$$\left[\frac{d^s f(x)}{dx^s}\right]_{n=x_n} = f_n^{(s-1)} \qquad . (4)$$

$S = 1, 2, \ldots, S$ where,

utilizing $y^1 = f(x,y),\ y(0) = \gamma$ (5)

$$F_n^{(S)} = \left[\frac{d^S f(s,y)}{dx^S}\right]_{X=X_n, Y=Y_n} = \left[\frac{\partial f^{(s-1)}(x,y)}{\partial x} + f(x,y)\frac{\partial f^{(s-1)}(x,y)}{\partial y}\right]_{X_n, Y_n} \qquad (6)$$

The above expressions hold provided all the derivatives concerned exist.

The eliminant of these undetermined coefficients from (2) then gives the required algorithm. If $F(x) = y(x)$ as defined by Lambert,.et.al, (1965; [4]), then $S = L + 1$, and the eliminant is found to be as presented below:

Let $F\left(x_n\right)=\sum_{j=0}^{L} a_j x_n^j + b\left|A+x_n\right|^N,\ N \notin \{0,1,\ldots,L\}$

Such that

$$F\left(x_n\right)=y_n \text{ and } F\left(x_{n+1}\right)=y_{n+1};$$
$$x_{n+1}=x+h \text{ for which } x_n=a+nh; \qquad (7)$$
$$F\left(x_{n+1}\right)-F\left(x_n\right)=y_{n+1}-y_n$$

Let $f^{(i)}$ denotes the $i^{th}$ total derivative of $f\left(x,y\right)$, $w.r.t.x$ such that

$$F^{(1)}\left(x_n\right)=f\left(x_n,y_n\right)=f_n \qquad (8)$$

$$F^{(2)}\left(x_n\right)=f^{(2)}\left(x_n,y_n\right)=f_n^{(1)} \qquad (9)$$

$$F^{(m)}\left(x_n\right)=f^{(m-1)}\left(x_n,y_n\right)=f_n^{(m-1)} \qquad (10)$$

$$y_{n+1}-y_n=\sum_{j=0}^{L} a_j\left[x_{n+1}^j-x_n^j\right]+b\left[\left(A+x_{n+1}\right)^N-\left(A+x_n\right)^N\right] \qquad (11)$$

### 2.1. *When L = 1 (The polynomial $P_j(x)$ is linear)*

$$F\left(x_n\right)=a_0+a_1x_n+b\left(A+x_n\right)^N \qquad (12)$$

In this case

$$a_1=f\left(x_n,y_n\right)-Nb\left(A+x_n\right)^{N-1} \qquad (13)$$

and

$$b=\frac{\left(A+x_n\right)^2 f_n^{(2)}}{N\left(N-1\right)\left(A+x_n\right)^N} \qquad (14)$$

With these, we derive the numerical integrator:

$$y_{n+1}=y_n+hf_n+\frac{\left(A+x_n\right)^2 f_n^{(1)}}{N\left(N-1\right)}\left[\left(1+\frac{h}{A+x_n}\right)^N-1-\frac{Nh}{A+x_n}\right] \qquad (15)$$

### 2.2. When L=2 (The polynomial $P_j\left(x\right)$ is a quadratic)

Differentiate to eliminate the undetermined coefficients

$$b=\frac{\left(A+x_n\right)^3 f_n^{(2)}}{N\left(N-1\right)\left(N-2\right)\left(A+x_n\right)^N} \qquad (16)$$

And,

$$a_2=\frac{1}{2}\left[f_n^{(1)}-\frac{\left(A+x_n\right)f_n^{(2)}}{\left(N-2\right)}\right] \qquad (17)$$

We also see that;

$$a_1=f_n-\left\{x_nf_n^{(1)}-x_n\frac{\left(A+x_n\right)f_n^{(2)}}{\left(N-2\right)}+\frac{\left(A+x_n\right)^3 f_n^{(2)}}{\left(N-1\right)\left(N-1\right)}\right\} \qquad (18)$$

and as such the integrator is given as;

$$y_{n+1}-y_n=\sum_{K=1}^{L}\frac{h^k}{k!}f_n^{(k-1)}+\frac{\left(A+x_n\right)^{L-1}}{\alpha_L^N}f_n^L\left[\left(1+\frac{h}{A+x_n}\right)^N-1-\sum_{K=1}^{L=N}\frac{K-1}{K!}\left(\frac{h}{A+X_n}\right)\right] \qquad (19)$$

### 2.3. *When L = 3 (The polynomials is order 3)*

Having observed all the steps that were applied in the first two cases, one sees that the undetermined coefficients are determined as follow:

$$b=\frac{f_n^{(3)}\left(A+x_n\right)^4}{N\left(N-1\right)\left(N-2\right)\left(N-3\right)\left(A+x_n\right)^N} \qquad (20)$$

and

$$a_3=\frac{1}{6}\left\{f_n^{(2)}-\frac{\left(A+x_n\right)f_n^{(3)}}{\left(N-3\right)}\right\} \qquad (21)$$

Such that:

$$a_2=\frac{1}{2}\left\{f_n^{(1)}-\left[x_nf_n^{(2)}-\frac{x_n\left(A+x_n\right)}{\left(N-3\right)}f_n^{(3)}+\frac{\left(A+x_n\right)^2 f_n^{(3)}}{\left(N-2\right)\left(N-3\right)}\right]\right\} \qquad (22)$$

In the same vein, one obtained

$$a_1=f_n-x_n f_n^{(1)}+\left(\frac{1}{2}x_n^2\right)f_n^2\left\{\left(x_n-\frac{\left(A+x_n\right)}{(N-1)}\right)\left(\frac{\left(A+x_n\right)}{(N-2)}\right)-\frac{x_n^2}{2}\right\}\left[\left(\frac{\left(A+x_n\right)}{(N-3)}\right)f_n^{(3)}\right] \quad (23)$$

### 2.4. RESULT ON THE GENERALIZATION OF THE NUMERICAL SCHEME

And at its end, we have the integrator of order three as:

$$y_{n+1}-y_n=\sum_{k=1}^{L}\frac{h^k}{k!}f_n^{(k-1)}+\frac{\left(A+x_n\right)^{L+1}}{\alpha_L^N}f_n^{(1)}\left\{\left(\left[1+\frac{h}{A+x_n}\right]\right)^N-1-\sum_{k=1}^{L}\frac{\alpha_{k-1}^N}{k!}\left(\frac{h}{A+x_n}\right)^k\right\} \quad (24)$$

where

$$\alpha_L^N=N\left(N-1\right)\left(N-2\right)\left(N-3\right),\ \text{for } L=3 \quad (25)$$

and

$$\Sigma\alpha_{k-1}^N=Nh+N\left(N-1\right)+N\left(N-1\right)\left(N-2\right),\ \text{for } k=3 \quad (26)$$

## 3. RESULT ON THE CONSISTENCY OF THE DERIVED SCHEME

$$\phi\left(x_n,y_n;0\right)=f\left(x_n,y_n\right) \quad (27)$$

If put $h=0$

$$y_{n+1}=y_n+\sum_{k=1}^{L}\frac{0^k}{k!}f_n^{(k-1)}+\frac{\left(A+x_n\right)^{L+1}}{\alpha_L^N}f_n^{(L)}\left\{1^N+0-1-0\right\} \quad (28)$$

$$y_{n+1}=y_n\Rightarrow f\left(x_n,y_n\right) \quad (29)$$

$$y_{n+1}\Rightarrow f\left(x_n,y_n\right) \quad (30)$$

## 4. RESULT ON THE LOCATION AND NATURE OF THE POINT OF SINGULARITY

To derive $A(n)$ and $N(n)$, we make use of the Taylor series expansion of (29). This gives the following expression for the truncation error(Lambert and Shaw, B. 1965; [4]):

$$T.E=y_{n+1}-y\left(x_{n+1}\right) \quad (31)$$

$$T.E=\sum_{q=1}^{\infty}\left[-f_n^{(L+q)}+\frac{\alpha_{q-1}^{N-L-1}}{\left(A+x_n\right)^q}f_n^{(L)}\right] \quad (32)$$

$$T_q=-f_n^{(L+q)}+\frac{\alpha_{q-1}^{N-L-1}}{\left(A+x_n\right)^q}f_n^{(L)} \quad (33)$$

The values of the parameters $A(n)$ and $N(n)$ are now chosen to satisfy $T_1=T_2$ (24)

So that :

$$N\left(n\right)=\left(L+1\right)\frac{\left[\left(f_n^{(L+1)}\right)^2\right]}{\left[\left(f_n^{(L+1)}\right)^2 f_n^{(L)} f_n^{(L+2)}\right]} \quad (34)$$

and

$$-A\left(n\right)=x_n-\frac{\left[\left(f_n^{(L+1)}\right)\right]f_n^{(L)}}{\left[\left(f_n^{(L+1)}\right)^2-f_n^{(L)}f_n^{(L+2)}\right]} \quad (35)$$

In the above derivation, $N(n)$ is the nature of singularity and $A(n)$ is the location of singularity (Lambert and Shaw, B. 1965; [5]).

## 5. RESULT ON THE REGION OF ABSOLUTE STABILITY OF THE SCHEME

$$y_{n+1}=y_n+h\left\{(B+C)f_n^L+\sum_{k=1}^{L}Df_n^{(L)}+Af_n^{(L)}+\sum_{k=1}^{L}Df_n^{(k-1)}\right\} \quad (36)$$

Let

$f(x,y)=\lambda y$, then

$$f_{(x,y)}^{(1)}=\lambda^2 y,\ldots,\ f_{(x,y)}^{(L)}=\lambda^{l+1}y$$

and

$$f_{(x,y)}^{(k)}=\lambda^{k+1}y,\ f_{(x,y)}^{(k-1)}=\lambda^{k}y$$

$$=y_n+\lambda^{L+1}(B+C)hy_n+hy_n\lambda^{L+1}\sum_{k=1}^{L}D+hy_n\lambda^k\sum_{k=1}^{L}A \quad (37)$$

Let $\lambda h=z$

$$y_{n+1}=\left[1+\left(B+C+\sum_{K=1}^{L}D\right)z\lambda^L+z\lambda^{k-1}\sum_{K=1}^{L}A\right]y_n \quad (38)$$

$$\frac{y_{n+1}}{y_n}=\left[1+\left(Q\lambda^L+G\lambda^{k-1}\right)z\right]=\mu(z) \quad (39)$$

Let

$$\left(Q\lambda^L+G\lambda^{k-1}\right)=\lambda\left(Q\lambda^{L-1}+G\lambda^{k-2}\right)$$

$$p=\frac{1}{h}\left(Q\lambda^{L-1}+G\lambda^{k-2}\right)z$$

$$\frac{y_{n+1}}{y_n}=\mu(z)=1+pz^2 \quad (40)$$

Putting $z=u+iv$

$$\mu(u+iv)=1+p(u+iv)^2 \quad (41)$$

$$|u(u+iv)|=\sqrt{\left(1+p\left(u^2-v^2\right)\right)^2+(2puvi)^2}$$
$$\sqrt{\left(1+p\left(u^2-v^2\right)\right)^2-4p^2u^2v^2} \quad (42)$$

If $|\mu(z)|<|$, then it follows that

$$\left(\left(1+pu^2-pv^2\right)^2-4u^2v^2p^2\right)^{1/2}<1 \quad (43)$$

The above equation is the RAS of the scheme is $\{z:|\mu(z)|\langle 1\}$.

## 6. DISCUSSION AND IMPLEMENTATION OF THE INTEGRATOR TO SYSTEMS OF INITIAL VALUE PROBLEMS

We shall make use of computer aided software in this implementation (Dahlquist and Bjorck,(1974); [8])

**6.1.**

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}=\begin{vmatrix} 0 & 1 \\ -\left(\frac{\beta^2}{x^2}\right) & -\left(\frac{1}{x^2}\right) \end{vmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y_1\left(e^2\right)=1 \text{ and } y_2\left(e^2\right)=\frac{\lambda}{e^2}, A=B=1$$

$$e^2<x<9, h=0.1,$$

where $\beta$ is a real constant.

The general solution to this differential equation is:

$$y_1(x) = A\sin(\beta.\log x) + B\cos(\beta.\log x)$$

$$y_2(x) = \beta\left[ACos(\beta.\log x) + B\cos(\beta.\log x)\right]/x$$

The numerical solution (Ibijola. 2006; [9]), to example 6 was obtained in the interval $e^2 \le x \le 9$ using uniform mesh size h=0.1. Each of the real numbers A, B and $\beta$ was set to one to give the initial value problem (Fatunla., 1988; [10]) and the parameters were obtained to six decimal places. Details of the experimental results are presented below. When this class of Integrators was used on scalar initial value problems, it was discovered that anytime the values of x and A are equal, it signals a singularity.

**Tab.6.1.** The Numerical Solutions of $y_{1,n}$ ,*A*, *N and the Truncation Error*

| *x* | *N* | *A* | $y_{1,n}$ | $T_{n+1,1}$ |
|---|---|---|---|---|
| 7.3890561 | 0.4686608 | -1.7871529 | 1.0000000 | 0.00000 |
| 7.4890561 | 0.4462486 | -1.6109432 | 1.0413273 | 0.52467 |
| 7.5890561 | 0.4246877 | -1.4393799 | 1.0802870 | 0.56521 |
| 7.6890561 | 0.4038547 | -1.2686623 | 1.1168977 | 0.52656 |
| 7.7890561 | 0.3836360 | -1.0980135 | 1.1511833 | 0.43429 |
| 7.8890561 | 0.3639247 | -0.9266579 | 1.1831711 | 0.28928 |
| 7.9890561 | 0.3446179 | -0.7537964 | 1.2128918 | 0.12429 |
| 8.0890561 | 0.3256132 | -0.5785795 | 1.2403792 | 0.07340 |
| 8.1890561 | 0.3068057 | -0.4000722 | 1.2656697 | 0.28838 |
| 8.2890561 | 0.2880840 | -0.2172096 | 1.2888020 | 0.51500 |
| 8.3890561 | 0.2693245 | -0.0287342 | 1.3098170 | 0.75915 |
| 8.4890561 | 0.2503841 | 0.1668964 | 1.3287572 | 1.01543 |
| 8.5890561 | 0.2310889 | 0.3716535 | 1.3456666 | 1.27425 |
| 8.6890561 | 0.2112150 | 0.5881757 | 1.3605905 | 1.54113 |
| 8.7890561 | 0.1904553 | 0.8201953 | 1.3735754 | 1.80920 |
| 8.8890561 | 0.1683542 | 1.0733758 | 1.3846685 | 2.08332 |
| 8.9890561 | 0.1441603 | 1.3571882 | 1.3939178 | 2.36462 |
| 9.0000000 | 0.0000002 | 1.0000210 | 1.4013717 | 2.64572 |

**Tab.6.2**. Numerical Solution of $y_{2,n}$ , with Uniform mesh-size *h* = 0.1

| *x* | *N* | *A* | $y_{2,n}$ | $T_{n+1,1}$ |
|---|---|---|---|---|
| 7.3890561 | 2.1071631 | -15.1163220 | 0.4251683 | 0.00000 |
| 7.4890561 | 1.7079582 | -12.2370060 | 0.4014065 | 1.88466 |
| 7.5890561 | 1.4630608 | -10.4633550 | 0.3778151 | 2.94642 |
| 7.6890561 | 1.2921509 | -9.2188272 | 0.3544389 | 3.73323 |
| 7.7890561 | 1.1634728 | -8.2755737 | 0.3313180 | 4.35879 |
| 7.8890561 | 1.0616019 | -7.5228248 | 0.3084884 | 4.86440 |
| 7.9890561 | 0.9780172 | -6.8995709 | 0.2859818 | 5.29862 |
| 8.0890561 | 0.9075732 | -6.3688757 | 0.2638267 | 5.66065 |
| 8.1890561 | 0.8469524 | -5.9069497 | 0.2420478 | 5.98193 |

| $x$ | $N$ | $A$ | $y_{2,n}$ | $T_{n+1,1}$ |
|---|---|---|---|---|
| 8.2890561 | 0.7939055 | -5.4976599 | 0.2206671 | 6.26031 |
| 8.3890561 | 0.7468446 | -5.1296142 | 0.1997035 | 6.49615 |
| 8.4890561 | 0.7046117 | -4.7944973 | 0.1791732 | 6.70459 |
| 8.5890561 | 0.6663388 | -4.4860675 | 0.1590902 | 6.89405 |
| 8.6890561 | 0.6313598 | -4.1995259 | 0.1394661 | 7.05444 |
| 8.7890561 | 0.5991532 | -3.9311043 | 0.1203106 | 7.19040 |
| 8.8890561 | 0.5693035 | -3.6777870 | 0.1016315 | 7.30976 |
| 8.9890561 | 0.5415739 | -3.4371180 | 0.0834346 | 7.40727 |
| 9.0000000 | 0.5214820 | -3.1105610 | 0.0657246 | 7.48581 |

**Table.6.3**. This is a Comparative Analysis of the truncation errors of the New Formula (24) when the order of the polynomial is two with the works of Krogh[8] and G-B-S[6,7] with order $6 \le k \le 8$.

| | | Krogh[8] | Formula (24) $m = 2$, $h = 0.05$ | G-B-S[6,7] $6 \le k \le 8$ | Krogh[8] | Formula (24) $m = 2$ $h = 0.05$ | G-B-S[6, 7] $6 \le k \le 8$ |
|---|---|---|---|---|---|---|---|
| $x$ | $h$ | | $T_{n+1,1}$ | | | $T_{n+1,2}$ | |
| 7.38905610 | 0.00625 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 7.48905610 | 0.01250 | 0.32222 | 0.33077 | 0.21529 | 0.28893 | 0.37823 | 0.61680 |
| 7.58905610 | 0.02500 | 0.29030 | 0.30289 | 0.32175 | 0.29546 | 0.44596 | 0.95133 |
| 7.68905610 | 0.05000 | 0.26328 | 0.27497 | 0.18698 | 0.30217 | 0.49852 | 1.28228 |
| 7.78905610 | 0.05000 | 0.24353 | 0.25029 | 0.43699 | 0.31370 | 0.54598 | 1.61252 |
| 7.88905610 | 0.05000 | 0.21596 | 0.21396 | 0.72587 | 0.31750 | 0.57941 | 1.92924 |
| 7.98905610 | 0.05000 | 0.19991 | 0.18873 | 1.03045 | 0.33042 | 0.61756 | 2.24703 |
| 8.08905610 | 0.05000 | 0.17667 | 0.15524 | 1.36791 | 0.33276 | 0.64134 | 2.54490 |
| 8.18905610 | 0.05000 | 0.15350 | 0.12203 | 1.72773 | 0.33537 | 0.66269 | 2.83207 |
| 8.28905610 | 0.05000 | 0.13606 | 0.09028 | 2.10128 | 0.34482 | 0.68845 | 3.11458 |
| 8.38905610 | 0.05000 | 0.11214 | 0.05103 | 2.49607 | 0.34691 | 0.70443 | 3.37703 |
| 8.48905610 | 0.05000 | 0.09748 | 0.01875 | 2.89631 | 0.35387 | 0.73263 | 3.63112 |
| 8.58905610 | 0.05000 | 0.07569 | 0.01799 | 3.31714 | 0.35153 | 0.73162 | 3.86164 |
| 8.78905610 | 0.05000 | 0.03985 | 0.08706 | 3.74680 | 0.36125 | 0.75029 | 4.08938 |
| 8.88905610 | 0.05000 | 0.02014 | 0.15622 | 4.16872 | 0.35339 | 0.75037 | 4.28416 |

**6.2.** The Van der Pol Oscillator with unknown theoretical solutions

$$y_1' = y_2, y_1(0) = 0$$

$$y_2' = 0.01(1 - y_1^2)y_2 - y_1, y_2(0) = 1$$

**Tab.6.4.** This is a Comparative Analysis of the numerical Solutions of the New Formula (24).

| | $y_1$ | | | $y_2$ | | |
|---|---|---|---|---|---|---|
| $x$ | Krogh[8] | Formula(24) | G-B-S[6, 7] | Krogh[8] | Formula (24) | G-B-S[6, 7] |

| | $y_1$ | | | $y_2$ | | |
|---|---|---|---|---|---|---|
| $x$ | Krogh[8] | Formula(24) | G-B-S[6, 7] | Krogh[8] | Formula (24) | G-B-S[6, 7] |
| 0.0 | 0.00000000 | 0.00000000 | 0.00000000 | 1.00000000 | 1.00000000 | 1.00000000 |
| 0.6 | 0.56624448 | 0.56624448 | 0.56624449 | 0.83005702 | 0.83005702 | 0.83005701 |
| 1.2 | 0.93663469 | 0.93663469 | 0.93663469 | 0.36677933 | 0.36677933 | 0.36677931 |
| 1.8 | 0.98015691 | 0.98015691 | 0.98015691 | -0.22611777 | -0.22611779 | -0.22611779 |
| 2.4 | 068113087 | 068113086 | 068113085 | -0.74094524 | -0.74094526 | -0.74094526 |
| 3.0 | 0.14270081 | 0.14270079 | 0.14270077 | -1.00026329 | -1.00026329 | -1.00026330 |
| 3.6 | -0.44872180 | -0.44872183 | -0.44872184 | -0.91138071 | -0.91138069 | -0.91138076 |
| 4.2 | -0.88581760 | -0.88581762 | -0.88581763 | -0.50098681 | -0.50098678 | -0.50098676 |
| 4.8 | -1.01403600 | -1.01403600 | -1.01403600 | 0.08658083 | 0.08658086 | 0.08658089 |
| 5.4 | -0.78798033 | -0.78798032 | -0.78798029 | 0.64443653 | 0.64443556 | 0.64443659 |
| 6.0 | -0.28565527 | -0.28565523 | -0.28565519 | 0.98012921 | 0.98012923 | 0.98012924 |

**6.3.** Considering the system of ODE,

$$x'(t) = x + 2y, \quad y'(t) = 3z + 2y;$$

$$x(0) = 6, \mathrm{y}(0) = 4.$$

The theoretical solutions are:

$$x(t) = 4e^{4t} + 2e^{-t};$$

$$y(t) = 6e^{4t} + 2e^{-t}.$$

**Tab.6.5.** The numerical Solutions of the Formula (24) as it is compared with the Actual Solution.

| $k$ | $h$ | $x(t_k)$ | Formula (24) $x_k$ | $T_{k+1,1}$ | $y(t_k)$ | Formula (24) $y_k$ | $T_{k+1,2}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.00 | 6.00000000 | 6.00000000 | 0.00000000 | 4.00000000 | 4.00000000 | 0.00000000 |
| 1 | 0.02 | 6.29354551 | 6.29354548 | 0.00000003 | 4.53932490 | 4.53932489 | 0.00000001 |
| 2 | 0.04 | 6.61562213 | 6.61562209 | 0.00000004 | 5.11948599 | 5.11948594 | 0.00000005 |
| 3 | 0.06 | 6.69852528 | 6.69852512 | 0.00000016 | 574396525 | 574396510 | 0.00000015 |
| 4 | 0.08 | 7.35474319 | 7.35474317 | 0.00000002 | 6.41653305 | 6.41653303 | 0.00000002 |
| 5 | 0.10 | 7.77697287 | 7.77697284 | 0.00000003 | 7.14127221 | 7.14127217 | 0.00000004 |
| 6 | 0.12 | 8.23813750 | 8.23813748 | 0.00000002 | 7.92260406 | 7.92260404 | 0.00000002 |
| 7 | 0.14 | 8.74140523 | 8.74140515 | 0.00000008 | 8.76531667 | 8.76531664 | 0.00000003 |
| 8 | 0.16 | 9.29020955 | 9.29020952 | 0.00000003 | 9.67459538 | 9.67459537 | 0.00000001 |
| 9 | 0.18 | 9.88827138 | 9.88827133 | 0.00000005 | 10.6560560 | 10.6560558 | 0.00000002 |
| 10 | 0.20 | 10.5396230 | 10.5396229 | 0.00000001 | 11.7157807 | 11.7157807 | 0.00000000 |

**6.4.** Considering the system of ODE,

$$x''(t) = \frac{2t}{1+t^2} x'(t) - \frac{2}{1+t^2} x(t) + 1;$$

$$x(0) = 1.25, x'(0) = 1.$$

Theoretical Solution is:

$$x(t) = 1.25 + 0.4860896526t - 2.25t^2 + 2\tan^{-1} t$$
$$- 0.5In\left(1+t^2\right) + 0.5t^2 In\left(1+t^2\right)$$

This second order initial value problem can be transformed into a matrix as:

$$\begin{bmatrix} x_2' \\ x_1' \end{bmatrix} = \begin{bmatrix} \frac{2t}{1+t^2} & -\frac{2t}{1+t^2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

**Tab.6.6**. The numerical Solutions of the Formula (24) as it is compared with the actual solution.

| $t_k$ | $x_1(t_k)$ | Formula (24) $x_{1k}$,$h = 0.2$ | $T_{k+1,1}$ | $x_2(t_k)$ | Formula (24) $x_{2k}$,$h = 0.2$ | $T_{k+1,2}$ |
|---|---|---|---|---|---|---|
| 0.0 | 1.250000 | 1.250000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 0.2 | 1.220131 | 1.220131 | 0.000000 | 0.097177 | 0.097173 | 0.000004 |
| 0.4 | 1.132073 | 1.132070 | 0.000003 | 0.194353 | 0.194350 | 0.000003 |
| 0.6 | 0.990122 | 0.990120 | 0.000002 | 0.291530 | 0.291529 | 0.000001 |
| 0.8 | 0.800569 | 0.800565 | 0.000004 | 0.388707 | 0.388705 | 0.000002 |
| 1.0 | 0.570844 | 0.570841 | 0.000003 | 0.485884 | 0.485882 | 0.000002 |
| 1.2 | 0.308850 | 0.308850 | 0.000000 | 0.583061 | 0.583060 | 0.000001 |
| 1.4 | 0.022522 | 0.022520 | 0.000002 | 0.680237 | 0.680234 | 0.000003 |
| 1.6 | -0.280424 | -0.280424 | 0.000002 | 0.777413 | 0.777411 | 0.000002 |
| 1.8 | -0.592609 | -0.592606 | 0.000003 | 0.874591 | 0.874590 | 0.000001 |
| 2.0 | -0.907039 | -0.907033 | 0.000006 | 0.971767 | 0.971766 | 0.000001 |
| 2.2 | -1.217121 | -1.217120 | 0.000001 | 1.068944 | 1.068942 | 0.000002 |
| 2.4 | -1.516639 | -1.516636 | 0.000003 | 1.166121 | 1.166120 | 0.000001 |
| 2.6 | -1.799740 | -1.799738 | 0.000002 | 1.263297 | 1.263295 | 0.000002 |
| 2.8 | -2.060904 | -2.060904 | 0.000003 | 1.360474 | 1.360472 | 0.000002 |
| 3.0 | -2.294916 | -2.294916 | 0.000002 | 1.457651 | 1.457650 | 0.000001 |
| 3.2 | -2.496842 | -2.496842 | 0.000001 | 1.554828 | 1.554828 | 0.000000 |
| 3.4 | -2.662004 | -2.662004 | 0.000001 | 1.652004 | 1.652004 | 0.000001 |
| 3.6 | -2.785960 | -2.785960 | 0.000002 | 1.749181 | 1.749181 | 0.000003 |
| 3.8 | -2.864481 | -2.864481 | 0.000001 | 1.846358 | 1.846358 | 0.000002 |
| 4.0 | -2.893535 | -2.893535 | 0.000002 | 1.943535 | 1.943535 | 0.000004 |

The numerical result for the autonomous system is quite impressive, even at h = 0.1.

## 7. CONCLUSION

We have successfully derived a numerical integrator for the solutions of systems of initial value problems for climate change model. The results obtained compared favorably with the results obtained in literature and when these numerical solutions are subjected to numerical and statistical analysis, we shall be able to give predictive insight to climatic and environmental behaviors.

The statistical understanding of the relations amongst the parameters in respective tables, will give us insight into the significance of the models and their numerical solution. This shall be demonstrated in a subsequent work.

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